

## Example 2

### Six bidders, three bidders collude

#### **The setup**

In this example we have 6 bidders with lognormal distributions

$$LN(\mu, \sigma^2) : F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \frac{1}{u} \exp\left(-\frac{1}{2} \left(\frac{\ln u - \mu}{\sigma}\right)^2\right) du$$

truncated on [1.5,6].

Two high type bidders:  $\mu_H = 1.35, \sigma_H = 0.35$

Four low type bidders:  $\mu_L = 0.75, \sigma_L = 0.35$

Two bidders of high type (H) collude with one bidder of low type (L) against three (non-collusive) bidders of low type (L).

The concepts of baseline (b-) and hybrid (h-) cumulative distribution functions (cdfs) play an important role in the interface of the program. A baseline cdf is a distribution function of at least one bidder participating in the auction. The actual bidders are labeled h-. Hence, h-player is either an individual bidder or a coalition of bidders. A hybrid cdf is a distribution function of an h-player. A coalition valuation distribution is the product of the baseline distributions of each of the coalition members. In this example we have two b-cdfs {H, L} and two h-cdfs {(2H, 1L), 1L}.

Multiple inputs should be separated by a space.

#### **Instructions**

**(1)** Enter 1 for auctions or 2 for procurements : **1**

The options available in the program are the same for the auctions and procurements environments.

**(2)** Enter the number of h-types : **2**

There are two h-cdfs {(2H, 1L) and (1L)}.

**(3)** Enter the order of the Taylor-series expansion: **2**

We recommend orders between 2 and 5. Increasing the order of the expansion leads to an increase in computational time. On the other hand, (theoretically) higher orders may produce better approximations.

**(4) Enter the number of grid-points : 200**

Increasing the number of grid-points may increase accuracy. The gains in accuracy depend on the local behavior of the distributions.

**(5) Enter the number of players (coalitions) for each h-type : 1 3**

We have one player (coalition) of h-type (2H, 1L) and three players of h-type (1L).

**(6) Enter the (common) lower bound of the support of the private-values cdfs : 1.5**

We took [1.5,6] as the support of the distributions. The lower bound is 1.5.

**(7) Enter the (common) upper bound of the support of the private-values cdfs : 6.0**

We took [1.5,6] as the support of the distributions. The upper bound is 6.0.

**(8) Enter a reserve price R : 1.5**

The reserve price should be within the support of the distributions. If you do not want to set up the reserve price or intend to compute the optimal reserve price, enter the lower bound of the support (upper bound for the procurements).

The b-cdfs currently available in the program are indexed in the following way:

1. Two parameter Weibull

$$W(\alpha, \beta) : F(x) = 1 - \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right);$$

2. Beta

$$BT(\alpha, \beta) : F(x) = \frac{1}{B(\alpha, \beta)} \int_0^x u^{\alpha-1} (1-u)^{\beta-1} du;$$

3. Normal

$$N(\mu, \sigma^2) : F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right) du;$$

4. Lognormal

$$LN(\mu, \sigma^2) : F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \frac{1}{u} \exp\left(-\frac{1}{2}\left(\frac{\ln u - \mu}{\sigma}\right)^2\right) du.$$

**(9) Enter the number of b-cdfs : 2**

There are two b-cdfs (H and L).

**(10) Enter the index of the b-cdfs, separated by a space : 4 4**

Both b-cdfs are lognormal distributions in this example. See the above list of the b-cdfs currently available in the program.

**(11)** Enter mu and sigma : **1.35 0.35**

Enter mu and sigma : **0.75 0.35**

The two parameters characterizing the selected cdfs as defined above.

**(12)** Enter the exponents of the b-cdf for h-type 1 : **2 1**

Enter the exponents of the b-cdf for h-type 2 : **0 1**

For each h-type player, list the number of bidders of each b-type composing this player. For example, the h-type 1 player consists of two b-type (H) bidders and one b-type (L) bidder exhibiting b-cdfs that we defined in steps (10) and (11).

After step (12), the program produces the following statistics for each h-type (on the defined support):

MEAN: mean of the distribution;

ST. DEV.: standard deviation of the distribution;

LPDF: value of the density function at the lower bound;

UPDF: value of the density function at the upper bound;

CRIT: lower bound of the cdf.

LPDF and UPDF allow us to verify that the densities are bounded away from zero as required by the uniqueness condition. Values of LPDF and UPDF lower than  $10^{-14}$  (0.1D-13) and values of CRIT lower than  $10^{-8}$  (0.1D-07) are likely to cause a problem in the convergence algorithm. If you choose to keep your specification and the program converges, the solution may be inaccurate. In such case we recommend that you compute the best responses and use them to check the accuracy of the solution.

The output for the current example reads as follows:

H-TYPE	MEAN	STD. DEV	LPDF	UPDF	CRIT
1	4.37929	0.85630	0.45D-12	0.19D+00	0.32D-11
2	2.43531	0.72407	0.56D+00	0.27D-02	0.13D-02

If any of the above conditions is violated you might consider shortening the support and/or adjusting the parameters of the b-distributions using the following option:

**(13)** Enter 1 to continue, 2 to adjust the parameters and/or support (return to step 6) or 3 to exit : **1**

**(14)** Enter 1 if you wish to compute the optimal reserve; Enter 2 if you wish to keep the reserve entered in step 8 : **1**

If 1 is entered, the program calculates the reserve price which maximizes the auctioneer's expected revenue.

**(15)** *Enter an output file name : **illustration.txt***

The output file will contain the equilibrium inverse bid functions. The first two columns contain the values drawn and the corresponding optimal bids for the h-type 1 player, the second pair of columns contains the same results for the h-type 2 player. The grid of equilibrium bids is common for all h-types. Waiting time depends on the order of Taylor-series expansion and the number of grid-points chosen as well as on whether you want to compute the optimal reserve.

**(16)** *Enter 1 to compute expected revenues, and probabilities of winning. Enter 2 otherwise : **1***

If 1 is entered, the program computes the auctioneer's expected revenue, each h-bidder's expected surplus and probability of winning, the probability of retention by the auctioneer, and the optimal reserve price if 1 was chosen in step 14. The expected surplus computed for a coalition is that of the coalition as a whole.

**(17)** *Enter a revenue file name : **rillustration.txt***

This step is skipped if 2 is entered in step 16. The file will contain all the statistics from the previous step.

**(18)** *Enter 1 to compute individual best responses. Enter 2 otherwise: **1***

If 1 is entered, the program calculates the best response function of each h-type by solving the original maximization problem pointwise given the computed equilibrium bid functions of all rival players. Note, in particular that if there are  $k_i > 1$  players with h-type  $i$ , then  $k_i - 1$  rivals are assigned their equilibrium bid functions from step 15.

**(19)** *Enter a best response file name : **brillustration.txt***

This step is skipped if 2 is entered in step 18. The output file will contain best response functions. The first two columns contain the values drawn and the corresponding optimal best responses for the h-type 1 player, the second pair of columns contains the same results for the h-type 2 player. Comparisons between equilibrium bid functions and individual best responses provide verification of the numerical accuracy of the former. To make such a comparison for, say h-type 1 player, depict first two columns of **illustration.txt** and first two columns of **brillustration.txt** on the same graph and check whether they overlap. If a divergence is revealed it is recommended that the user increase both the order of the Taylor series expansion as well as the number of grid points.

All output files will be saved in the directory containing the executable file.