# Theoretical Overview

The following discussion is divided into 5 sections: algorithm and the economic model underlying it, treatment of coalitions in the program, conditions on the support of cdf, and potential applications of the FORTRAN program. Please refer to the paper for a more extensive explanation of the algorithm.

## The model

We consider an asymmetric single object independent private values first-price auction. Specifically, bidders simultaneously submit sealed bids for a single object where the highest bidder wins and pays his bid price. There are *N* potential bidders. A player is either an individual bidder or a coalition of bidders. Only those players with private valuations above the reserve price $R$ set by the auctioneer submit competitive bids. Bidders are ex-ante heterogeneous. Each bidder belongs to one of *n* types. Each type is characterized by a cumulative distribution function (CDF) $F_i$ on a common support [$\underline{\nu}$,$\bar{\nu}$]. There are $k_i$ bidders of type *i* for a total of $N = \sum_{i=1}^{n} k_i$ (potential) bidders. Bidders are assumed to be risk neutral with utility from winning the auction with a bid $b$ given a valuation $\nu$ defined as $U_i(\nu - b) = \nu - b$. Clearly, utility from winning the auction is increasing in the individual's valuation. Under these assumptions, Proposition of Maskin and Riley (2000) establishes the existence of monotonic pure-strategy equilibrium in the standard first-price auction. Lebrun (1996) has shown that these bid functions are strictly monotone and increasing, therefore, invertible. Under the assumption that $F_i$ is twice continuously differentiable with a density $f_i$ bounded away from zero on [$\underline{\nu}$,$\bar{\nu}$] Lebrun (1999) proves that the equilibrium is unique in the general *n*-bidder case, and that the inverse bid functions have a common support $[R, t^*]$, where $t^*$ is the bid associated with the valuation $\bar{\nu}$, and $R$ is the reserve price set by the auctioneer.

## The algorithm

With a very few exceptions $t^*$ cannot be found analytically and its numerical determination is a critical component of the problem to be solved. For any tentative value of $t^*$ the program solves backward the ordinary differential equations associated with the first order equilibrium conditions relying upon piecewise polynomial approximations generated from the Taylor series expansions of the inverse cdfs. It checks whether these solutions satisfy the lower boundary conditions and if not $t^*$ is iterated as needed (in line with Lebrun's 1999 theoretical result). Analytical Taylor-series expansions for inverse CDF's are available for a number of standard distributions (such as the extreme value distribution) which are commonly assumed in empirical applications. On the other hand, there are situations where this is not the case. One such important situation is analysis of less than all-inclusive coalitions (size of coalition strictly less than *N*). Other important examples would be applications where empirical and/or nonparametric CDFs have been numerically evaluated. To accommodate such situations, the program includes a fully-automated numerical procedure for the computation of piecewise Taylor-series expansion for the inverse of arbitrary CDF.

## Conditions on the Support of Cumulative Distribution Function

Current program implementation requires that tail areas of very low probability be truncated away. Such truncations are commonly imposed in empirical applications since most estimation techniques for auction models critically rely upon the invertibility of bid functions and lack robustness relative to the tail area behavior of the latter. However, many distributions of interest with tractable higher order statistics (e.g., exponential, Weibull or extreme value distribution) have unbounded support. In practice, any such distribution $F_i$ with unbounded support is replaced in the algorithm by a truncated version thereof:

$$F_i^*(\nu) = \frac{F_i(\nu) - F_i(\underline{\nu})}{F_i(\bar{\nu}) - F_i(\underline{\nu})} \quad \underline{\nu} \leq \nu \leq \bar{\nu}$$

Transforming the Taylor-series expansion of $F_i^{-1}$ into that of $F_i^{*-1}$ is fully automated in the computer program.

## Coalitions

The algorithm can be used to investigate numerically whether less than all-inclusive coalitions within a first-price asymmetric framework could potentially be incentive compatible, and also whether a strategic auctioneer could reduce the profitability of collusions. The program can be used to evaluate bid functions and expected revenues in the presence of less than all-inclusive cartels, as long as one treats such a cartel as a single representative bidder. At a minimum, such computations can provide useful insight on potential incentives to defect and on the auctioneer's capability to reduce a cartel's profitability. For example, Marshall, R.C., Meurer, M.J., Richard, J.F. , and Stromquist, W., (1994), in "*Numerical Analysis of Asymmetric First Price Auctions"* have already illustrated the fact that within an ex-ante (uniform) symmetric framework outsiders can benefit more than insiders (on a per capita basis) from the presence of a less than all-inclusive coalition. One would not expect such findings to generalize to asymmetric scenarios. In particular, there exist numerous real life illustrations of the viability of less than all-inclusive cartels consisting, for example, of better informed bidders.

Specifically, in the context of the program, an arbitrary cartel consisting of $u = \sum_{i=1}^{n} u_i$ bidders, where $u_i$ denotes the number of bidders of type *i*, is treated as a single player drawing her signal from the corresponding highest order statistics CDF:

$$F^*(\nu) = \prod_{j=1}^{n} \left[ \frac{F_j(\nu) - F_j(\underline{\nu})}{F_j(\bar{\nu}) - F_j(\underline{\nu})} \right]^{u_j} \qquad \underline{\nu} \leq \nu \leq \bar{\nu}$$

One can therefore choose the exponents $u_j$ to generate hybrid distribution functions from those available in the program. In particular, these exponents can be used to define coalitions of symmetric and asymmetric bidders.

Taylor-series expansions for the inverse of $F^*$ are automatically produced by application of the numerical procedure. Calculations of all probabilities and expected revenues for the baseline model remain valid under such scenarios, with the only modification being that the revenue computed represents the cartel's total expected revenue. Allocation rules among cartel's members are not discussed in the paper. Therefore, expected revenue of the cartel only allows comparison across alternative collusive scenarios on a per capita basis

## Applications

One of the intended uses of the algorithm is that of running comparisons between first and second-price auctions under a variety of asymmetric environments. In order to do so operational expressions for expected revenues under second-price auctions need to be derived. While Vickrey's logic still applies, whereby bidders bid their private values, expected revenue calculations are more complex than under first-price auctions due to a wider range of scenarios for the price paid by the winner. All probabilities and expected revenue calculations for the first-price and second-price auctions have been incorporated in the algorithm allowing for automated comparisons between first and second-price auctions under a wide variety of asymmetric scenarios. In addition to inverse bid functions, the algorithm computes a full range of auxiliary statistics of interest, namely expected revenue of the auctioneer; expected surpluses of bidders; probabilities of winning; probability of retention under reserve pricing and; on request, optimal reserve price. Comparisons between bidders' expected surpluses under alternative collusive agreements provide key insights on their willingness to collude. Such analysis is a staple of the auctioneer's selection of auction format when collusion is suspected, a rather common real life occurrence. It is also directly relevant for the selection of an optimal reserve price as an effective tool to minimize the impact of collusion. The program is an effective tool for investigation whether classic results (revenue equivalence, etc.) extend to situations where symmetry or stochastic dominance is no longer assumed. The algorithm can be applied to a broad class of first-price, asymmetric, independent private values auction and procurement problems, allowing for arbitrary numbers of (subgroups of) bidders independently drawing their valuations from arbitrary distributions. Common distributions (normal, lognormal, Weibull (including exponential distribution), Beta (including uniform and power distributions)) are offered as options in the program. Additional distributions can be added by users in the form of a subroutine. The only restriction is that these distributions have common support.