Semi-nonparametric estimation of independently and identically repeated first-price auctions via an integrated simulated moments method

Herman J. Bierens[a] and Hosin Song[b]
[a]Department of Economics and CAPCP*
Pennsylvania State University,
University Park, PA 16802, USA
[b]Korea Institute of Public Finance, Seoul, Korea

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Abstract

In this paper we propose to estimate the value distribution of independently and identically repeated first-price auctions directly via a semi-nonparametric integrated simulated moments sieve approach. Given a candidate value distribution function in a sieve space, we simulate bids according to the equilibrium bid function involved. We take the difference of the empirical characteristic functions of the actual and simulated bids as the moment function. The objective function is then the integral of the squared moment function over an interval. Minimizing this integral to the distribution functions in the sieve space involved and letting the sieve order increase to infinity with the sample size then yield a uniformly consistent semi-nonparametric estimator of the actual value distribution. Also, we propose an integrated moment test for the validity of the first-price auction model,

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and an data-driven method for the choice of the sieve order. Finally, we conduct a few numerical experiments to check the performance of our approach.
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1 Introduction

As Laffont and Vuong (1993) point out, the distribution of bids determines the structural elements of auction models, provided identification is achieved. In the first-price auction model with symmetric independent private values, the structural element of interest is the value distribution. Much research has been done on the identification and the estimation of the value distribution. Donald and Paarsch (1996) apply ML estimation to first-price auctions and Dutch auctions. They use a parametric specification for the value distribution to implement ML estimation. In particular, they assume in a numerical example that the value distribution is a uniform distribution on the interval $[0, \overline{v}]$ where $\overline{v}$ is a parametric function of auction-specific covariates. Since in this case the support of the bid distribution involved depends on parameters, the standard consistency proof of ML estimators does no longer apply. Another difficulty with ML estimation of first-price auction models is that the equilibrium bid function is highly non-linear in the value and its distribution, so that the computation of the ML estimators involved is challenging. The same applies to descending price (Dutch) auctions, because they are strategically equivalent to first-price auctions. Because of the difficulty of ML estimation of these models, Laffont and Vuong (1993) suggest Simulated Non-Linear Least Squares (SNLLS) estimation for first-price auction models and Simulated Method of Moment (SMM) estimation for descending price auction models with symmetric independent private values. Their SNLLS approach requires to replace the expectation of the winning bid with a simulated one. They also suggest that the expectation of higher moments of winning bids can be used for SMM estimation if the expectation of the winning bids itself is not sufficient to identify all parameters. Both SNLLS and SMM approaches require a parametric specification for the value distribution. Laffont et al. (1995) apply the SNLLS approach suggested by Laffont and Vuong (1993) to the egg plant auction, which is a descending price auction. They specify a log-normal value distribution conditional on covariates. Li (2005) considers first-price auctions with entry and binding reserve price. This auction consists of two stages. In the first stage the potential bidder decides whether he or she enters the auction, with payment of entry cost. In the second stage, the bidder gets to know his or her value and then decides to bid according to the equilibrium bid function, which is the same function as for the first-price auction model. Li (2005) proposes a SMM approach to estimate the entry probability and the parameters of the value distribution. One of the conditional moments is a function of the upper bound of the bid support, which can be computed via the simulation approach in Laffont et al. (1995). The other moment conditions are related to the number of active bidders, i.e., potential bidders who decides to participate in the auction.

For general nonparametric identification results of first-price auctions models with symmetric independent private values, see Guerre et al. (1995) and Athey and Haile (2005). In particular, Guerre et al. (2000) show the nonparametric identification of value distributions with bounded support $[\underline{v}, \overline{v}]$, $\overline{v} < \infty$, and propose an indirect nonparametric kernel estimation approach. Their approach is based on the inverse bid function $v = b + (I-1)^{-1} \Lambda(b)/\lambda(b)$, where I is the number of potential bidders, v is a private value, b is a corresponding bid, and $\Lambda(b)$ is the distribution function of bids with density $\lambda(b)$. The latter two functions are estimated via nonparametric kernel methods, as $\hat{\Lambda}(b)$ and $\hat{\lambda}(b)$, respectively. Using the pseudo-private values $\widetilde{V} = B + (I-1)^{-1} \hat{\Lambda}(B)/\hat{\lambda}(B)$, where each B is an observed bid, the density of the private value distribution can now be estimated by kernel density estimation. However, the ratio $\hat{\Lambda}(b)/\hat{\lambda}(b)$ may be an unreliable estimate of $\Lambda(b)/\lambda(b)$ near the boundary of the support of $\lambda(b)$. To solve this problem, Guerre et al. (2000) use a trimming procedure which amounts to discarding pseudo-private values $\widetilde{V}$ corresponding to bids B that are too close to the boundary of the (known) support of the bid distribution.

Throughout this paper we confine our analysis to first-price sealed bid auctions where values are independent, private and bidders are symmetric and risk-neutral. Moreover, our asymptotic results are based on the assumption that the observe the bids are generated by independently repeated identical first-price auctions. Admittedly, this type of repeated auctions is rare in practice. The reason for considering this case is to lay the groundwork for the more realistic case of first-price auctions with auction-specific covariates and different numbers of potential bidders and reservation prices, like the well-known US Forest Service timber auctions.

In this paper we propose a direct Semi-Nonparametric Integrated Simulated Moments (SNP-ISM) estimation approach, as an alternative to the two-step nonparametric kernel estimation approach of Guerre et al. (2000). This approach does not require that the support of the value distribution is bounded. In stead of the latter, we require that the value distribution is absolutely continuous with finite expectation. This condition implies that the corresponding bids are bounded random variables.

Our SNP-ISM methodology is different from the SMM approach of Laffont and Vuong (1993) and Li (2005) in that the latter two approaches require parametric specification for the value distribution whereas ours does not. Instead, we treat the unknown value distribution function itself as a parameter, contained in a compact metric space $\mathcal{F}$ of absolutely continuous distribution functions endowed with the "sup" metric.

Based on the approach in Bierens (2007), we construct an increasing sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ (the sieve[1]) of subspaces of our "parameter space" $\mathcal{F}$, where the distribution functions in each subspace $\mathcal{F}_n$ can be represented by parametric functions of Legendre polynomials of order up to n. Given a distribution function $F \in \mathcal{F}_n$, we simulate bids according to the equilibrium bid function involved. Motivated by the well-known fact that distributions are equal if and only if their characteristic functions are identical, we take the difference of the empirical characteristic functions of the actual and simulated bids as the simulated moment function. Thus, our approach uses uncountable many moment conditions whereas Laffont and Vuong (1993) and Li (2005) use only a finite number of moment conditions. Since characteristic functions of bounded random variables coincide everywhere if and only if they coincide on an arbitrary interval around zero, we take the integral of the squared simulated moment function over such an interval as our objective function, similar to the Integrated Conditional Moment (ICM) test statistic of Bierens (1982) and Bierens and Ploberger (1997). Minimizing this objective function to the distribution functions in $\mathcal{F}_n$ and letting n increase with the sample size N then yield a uniformly consistent SNP sieve estimator of the actual value distribution. This approach yields as by-products an Integrated Moment (IM) test of the validity of the first-price auction model, together with an

[1] A sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of subspaces of a metric space $\mathcal{F}$ is called a sieve if the closure of $\cup_{n=0}^{\infty}\mathcal{F}_n$ is equal to $\mathcal{F}$. See for example Shen(1997) and Chen (2004).

information criterion similar the information criteria of Hannan-Quinn (1979) and Schwarz (1978) for likelihood models, which can be used to estimate the sieve order n consistently if it is finite, and otherwise yields a data-driven sequence n_N for which the SNP sieve estimator is uniformly consistent as well. Finally, we conduct a few numerical experiments to check the performance of our SNP-ISM approach.

The paper is organized as follows. In section 2 we review the first-price auction model and show that this model is nonparametrically identified as long as the value distribution is absolutely continuous, and that the bids are bounded random variables if the value distribution has a finite expectation. In section 3 we introduce our estimation methodology and we set forth further conditions for the uniform strong consistency of the SNP-ISM sieve estimator involved. In section 4 we show how to construct the metric space $\mathcal{F}$ and corresponding sieve spaces $\mathcal{F}_n$ of absolutely continuous distribution functions on $(0, \infty)$. In section 5 we propose a consistent integrated moment (IM) test of the validity of the first-price auction model with symmetric independent values and risk neutrality. In section 6 we propose an information criterion for the data-driven selection of the sieve order n, similar to the order selection of autoregressions via the Hannan-Quinn (1979) and Schwarz (1978) information criteria. In section 7 we show the performance of our proposed SNP-SMM estimation and IM testing approach via a few numerical experiments. In section 8 we make some concluding remarks. Most of the proofs of our results are given in the Appendix (section 9).

Throughout the paper, we denote a random variable in upper-case and a non-random variable in lower-case. The indicator function is denoted by $\mathbf{1}(\cdot)$.[2] Almost sure (a.s.) convergence is denoted by $X_n \to X$ a.s.[3] Similarly, convergence in probability will be denoted by $X_n \to_p X$ or $p\lim_{n\to\infty} X_n = X$, and $X_n \to_d X$ indicates that X_n converges in distribution to X. In the case that X_n and X are random functions we use the notation $X_n \Rightarrow X$ to indicate that $X_n(\cdot)$ converges weakly to $X(\cdot)$. See for example Billingsley (1999) for the meaning of the notion of weak convergence.

[2] $\mathbf{1}(True) = 1$, $\mathbf{1}(False) = 0$.

[3] This means that $P\left[\lim_{n\to\infty} X_n = X\right] = 1$.

2 First-Price Auctions

A first price sealed bids auction (henceforth called *first-price auction*) is an auction with I potential bidders, where the potential bidder's values for the item to be auctioned off are independent and private, and the bidders are symmetric and risk neutral. The reservation price p_0, if any, is announced in advance and the number $I \geq 2$ of potential bidders is known to each potential bidder.

2.1 The Bid Distribution

As is well-known, the equilibrium bid function of a first-price auction takes the form

$$\beta\left(v|F_0\right) = v - \frac{1}{F_0(v)^{I-1}} \int_{p_0}^{v} F_0(x)^{I-1} dx \text{ for } v > p_0, \tag{1}$$

if the reservation price p_0 is binding, and

$$\beta\left(v|F_0\right) = v - \frac{1}{F_0(v)^{I-1}} \int_{0}^{v} F_0(x)^{I-1} dx \text{ for } v > \underline{v}, \tag{2}$$

if the reservation price p_0 is non-binding, where $F_0(v)$ is the value distribution and $\underline{v} \geq 0$ is the lower bound of its support. See for example Riley and Samuelson (1981) or Krishna (2002).

If p_0 is binding, only potential bidders whose values are greater than p_0 participate in the auction, and issue a bid $B > p_0$. For notational convenience we will assume that the other potential bidders issue a zero bid: $B = 0$. Then the bid function involved becomes

$$\begin{aligned}\beta_0\left(v|F_0\right) &= v.\mathbf{1}\left(F_0\left(v\right) \geq F_0\left(p_0\right)\right) \\ &- \frac{\mathbf{1}\left(F_0\left(v\right) \geq F_0\left(p_0\right)\right) \int_{p_0}^{v} F_0(x)^{I-1} dx}{\left(F_0\left(v\right)\mathbf{1}\left(F_0\left(v\right) \geq F_0\left(p_0\right)\right) + F_0\left(p_0\right).\mathbf{1}\left(F_0\left(v\right) < F_0\left(p_0\right)\right)\right)^{I-1}},\end{aligned} \tag{3}$$

which applies to the non-binding reservation price case as well. Consequently, if V is a random drawing from F_0 then the corresponding (possibly zero) bid B is

$$\begin{aligned}B = \beta_0\left(V|F_0\right) &= V.\mathbf{1}\left(F_0\left(V\right) \geq F_0\left(p_0\right)\right) \\ &- \frac{\mathbf{1}\left(F_0\left(V\right) \geq F_0\left(p_0\right)\right) \int_{p_0}^{V} F_0(x)^{I-1} dx}{\left(F\left(V_0\right)\mathbf{1}\left(F\left(V_0\right) \geq F_0\left(p_0\right)\right) + F_0\left(p_0\right).\mathbf{1}\left(F_0\left(V\right) < F_0\left(p_0\right)\right)\right)^{I-1}}.\end{aligned} \tag{4}$$

It is well-known that if

Assumption 1. *The true value distribution F_0 is absolutely continuous, with density f_0*

then, with V a random drawing from F_0, $U = F_0(V)$ is uniformly $[0,1]$ distributed. Moreover, if in addition F_0 is invertible, i.e., for each $u \in [0,1]$ there exists a unique $v = F_0^{-1}(u)$ such that $F_0(v) = u$, then (4) is distributed as

$$B \sim \eta(U|F_0), \ U \sim \text{Uniform } [0,1], \tag{5}$$

with

$$\begin{aligned}
&\eta(u|F) = F^{-1}(u).\mathbf{1}(u \geq \alpha) - \frac{\mathbf{1}(u \geq \alpha)\int_{p_0}^{F^{-1}(u)} F(x)^{I-1}dx}{(u.\mathbf{1}(u \geq \alpha) + \alpha.\mathbf{1}(u < \alpha))^{I-1}} \\
&= F^{-1}(u).\mathbf{1}(u \geq \alpha) \\
&\quad - \frac{\mathbf{1}(u \geq \alpha)(F^{-1}(u) - p_0)\int_0^1 F(p_0 + y(F^{-1}(u) - p_0))^{I-1} dy}{(u.\mathbf{1}(u \geq \alpha) + \alpha.\mathbf{1}(u < \alpha))^{I-1}} \\
&\alpha = F(p_0), \ u \in [0,1],
\end{aligned} \tag{6}$$

where the second equality in (6) follows by substituting $x = p_0 + y(F^{-1}(u) - p_0)$. However, invertibility of F_0 is not necessary for (5), due to the following lemma.

Lemma 1. *Let $F(v)$ be an absolutely continuous distribution function. If V is a random drawing from $F(v)$ then $F(V)$ is uniformly $[0,1]$ distributed. Moreover, the set S of points $u \in (0,1)$ for which the solution v_u of $u = F(v_u)$ is* ***not*** *unique is either empty or countable. Hence, given a random drawing U from the uniform $[0,1]$ distribution, the solution V_U of $F(V_U) = U$ is* ***a.s.*** *unique, and $P[V_U \leq v] = F(v)$.*

Proof: Although this is a known result, the proof will be given in the Appendix.

2.2 Nonparametric Identification

There are two seminal papers on the identification of first-price auction models, namely Donald and Paarsch (1996) and Guerre et al. (2000). Of course, parametric identification has been established earlier, in particular by Laffont et al. (1995). Donald and Paarsch (1996) show the nonparametric identification of first-price auction models under the assumption that the support of the distribution $F(v)$ of the values is a known bounded interval $[\underline{v}, \overline{v}]$, i.e., $F(v)$ is absolutely continuous with density f such that $f(v) > 0$ on $(\underline{v}, \overline{v})$, and $F(\underline{v}) = 0$, $F(\overline{v}) = 1$. They identify the value distribution and the risk aversion parameter using the family of Hara utility functions. Particularly they identify the distribution $F(v)$ using a fixed upper bound.

As said before, Guerre et al. (2000) also show the nonparametric identification of first-price auction models under the bounded support assumption, via the inverse bid function.

In this subsection we will show that these identification results carry over to general absolutely continuous value distributions.

If the reservation price p_0 is binding, the number of actual bidders, I_0 say, may be less than the number I of potential bidders. Then $I - I_0$ is the number of zero bids, which has a Binomial$(I, F_0(p_0))$ distribution, hence

$$F_0(p_0) = E\left[(I - I_0)/I\right].$$

Since both I and I_0 are observable, it follows therefore that $\alpha = F_0(p_0)$ is identified.

Now suppose that there exists another absolutely continuous value distribution F_* with density f_* satisfying $\alpha = F_0(p_0) = F_*(p_0)$ such that with $U \sim$ Uniform $[0,1]$, $B \sim \eta(U|F_0) \sim \eta(U|F_*)$. Similar to S in Lemma 1, let S_* be the set of points $u \in (0,1)$ for which the solution v_u^* of $u = F_*(v_u^*)$ is not unique. Then it follows from (6) that for $u \in (\alpha, 1) \setminus (S \cup S_*)$,

$$u^{I-1}F_0^{-1}(u) - \int_{p_0}^{F_0^{-1}(u)} F_0(x)^{I-1}dx = u^{I-1}F_*^{-1}(u) - \int_{p_0}^{F_*^{-1}(u)} F_*(x)^{I-1}dx.$$

Since $S \cup S_*$ is countable, it follows that both sides of this equation are differentiable in every $u \in (\alpha, 1) \setminus (S \cup S_*)$, with derivatives

$$\begin{aligned}
&(I-1)u^{I-2}F_0^{-1}(u) \\
&= (I-1)u^{I-2}F_0^{-1}(u) + u^{I-1}\frac{dF_0^{-1}(u)}{du} - (F_0(F_0^{-1}(u)))^{I-1}\frac{dF_0^{-1}(u)}{du} \\
&= (I-1)u^{I-2}F_*^{-1}(u) + u^{I-1}\frac{dF_*^{-1}(u)}{du} - (F_*(F_*^{-1}(u)))^{I-1}\frac{dF_*^{-1}(u)}{du} \\
&= (I-1)u^{I-2}F_*^{-1}(u).
\end{aligned}$$

Hence $F_0^{-1}(u) = F_*^{-1}(u)$ on $(\alpha, 1) \setminus (S \cup S_*)$, which implies that

$$u = F_0\left(F_*^{-1}(u)\right) \in (\alpha, 1) \setminus (S \cup S_*) \tag{7}$$

Taking the derivative to $u \in (\alpha, 1) \setminus (S \cup S_*)$ again, it follows from (7) that

$$1 = \frac{dF_0\left(F_*^{-1}(u)\right)}{du} = \frac{f_0\left(F_*^{-1}(u)\right)}{f_*\left(F_*^{-1}(u)\right)}$$

hence $f_0(v) = f_*(v)$ for $v \in (p_0, \infty) \cap \{v : F_*(v) \notin S \cup S_*\}$. It is now easy to verify that the latter equality implies that $F_*(v) = F_0(v)$ for all $v \in [p_0, \infty)$.

The result for the non-binding reservation price case follows by letting $p_0 = 0$. Thus,

Lemma 2. *Under Assumption 1 the value distribution $F_0(v)$ is nonparametrically identified on $[p_0, \infty)$ if the reservation price p_0 is binding, and on $[0, \infty)$ in the non-binding case.*

2.3 Nonparametric Identification via Characteristic Functions

It follows trivially from (3) and Assumption 1 that[4] $\lim_{v\to\infty} \beta_0(v|F_0) = (I-1)\int_{p_0}^{\infty} x.F_0(x)^{I-2} f_0(x)\,dx + p_0 F(p_0)^{I-1}$, which implies that

Lemma 3. *Under Assumption 1,* $\sup_{v>0} \beta_0(v|F_0) < \infty$ *if and only if* $\int_0^{\infty} v f_0(v)\,dv < \infty$. [5]

Since the boundedness of the bids plays a key-role in our semi-nonparametric estimation approach, we will from now on assume that

Assumption 2. *The true value distribution F_0 has finite expectation.*

The significance of the boundedness of an actual bid B is that the bid distribution $\Lambda_0(b) = P[B \leq b]$ is then completely determined by the shape

[4] See Laffont et al. (1995) and Guerre et al. (2000, Footnote 8).

[5] This result also follows from equation (6) in Li and Vuong (1997). Moreover, note that under the conditions of Lemma 3 the expected revenue of the seller, $I.\int_{p_0}^{\infty} \beta(v|F_0) f_0(v)\,dv$, is finite too.

of its characteristic function

$$\varphi_0(t) = E\left[\exp(i.t.B)\right] = \int_0^\infty \exp\left(i.t.b\right) d\Lambda_0(b), \; i = \sqrt{-1}, \tag{8}$$

in an arbitrary neighborhood of $t = 0$. More formally:

Lemma 4. *Let B be a bounded random variable with distribution function $\Lambda_0(b)$ and characteristic function $\varphi_0(t)$. Let $\psi\left(t\right)$ be the characteristic function of a distribution function $\Lambda(b)$. Then $\Lambda(b) = \Lambda_0(b)$ for all $b \in \mathbb{R}$ if and only if for an arbitrary $\kappa > 0$, $\varphi_0(t) = \psi\left(t\right)$ for all $t \in \left(-\kappa, \kappa\right)$.*

This is a well-known result[6], which is based on the fact that due to the boundedness condition $\varphi_0(t)$ can be written as $\varphi_0(t) = \sum_{m=0}^{\infty} \frac{i^m}{m!} t^m E\left[B^m\right]$, hence $\varphi_0(t) = \psi\left(t\right)$ on $\left(-\kappa, \kappa\right)$ implies that $i^{-m} \; d^m \psi\left(t\right) / \left(dt\right)^m\big|_{t=0} = i^{-m} d^m \varphi\left(t\right) / \left(dt\right)^m\big|_{t=0} = E\left[B^m\right]$ for $m = 0, 1, 2, ...$, so that $\psi(t) = \varphi_0(t) = \sum_{m=0}^{\infty} \frac{i^m}{m!} t^m . E\left[B^m\right]$ for all $t \in \mathbb{R}$. As is well-known, the latter implies that the two distributions involved are identical.

Note that we do not need to assume from the outset that $\Lambda(b)$ is a distribution function of a bounded random variable. The condition $\varphi_0(t) = \psi\left(t\right)$ on $\left(-\kappa, \kappa\right)$ automatically implies boundedness of this distribution.

Now let F be an absolutely continuous distribution function on $(0, \infty)$ with density and let U be a random drawing from the Uniform $[0, 1]$ distribution. Moreover, let

$$\widetilde{B}\left(F\right) = \eta\left(U|F\right) \tag{9}$$

[c.f. (6)]. Furthermore, let $\Lambda\left(b|F\right) = P\left[\widetilde{B}\left(F\right) \leq b\right]$ and

$$\psi\left(t|F\right) = E\left[\exp\left(i.t.\widetilde{B}\left(F\right)\right)\right]. \tag{10}$$

Then it follows from Assumptions 1-2 and Lemma 4 that $\psi\left(t|F\right) = \varphi_0(t)$ for all $t \in \left(-\kappa, \kappa\right)$, with $\kappa > 0$ arbitrary, if and only if $\Lambda\left(b|F\right) = \Lambda_0\left(b\right)$ for all b, where Λ_0 is now the actual bid distribution. In its turn it follows from Lemma 2 that the latter implies that $F(v) = F_0\left(v\right)$ for all $v \in [p_0, \infty)$ if the reservation price is binding, and $F(v) = F_0\left(v\right)$ for all $v \in [0, \infty)$ if the reservation price is non-binding. Thus:

[6] Usually stated for moment generating functions rather than characteristic functions.

Lemma 5. *Denote*

$$\Psi(t|F) = E\left[\exp\left(i.t.B\right)\right] - E\left[\exp\left(i.t.\widetilde{B}\left(F\right)\right)\right] \tag{11}$$

where B is an actual bid corresponding to the true value distribution F_0, and $\widetilde{B}\left(F\right)$ is a simulated bid generated by (9). *Let for an arbitrary constant* $\kappa > 0$,

$$\overline{Q}(F) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} |\Psi(t|F)|^2 \, dt \tag{12}$$

Under Assumptions 1-2, $\overline{Q}\left(F\right) = 0$ if and only if $F\left(v\right) = F_0\left(v\right)$ on $[p_0, \infty)$ if the reservation price p_0 is binding, and $F\left(v\right) = F_0\left(v\right)$ on $[0, \infty)$ if the reservation price p_0 is non-binding.

3 Integrated Simulated Moments Sieve Estimation.

3.1 The Objective Function

As said before, in this paper we will consider the case where a first-price auction is repeated independently L times, with the same true value distribution $F_0(v)$, the same fixed number of potential bidders I, and the same reservation price p_0. The asymptotic results will be derived for $L \to \infty$. Thus, we observe $N = I \times L$ bids B_j generated independently from the distribution (5).

Let F be a potential candidate (henceforth called a *candidate value distribution*) for the true value distribution F_0, and let $\{U_j\}_{j=1}^N$ be a random sample drawn from the uniform $[0, 1]$ distribution. Then similar to (9) we can generate independent simulated bids $\widetilde{B}_j\left(F\right)$ according to

$$\widetilde{B}_j\left(F\right) = \eta\left(U_j|F\right), \ j = 1, 2, ..., N = I \times L, \tag{13}$$

where $\eta(u|F)$ is defined by (6).

Next, denote

$$\widehat{\Psi}(t|F) = \widehat{\varphi}_0(t) - \widehat{\psi}(t|F), \ t \in \mathbb{R}, \tag{14}$$

where

$$\widehat{\varphi}_0(t) = \frac{1}{N}\sum_{j=1}^{N} \exp(i.t.B_j) \tag{15}$$

is the empirical characteristic function of the actual bids and

$$\widehat{\psi}(t|F) = \frac{1}{N}\sum_{j=1}^{N}\exp\left(i.t.\widetilde{B}_j\left(F\right)\right) \tag{16}$$

is the empirical characteristic function of the simulated bids, and let for an arbitrary constant $\kappa > 0$,

$$\widehat{Q}(F) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widehat{\Psi}(t|F)\right|^2 dt \tag{17}$$

By Kolmogorov's strong law of large numbers for i.i.d. random variables, $\widehat{\Psi}(t|F) \to \Psi(t|F)$ a.s., pointwise in t and F, where $\Psi(t|F)$ is defined by (11) in Lemma 5, hence by the bounded convergence theorem, $\widehat{Q}(F) \to \overline{Q}(F)$ a.s., pointwise in F,where $\overline{Q}(F)$ is defined by (12) in Lemma 5. Now Lemma 5 suggests that F_0 can be estimated consistently by minimizing $\widehat{Q}(F)$ to F, in some way to be discussed in the next subsection.

Note that the objective function $\widehat{Q}(F)$ has the following closed form expression in terms of the actual bids B_j and the simulated bids $\widetilde{B}_j\left(F\right)$:

$$\begin{aligned}\widehat{Q}(F) &= \frac{2}{N^2}\sum_{j_1=1}^{N-1}\sum_{j_2=j_1+1}^{N}\frac{\sin\left(\kappa.\left(B_{j_1} - B_{j_2}\left(F\right)\right)\right)}{\kappa.\left(B_{j_1} - B_{j_2}\left(F\right)\right)} \\ &\quad + \frac{2}{N^2}\sum_{j_1=1}^{N-1}\sum_{j_2=j_1+1}^{N}\frac{\sin\left(\kappa.\left(\widetilde{B}_{j_1}\left(F\right) - \widetilde{B}_{j_2}\left(F\right)\right)\right)}{\kappa.\left(\widetilde{B}_{j_1}\left(F\right) - \widetilde{B}_{j_2}\left(F\right)\right)} \\ &\quad - \frac{2}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{\sin\left(\kappa.\left(B_{j_1} - \widetilde{B}_{j_2}\left(F\right)\right)\right)}{\kappa.\left(B_{j_1} - \widetilde{B}_{j_2}\left(F\right)\right)}.\end{aligned} \tag{18}$$

3.2 Conditions for Strong Consistency of Sieve Estimators

The standard consistency proof for parameter estimators of nonlinear parametric models requires that the parameters are confined to a compact subset of a Euclidean space. Since the true value distribution F_0 plays now the role of parameter, we need to construct a compact metric space $\mathcal{F}$ of absolutely continuous distribution functions F on $(0,\infty)$ containing the true value distribution F_0: $F_0 \in \mathcal{F}$. Because F_0 is unknown, the latter has to be assumed, though. We will endow the space $\mathcal{F}$ with the "sup" metric, i.e., the metric

$$||F_1 - F_2|| = \sup_{v>0} |F_1(v) - F_2(v)| \tag{19}$$

if the reservation price p_0 is non-binding, and

$$||F_1 - F_2|| = \sup_{v \geq p_0} |F_1(v) - F_2(v)| \tag{20}$$

if the reservation price p_0 is binding. Recall that in the latter case only the shape of F_0 on $[p_0, \infty)$ matters, and the same holds for any candidate distribution function F, so that without of loss of generality may declare two candidate value distributions F_1 and F_2 "equal" (or equivalent) if they coincide on $[p_0, \infty)$.

Suppose it were possible to compute

$$\widehat{F} = \arg\min_{F\in\mathcal{F}} \widehat{Q}(F). \tag{21}$$

Then it follows from Theorem 2 below that

$$||\widehat{F} - F_0|| \to 0 \text{ a.s. for } N \to \infty \tag{22}$$

if

$$\sup_{F\in\mathcal{F}} \left|\widehat{Q}(F) - \overline{Q}(F)\right| \to 0 \text{ a.s. for } N \to \infty \tag{23}$$

and

$$\overline{Q}(F) \text{ is continuous on } \mathcal{F}. \tag{24}$$

A third condition is that F_0 is unique: if $F_* = \arg\min_{F\in\mathcal{F}} \overline{Q}(F)$ then $||F_* - F_0|| = 0$, but this condition follows from Lemma 5.

The uniform convergence condition (23) follows by bounded convergence if pointwise in $t \in (-\tau, \tau)$,

$$\sup_{F\in\mathcal{F}} \left|\widehat{\psi}(t|F) - \psi(t|F)\right| \to 0 \text{ a.s. for } N \to \infty, \tag{25}$$

where $\widehat{\psi}(t|F)$ is defined by (16) and $\psi(t|F)$ by (10). To prove (25), we need the following generalization of Jennrich's (1969) uniform strong law of large numbers to random functions on compact metric spaces:

Theorem 1. *Let* Θ *be a compact metric space, and let* $\mu_j(\theta), j = 1, 2, ..., N, ...$ *be a sequence of real valued almost surely continuous i.i.d. random functions on* Θ. *If in addition*

$$E\left[\sup_{\theta\in\Theta}|\mu_1(\theta)|\right] < \infty \tag{26}$$

then for $N \to \infty$,

$$\sup_{\theta\in\Theta}\left|\frac{1}{N}\sum_{j=1}^{N}\mu_j(\theta) - \overline{\mu}(\theta)\right| \to 0 \text{ } a.s., \tag{27}$$

where $\overline{\mu}(\theta) = E\left[\mu_1(\theta)\right]$. [7] *This result carries over to complex-valued random functions* $\mu_j(\theta)$ *if the conditions involved hold for* $\mathrm{Re}\left[\mu_j(\theta)\right]$ *and* $\mathrm{Im}\left[\mu_j(\theta)\right]$.

Proof: Appendix.

In our case (25), Θ is $\mathcal{F}$, $\theta = F$ and $\mu_j(\theta) = \exp\left(i.t.\widetilde{B}_j(F)\right)$ for fixed t. Since $|\exp(i.t.\widetilde{B}_j(F))| \equiv 1$, condition (26) is trivially satisfied. Thus, to conclude (25) from Theorem 1 it remains to show that $\widetilde{B}_j(F)$ is a.s. continuous in F, which will be done in Lemma 6 below. Given this continuity condition, (24) then holds as well, and then (22) follows from the following generalization of the consistency results in Jennrich's (1969) for nonlinear least squares estimators.

Theorem 2. *Let* $\widehat{Q}_N(\theta)$ *be a sequence of real valued random functions on a compact metric space* Θ *with metric* $\rho(\theta_1, \theta_2)$, *such that* $\sup_{\theta\in\Theta}|\widehat{Q}_N(\theta) - \overline{Q}(\theta)| \to 0$ *a.s., where* $\overline{Q}(\theta)$ *is a continuous real function on* Θ. *Let* $\widehat{\theta}_N = \arg\min_{\theta\in\Theta} \widehat{Q}_N(\theta)$ *and* $\theta_0 = \arg\min_{\theta\in\Theta}\overline{Q}(\theta)$. *Then for* $N \to \infty$,

$$\overline{Q}(\widehat{\theta}_N) \to \overline{Q}(\theta_0) \text{ } a.s. \tag{28}$$

If θ_0 *is unique then* (28) *implies* $\rho(\widehat{\theta}_N, \theta_0) \to 0$ *a.s.*

Proof: Appendix.

Of course, in practice it is not possible to compute (21). The standard approach to get around this problem is sieve estimation. The general consistency result for sieve estimators is the following:

[7] Note that $\overline{\mu}(\theta)$ is continuous because $\mu_1(\theta)$ is a.s. continuous.

Theorem 3. *Let the conditions of Theorem 2 be satisfied, including the uniqueness of* θ_0. *Let* $\{\Theta_n\}_{n=0}^{\infty}$ *be an increasing sequence of compact subspaces of* Θ *which is dense in* Θ, *i.e.,* $\Theta = \overline{\cup_{n=0}^{\infty}\Theta_n}$,[8] *and for which the computation of* $\widetilde{\theta}_{n,N} = \arg\min_{\theta\in\Theta_n} \widehat{Q}_N(\theta)$ *is feasible. Let* n_N *be an arbitrary subsequence of* n *satisfying* $\lim_{N\to\infty} n_N = \infty$, *and denote the sieve estimator involved by* $\widetilde{\theta}_N = \widetilde{\theta}_{n_N,N}$. *Then* $\rho(\widetilde{\theta}_N, \theta_0) \to 0$ *a.s.*

Proof: Appendix.

Thus, in our case we need construct an increasing sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of compact subspaces of $\mathcal{F}$ which is dense in $\mathcal{F}$, and for which the computation of

$$\widetilde{F}_n = \arg\min_{F\in\mathcal{F}_n} \widehat{Q}(F) \tag{29}$$

is feasible. Then for any subsequence n_N of N satisfying $\lim_{N\to\infty} n_N = \infty$ the sieve estimator $\widetilde{F} = \widetilde{F}_{n_N}$ is strongly consistent: $||\widetilde{F} - F_0|| \to 0$ a.s. as $N \to \infty$.

3.3 Continuity of Simulated Values and Bids in the Candidate Value Distribution

The simulation procedure (13) can be conducted as follows. First,

Assumption 3. *Given a sequence* $U_1, ..., U_N, ...$ *of independent random drawings from the uniform* $[0,1]$ *distribution, for each candidate value distribution* F *the corresponding simulated values* $\widetilde{V}_j(F) = F^{-1}(U_j)$ *are generated by solving* $U_j = F\left(\widetilde{V}_j(F)\right)$ *for* $j = 1, ..., N$.

Then it follows from (6) and (13) that the corresponding simulated bids can be generated according to

$$\begin{aligned}\widetilde{B}_j(F) &= \left(\widetilde{V}_j(F) - \frac{\widetilde{V}_j(F) - p_0}{U_j^{I-1}}\int_0^1 F\left(p_0 + y(\widetilde{V}_j(F) - p_0)\right)^{I-1} dy\right) \\ &\quad \times \mathbf{1}\left(\widetilde{V}_j(F) > p_0\right)\end{aligned} \tag{30}$$

[8] The bar denotes the closure of the set involved. The condition $\Theta = \overline{\cup_{n=1}^{\infty}\Theta_n}$ is equivalent to the statement that for each $\theta \in \Theta$ there exists a sequence $\theta_n \in \Theta_n$ such that $\lim_{n\to\infty} \rho(\theta_n, \theta) = 0$.

if the reservation price p_0 is binding, and

$$\widetilde{B}_j(F) = \widetilde{V}_j(F) - \frac{\widetilde{V}_j(F)}{U_j^{I-1}} \int_0^1 F\left(y.\widetilde{V}_j(F)\right)^{I-1} dy \tag{31}$$

if the reservation price p_0 is non-binding. The the integral involved can be computed numerically or via Monte Carlo integration. However, the asymptotic theory in this paper will be based on the assumption that this integral is computed exactly.

Note that to guarantee smoothness of the empirical characteristic function $\widehat{\psi}(t|F)$ of the simulated bids $\widetilde{B}_j(F)$ in t and F, the same sequence $\{U_k\}_{k=1}^N$ of independent uniformly $[0,1]$ distributed random variables should be used to generate the simulated bids for different candidate value distributions F.

An alternative approach to generate simulated values $\widetilde{V}_j(F)$ from F is the well-known accept-reject method. See for example Devroye (1986) or Rubinstein (1981). However, the simulation procedure in Assumption 3 has the advantage that it is easier to prove that the simulated values and bids involved are continuous in F, in the following sense:

Lemma 6. *Let F_n and F be candidate value distributions such that*

$$\lim_{n\to\infty} ||F_n - F|| = 0 \tag{32}$$

For a given random drawing U from the uniform $[0,1]$ distribution, let $\widetilde{V}(F_n) = F_n^{-1}(U)$ and $\widetilde{V}(F) = F^{-1}(U)$, with corresponding simulated bids $\widetilde{B}(F_n)$ and $\widetilde{B}(F)$, respectively. Then

$$P\left[\lim_{n\to\infty} \widetilde{V}(F_n) = \widetilde{V}(F)\right] = 1 \tag{33}$$

and

$$P\left[\lim_{n\to\infty} \widetilde{B}(F_n) = \widetilde{B}(F)\right] = 1. \tag{34}$$

Proof: Appendix.

4 Construction of a Compact Metric Space of Absolutely Continuous Distribution Functions and Its Sieve Spaces

4.1 Representation of Distribution by Distribution on the Unit Interval

Any absolutely continuous distribution function $F(v)$ can be expressed as

$$F(v) = H(G(v)), \tag{35}$$

where $G(v)$ is a given absolutely continuous distribution function with connected support[9] containing the support of F, and H is an absolutely continuous distribution function on the unit interval, namely $H(u) = F\left(G^{-1}(u)\right)$. The density $f(v)$ of $F(v)$ then takes the form

$$f(v) = h(G(v))g(v), \tag{36}$$

where $g(v)$ is the density of $G(v)$ and $h(u)$ is the density of $H(u)$, i.e.,

$$H(u) = \int_0^u h(x)dx. \tag{37}$$

Therefore, we can estimate f and F by estimating h given G.

In our case, where F is candidate value distribution, it is advisable to choose for G a distribution function with support $(0, \infty)$, for example the exponential distribution, because in general the support of $F(v)$ is unknown.

4.2 Modeling Density Functions on the Unit Interval via Legendre Polynomials

Following Bierens (2007), we now show how to approximate any density function $h(u)$ on the unit interval arbitrary close by using orthonormal Legendre polynomials.

[9] So that $G(v)$ is invertible: $v = G^{-1}(u)$ for all $u \in [0,1]$, with support $(\underline{v}, \overline{v})$, where $\underline{v} = \lim_{u \downarrow 0} G^{-1}(u)$ and $\overline{v} = \lim_{u \uparrow 1} G^{-1}(u)$.

Legendre polynomials $\rho_n(u)$ of order $n \geq 2$ on the unit interval $[0,1]$ can be constructed recursively by

$$\rho_n(u) = \frac{\sqrt{2n-1}\sqrt{2n+1}}{n}(2u-1)\rho_{n-1}(u) - \frac{(n-1)\sqrt{2n+1}}{n\sqrt{2n-3}}\rho_{n-2}(u)$$

starting from

$$\rho_0(u) = 1,\ \rho_1(u) = \sqrt{3}(2u-1).$$

These polynomials are orthonormal, in the sense that

$$\int_0^1 \rho_m(u)\rho_k(u)du = \begin{cases} 1 & \text{for } m = k \\ 0 & \text{otherwise} \end{cases}$$

Theorem 1 in Bierens (2007) states that the Legendre polynomials $\rho_k(u)$ form a complete orthonormal basis for the Hilbert space $L^2_B(0,1)$ of square-integrable Borel measurable real functions on $[0,1]$, endowed with the inner product $\langle f, g\rangle = \int_0^1 f(u)g(u)du$ and associated norm $||f||_2 = \sqrt{\langle f, f\rangle}$ and metric $||f-g||_2$. Hence, any square-integrable Borel measurable real function $q(u)$ on $[0,1]$ can be represented as $q(u) = \sum_{k=0}^{\infty}\gamma_k\rho_k(u)$ a.e. on $[0,1]$ where the γ_k's are the Fourier coefficients: $\gamma_k = \int_0^1 \rho_k(u)q(u)du$, satisfying $\sum_{k=0}^{\infty}\gamma_k^2 < \infty$. Therefore, every density function $h(u)$ on $[0,1]$ can be written as $h(u) = q(u)^2$ where $q(u) \in L^2_B(0,1)$, with $\int_0^1 q(u)^2du = \sum_{k=0}^{\infty}\gamma_k^2 = 1$.

Without loss of generality we may assume that $\gamma_0 \in (0,1)$, because given $h(u)$ we may assume that $q(u) = \sqrt{h(u)}$. Therefore, the restriction $\sum_{k=0}^{\infty}\gamma_k^2 = 1$ can be imposed by reparametrizing the γ_k's as

$$\gamma_0 = \frac{1}{\sqrt{1+\sum_{k=1}^{\infty}\delta_k^2}},\ \gamma_k = \frac{\delta_k}{\sqrt{1+\sum_{k=1}^{\infty}\delta_k^2}} \text{ for } k = 1,2,3,\ldots$$

Thus, any density function $h(u)$ on $[0,1]$ can be represented as

$$h(u) = \frac{(1+\sum_{k=1}^{\infty}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{\infty}\delta_k^2},\ \text{where } \sum_{k=1}^{\infty}\delta_k^2 < \infty, \tag{38}$$

and therefore any absolutely continuous distribution function $H(u)$ on $[0,1]$ takes the form (37), where $h(u)$ is of the form (38).

4.3 Compact Spaces of Density and Distribution Functions

Since, indirectly, the density h in (36) plays the role of unknown parameter, we will first construct a compact metric space of densities on the unit interval. This can be done by imposing restrictions on the parameters δ_k in (38), as follows.

Lemma 7. *Let $\mathcal{D}(0,1)$ be the space of density function $h(u)$ on $[0,1]$ of the form* (38), *where the parameters δ_k are restricted by the inequality*

$$|\delta_k| \leq c\left(1+\sqrt{k}\ln k\right)^{-1},\ k=1,2,3,.... \tag{39}$$

for an a priori chosen constant $c>0$. If we endow $\mathcal{D}(0,1)$ with the L^1 metric

$$||h_1-h_2||_1 = \int_0^1 |h_1(u)-h_2(u)|\,du, \tag{40}$$

then $\mathcal{D}(0,1)$ is a compact metric space. Consequently, the corresponding space of absolutely continuous distribution functions on $[0,1]$,

$$\mathcal{H}(0,1) = \left\{H(u) = \int_0^u h(x)dx,\ h\in\mathcal{D}(0,1)\right\},$$

endowed with the "sup" metric $\sup_{0\leq u\leq 1}|H_1(u)-H_2(u)|$, is a compact metric space as well.

Proof: Bierens (2007, Theorem 8).

To construct compact spaces of densities and distribution functions on $(0,\infty)$,

Assumption 4. *Choose an absolutely continuous distribution function $G(v)$ with density $g(v)$, finite expectation $\int_0^\infty vg(v)dv<\infty$, and support $(0,\infty)$, as initial guess of the true value distribution F_0.*

The reason for requiring that $\int_0^\infty vg(v)dv<\infty$ is that if the initial guess $G(v)$ of $F_0(v)$ is right then $\int_0^\infty vdF_0(v)<\infty$. C.f. Assumption 2.

It follows now straightforwardly from Lemma 7 that:

Lemma 8. *With $G(v)$ and $g(v)$ as in Assumption 4, the space*

$$\mathcal{D}(G) = \{f(v) = h(G(v))g(v),\ h \in \mathcal{D}(0,1)\} \tag{41}$$

of densities on $(0,\infty)$, endowed with the L^1 metric

$$\int_0^\infty |f_1(v) - f_2(v)|\, dv, \tag{42}$$

is a compact metric space. Moreover, the corresponding space

$$\mathcal{F} = \left\{F(v) = \int_0^v f(x)dx,\ f \in \mathcal{D}(G)\right\} \tag{43}$$

of absolutely continuous distribution functions on $(0,\infty)$, endowed with one of the sup *metrics* (19) *or* (20), *is a compact metric space as well.*

Remark. The compactness of the spaces $\mathcal{D}(0,1)$ and $\mathcal{D}(G)$ follows from Bierens (2007, Lemmas A.1-A.3) and the fact that for metric spaces compactness is equivalent to sequential compactness. See Royden (1968, Corollary 14, p. 163). Sequential compactness means that any infinite sequence in the metric space has a convergent subsequence which converges to an element in this space. Thus, since $\mathcal{D}(G)$ is compact, any sequence $f_n \in \mathcal{D}(G)$ has a further subsequence f_{n_m} such that for some $f_* \in \mathcal{D}(G)$, $\lim_{m\to\infty} \int_0^\infty |f_{n_m}(v) - f_*(v)|dv = 0$. Consequently, any sequence of distribution functions $F_n(v) = \int_0^v f_n(x)dx \in \mathcal{F}$ has a further subsequence $F_{n_m}(v) = \int_0^v f_{n_m}(x)dx$ such that, with $F_*(v) = \int_0^v f_*(x)dx \in \mathcal{F}$,

$$\begin{aligned} \limsup_{m\to\infty} \sup_{v\geq p_0} |F_{n_m}(v) - F_*(v)| &\leq \limsup_{m\to\infty} \sup_{v>0} |F_{n_m}(v) - F_*(v)| \\ &\leq \limsup_{m\to\infty} \int_0^\infty |f_{n_m}(v) - f_*(v)|\, dv = 0. \end{aligned}$$

Therefore, $\mathcal{F}$ is sequentially compact and thus compact, regardless whether we endow $\mathcal{F}$ with the metric (19) or the metric (20).

Now $\mathcal{F}$ is the "parameter" space of candidate value distributions F, provided that

Assumption 5. *The constant $c > 0$ in* (39) *is chosen so large that the density f_0 of the true value distribution F_0 is contained in $\mathcal{D}(G)$.*

Note that not all the densities in $\mathcal{D}(G)$ will have finite expectations. The reason is that it is always possible to select a sequence $f_n \in \mathcal{D}(G)$ with finite expectations such that for a density $f \in \mathcal{D}(G)$ with infinite expectation, $\lim_{n\to\infty} \int_0^\infty |f_n(v) - f(v)|dv = 0$. However, this is of no consequence, as long as the true value distribution F_0 has finite expectation and Assumption 5 holds, because then the true bid distribution $\Lambda_0(b)$ has bounded support, so that Lemma 4 is applicable[10] for all bid distributions $\Lambda(b)$ corresponding to an $F \in \mathcal{F}$.

4.4 The Sieve Spaces and the Uniform Strong Consistency of the Sieve Estimator

For a density function $h(u)$ in (38) and its associated parameter sequence $\{\delta_k\}_{k=1}^\infty$, let

$$h_n(u) = h(u|\boldsymbol{\delta}_\mathbf{n}) = \frac{(1 + \sum_{k=1}^n \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^n \delta_k^2}, \text{ where } \boldsymbol{\delta}_\mathbf{n} = (\delta_1, \ldots, \delta_n)', \quad (44)$$

be the n-th order truncation of $h(u)$. The case $n = 0$ corresponds to the uniform density: $h_0(u) = 1$. Following Gallant and Nychka (1987) we will call this truncated density a SNP density function. It has been shown by Bierens (2007) that

$$\lim_{n\to\infty} \int_0^1 |h_n(u) - h(u)|\, du = 0. \quad (45)$$

Thus, defining the space of n-th order truncations of $h(u)$ by

$$\mathcal{D}_n(0,1) \quad (46)$$
$$= \left\{ h_n(u) = \frac{(1 + \sum_{k=1}^n \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^n \delta_k^2},\ |\delta_k| \le c\left(1 + \sqrt{k}\ln k\right)^{-1},\ k \ge 1 \right\},$$

it follows that for each $h \in \mathcal{D}(0,1)$ there exists a sequence $h_n \in \mathcal{D}_n(0,1)$ of SNP densities such that (45) holds. Consequently, defining

$$\mathcal{H}_n(0,1) = \left\{ H_n(u) = \int_0^u h_n(v)du,\ h_n \in \mathcal{D}_n(0,1) \right\} \quad (47)$$

[10] Recall that Lemma 4 does not require that the other bid distribution $\Lambda(b)$ has bounded support.

it follows that for each distribution function $H \in \mathcal{H}(0,1)$ there exists a sequence of SNP distribution functions $H_n \in \mathcal{H}_n(0,1)$ such that

$$\lim_{n\to\infty} \sup_{0\leq u\leq 1} |H_n(u) - H(u)| = 0. \tag{48}$$

Note that the SNP distribution functions $H_n(u)$ can be computed as a ratio of two quadratic forms in $\boldsymbol{\delta}_\mathbf{n} = (\delta_1, \ldots, \delta_n)'$, using the approach in Bierens (2007).

The densities $h_n \in \mathcal{D}_n(0,1)$ will be used to construct SNP densities $f_n(v) = h_n(G(v))g(v)$ of candidate value distributions, where G and its density g are chosen in advance according to Assumption 4. The latter implies that

$$\int_0^\infty v f_n(v)dv = \int_0^\infty v h_n(G(v))g(v)dv \leq \sup_{0\leq u\leq 1} h_n(u) \int_0^\infty v g(v)dv < \infty$$

because each $h_n(u) \in \mathcal{D}_n$ is a squared polynomial of order n with bounded coefficients and is therefore uniformly bounded,

$$\overline{h}_n = \sup_{h_n\in\mathcal{D}_n} \sup_{0\leq u\leq 1} h_n(u) < \infty,$$

although it is possible that $\lim_{n\to\infty} \overline{h}_n = \infty$.

Similar to (41) and (43), define the increasing sets

$$\begin{aligned} \mathcal{D}_n(G) &= \{f_n(v) = h_n(G(v))g(v),\ h_n \in \mathcal{D}_n(0,1)\}, & (49)\\ \mathcal{F}_n &= \{F_n(v) = H_n(G(v)),\ H_n \in \mathcal{H}_n(0,1)\}. & (50) \end{aligned}$$

Then

Lemma 9. *Choose G as in Assumption 4. Then all the densities $f_n \in \mathcal{D}_n(G)$ have finite expectation. Moreover, for each density $f \in \mathcal{D}(G)$ there exists a sequence of densities $f_n \in \mathcal{D}_n(G)$ such that $\lim_{n\to\infty} \int_0^\infty |f_n(v) - f(v)|dv = 0$, and for each distribution function $F \in \mathcal{F}$ there exists a sequence of distribution functions $F_n \in \mathcal{F}_n$ such that $\lim_{n\to\infty} \sup_{v>0} |F_n(v) - F(v)| = 0$. Consequently, $\{\mathcal{D}_n(G)\}_{n=0}^\infty$ is dense in $\mathcal{D}(G)$, and $\{\mathcal{F}_n\}_{n=0}^\infty$ is dense in $\mathcal{F}$.*

The sequence of spaces $\mathcal{F}_n$ now forms the sieve. Since the distribution functions in $\mathcal{F}_n$ are parametric, with parameters $\boldsymbol{\delta}_\mathbf{n} = (\delta_1, \ldots, \delta_n)'$, the computation of $\widehat{F}_n = \arg\min_{F\in\mathcal{F}_n} \widehat{Q}(F)$ is feasible. In particular, $\widehat{F}_n$ can be computed via the simplex method of Nelder and Mead (1965).

Summarizing, it has been shown that,

Theorem 4. *With $\mathcal{F}$ defined by* (43) *with sieve spaces $\mathcal{F}_n$ defined by* (50), *it follows from Assumptions 1-5 that for an arbitrary subsequence n_N satisfying* $\lim_{N\to\infty} n_N = \infty$ *the SNP-ISM sieve estimator*

$$\widetilde{F} = \arg\min_{F\in\mathcal{F}_{n_N}} \widehat{Q}(F)$$

is uniformly strongly consistent. In particular, $\sup_{v\geq p_0} |\widetilde{F}(v) - F_0(v)| \to 0$ *a.s. if the reservation price p_0 is binding, and* $\sup_{v\geq 0} |\widetilde{F}(v) - F_0(v)| \to 0$ *a.s. if not.*

5 An Integrated Moment Test of the Validity of the First-Price Auction Model

5.1 The Test

If the independent private values paradigm and/or the risk neutrality assumption do not hold, the bid functions (1) and (2) no longer apply to the actual bids. Since the simulated bids are derived from these bid functions, we then have, by Lemma 4, that

$$\widehat{Q}(\widetilde{F}) \to \inf_{F\in\mathcal{F}} \overline{Q}(F) > 0 \text{ a.s.}, \tag{51}$$

where $\widetilde{F}$ is the sieve estimator. This suggests to use $\widehat{Q}(\widetilde{F})$ as a basis for a consistent integrated moment (IM) test of the null hypothesis that

H_0: *the actual bids come from a first-price sealed bid auction where values are independent, private and bidders are symmetric and risk-neutral,*

against the general alternative that

H_1: *the null hypothesis H_0 is false.*

The IM test we will propose is based on the fact that similar to the results in Bierens (1990) and Bierens and Ploberger (1997) for the Integrated Conditional Moment (ICM) test, the following result holds.

Theorem 5. *Under H_0 and Assumptions 1-2,*

$$\widehat{W}_N(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left(\exp\left(i.t.\widetilde{B}_j\left(F_0\right)\right) - \exp\left(i.t.B_j\right)\right) \Rightarrow W(t)$$

on $[-\kappa,\kappa]$, where $W(t)$ is a complex-valued zero-mean Gaussian process on $[-\kappa,\kappa]$ with covariance function[11]

$$\Gamma(t_1,t_2) = E\left[W(t_1)\overline{W(t_2)}\right] = E\left[\widehat{W}_N(t_1)\overline{\widehat{W}_N(t_2)}\right].$$

Hence[12]

$$N.\widehat{Q}(F_0) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} \left|\widehat{W}_N(t)\right|^2 dt \to_d \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} |W(t)|^2 \, dt.$$

Note that this result does not imply that $N.\widehat{Q}(\widetilde{F}) \to_d \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} |W(t)|^2 dt$, because this requires that $\sqrt{N}\left(\widehat{\psi}(t|\widetilde{F}) - \widehat{\psi}(t|F_0)\right) \to_d 0$, which in view of the proof of Lemma 6 (see the Appendix) requires that the subsequence n_N in Theorem 4 is chosen such that $||\widetilde{F} - F_0|| = o_p\left(1/\sqrt{N}\right)$. However, if

Assumption 6. *The true value distribution F_0 is of the SNP type itself*: $F_0 \in \cup_{n=1}^{\infty}\mathcal{F}_n$,

then there exists a smallest natural number n_0 such that $F_0 \in \mathcal{F}_{n_0}$, so that

$$N.\widehat{Q}(\widetilde{F}) \le N.\widehat{Q}(F_0) \text{ for } n_N \ge n_0.$$

This suggests that upper bounds of the critical values of the test can be based on the limiting distribution of $N.\widehat{Q}(F_0)$. The consistency of this IM tests then follows from (51).

The result under H_0 follows from the fact that

Lemma 10. *Under H_0 and Assumptions 1-2 the process $\widehat{W}_N(.)$ is tight*[13] *on $[-\kappa,\kappa]$.*

Proof: Appendix.

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[11] Now the bar denotes the complex conjugate: $\overline{a+i.b} = a - i.b$.

[12] By the continuous mapping theorem.

[13] See Billingsley (1999) for the definition of tightness.

5.2 Bootstrap Critical Values

The problem in approximating the limiting process $W(t)$ by bootstrapping is two-fold, namely that we cannot increase $N \to \infty$ because the B_j's are only observable for $j = 1, ..., N$, and F_0 is unknown. To overcome these problems, generate for large M simulated bids $\widetilde{B}_j$, $j = 1, 2, ..., 2.M$, from the bid distribution corresponding to the sieve estimator $\widetilde{F}$ of F_0, according to the approach in Assumption 2. Thus, draw U_j, $j = 1, 2, ..., 2.M$, independently from the uniform $[0, 1]$ distribution, and generate the corresponding simulated bids $\widetilde{B}_j$ by $\widetilde{B}_j = \eta\left(U_j|\widetilde{F}\right)$. C.f. (6). Denote

$$\begin{aligned}\widetilde{W}_M\left(t|\widetilde{F}\right) &= \frac{1}{\sqrt{M}}\sum_{j=1}^{M}\exp\left(i.t.\widetilde{B}_j\right) - \frac{1}{\sqrt{M}}\sum_{j=M+1}^{2M}\exp\left(i.t.\widetilde{B}_j\right) \\ &= \frac{1}{\sqrt{M}}\sum_{j=1}^{M}\exp\left(i.t.\eta\left(U_j|\widetilde{F}\right)\right) - \frac{1}{\sqrt{M}}\sum_{j=M+1}^{2M}\exp\left(i.t.\eta\left(U_j|\widetilde{F}\right)\right).\end{aligned}$$

Then similar to Lemma 10, $\widetilde{W}_M\left(t|\widetilde{F}\right)$ is tight on $[-\kappa, \kappa]$, conditional on $\widetilde{F}$. Hence for $M \to \infty$,

$$\widetilde{W}_M\left(.|\widetilde{F}\right) \Rightarrow W\left(.|\widetilde{F}\right) \text{ on } [-\kappa, \kappa], \text{ conditional on } \widetilde{F}$$

where $W\left(.|\widetilde{F}\right)$ is a complex-valued zero-mean Gaussian process with conditional covariance function

$$\widetilde{\Gamma}(t_1, t_2|\widetilde{F}) = E\left[\widetilde{W}_M\left(t_1|\widetilde{F}\right)\overline{\widetilde{W}_M\left(t_2|\widetilde{F}\right)}\middle|\widetilde{F}\right]$$

Lemma 11. *Under H_0 and the conditions of Theorem 4,*

$$\sup_{(t_1,t_2)\in[-\kappa,\kappa]\times[-\kappa,\kappa]}\left|\widetilde{\Gamma}(t_1, t_2|\widetilde{F}) - \Gamma\left(t_1, t_2\right)\right| \to 0 a.s. \tag{52}$$

as $N \to \infty$, and consequently

$$\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|W\left(t|\widetilde{F}\right)\right|^2 dt \to_d \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|W\left(t\right)\right|^2 dt. \tag{53}$$

Hence, for $M \to \infty$ first, and then $N \to \infty$,

$$\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widetilde{W}_M\left(t|\widetilde{F}\right)\right|^2 dt \to_d \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2 dt. \tag{54}$$

Proof: Appendix

Therefore, bootstrap critical values of $\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2 dt$ can be computed as follows. First, choose a large M, say $M = 1000$. Next, generate $\widetilde{T}_k = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widetilde{W}_M\left(t|\widetilde{F}\right)\right|^2 dt$ independently for $k = 1, ..., K$, say $K = 500$, and sort the statistics $\widetilde{T}_k$ in increasing order. The $\alpha \times 100\%$ bootstrap critical value is then $\widetilde{T}_{(1-\alpha)K}$.

5.3 Critical Values Based on a Further Upper Bound

It has been shown by Bierens and Ploberger (1997) that

$$\frac{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2 dt}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\Gamma(t,t)dt} \le \sup \frac{1}{m}\sum_{k=1}^{m}\varepsilon_j^2 = \overline{T},$$

say, where the ε_j's are independently $N(0,1)$ distributed. Therefore, with $\widehat{\Gamma}(t,t)$ a consistent estimator of $\Gamma(t,t)$, we have

$$\widetilde{T} = \frac{N.\widehat{Q}(\widetilde{F})}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\widehat{\Gamma}(t,t)dt} \le \frac{N.\widehat{Q}(F_0)}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\widehat{\Gamma}(t,t)dt} \to_d \frac{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2 dt}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\Gamma(t,t)dt} \le \overline{T}.$$

The 5% and 10% critical values based on $\overline{T}$ with are 4.26 and 3.23, respectively.

As to the choice of $\widehat{\Gamma}(t,t)$, note that

Lemma 12. $\Gamma(t,t) = 2 - 2|\varphi(t)|^2$, *where* $\varphi(t)$ *is the characteristic function of the actual bid distribution. Then* $\widehat{\Gamma}(t,t) = 2 - 2|\widehat{\varphi}_0(t)|^2 \to \Gamma(t,t)$ *a.s., pointwise in* t, *where* $\widehat{\varphi}_0(t)$ *is the empirical characteristic function of the observed bids.*

6 Determination of the Sieve Order Via an Information Criterion

Recall that under Assumption 6 there exists a smallest natural number n_0 such that $F_0 \in \mathcal{F}_{n_0}$. The question now arises how to estimate n_0 consistently.

For nested likelihood models this can be done via information criteria, for example the Hannan-Quinn (1979) or Schwarz (1978) information criteria. These information criteria are of the form

$$C_N(n) = \frac{-2}{N} \ln\left(L_N(n)\right) + n.\frac{\phi(N)}{N}$$

where $L_N(n)$ is the maximum likelihood of a model with n parameters, with $\phi(N) = \ln(N)$ for the Schwarz criterion and $\phi(N) = 2.\ln(\ln(N))$ for the Hannan-Quinn criterion. Then for $2 \leq n \leq n_0$,

$$\begin{aligned} p\lim_{N\to\infty} \left(C_N(n) - C_N(n-1)\right) &= p\lim_{N\to\infty} \frac{2}{N} \ln\left(L_N(n-1)\right) \\ &\quad - p\lim_{N\to\infty} \frac{2}{N} \ln\left(L_N(n)\right) < 0 \end{aligned}$$

whereas for $n > n_0$, $-2\left(\ln\left(L_N(n_0)\right) - \ln\left(L_N(n)\right)\right) \to_d \chi^2_{n-n_0}$, hence

$$p\lim_{N\to\infty} \frac{N}{\phi(N)} \left(C_N(n) - C_N(n_0)\right) = n - n_0$$

Note that the latter result only hinges on $-2\left(\ln\left(L_N(n_0)\right) - \ln\left(L_N(n)\right)\right) = O_p(1)$.

Since by Theorem 5,

$$N\left(\inf_{F\in\mathcal{F}_n} \widehat{Q}(F) - \inf_{F\in\mathcal{F}_{n_0}} \widehat{Q}(F)\right) = O_p(1) \text{ if } n > n_0,$$

whereas for $2 \leq n \leq n_0$,

$$p\lim_{N\to\infty} \left(\inf_{F\in\mathcal{F}_n} \widehat{Q}(F) - \inf_{F\in\mathcal{F}_{n-1}} \widehat{Q}(F)\right) < 0$$

it seems that in our case we may replace $\frac{-2}{N}\ln\left(L_N(n)\right)$ by $\inf_{F\in\mathcal{F}_n} \widehat{Q}(F)$:

$$\widehat{C}_N(n) = \inf_{F\in\mathcal{F}_n(G)} \widehat{Q}(F) + n.\frac{\phi(N)}{N},$$

and estimate n_0 by $\widehat{n}_N = \arg\min \widehat{C}_N(n)$. Asymptotically that will work: $\lim_{N\to\infty} P\left[\widehat{n}_N = n_0\right] = 1$. However, in practice it will not, due to the fact that $\widehat{Q}(F)$ is bounded: $\sup_F \widehat{Q}(F) \leq 4$, and that $\inf_{F\in\mathcal{F}_n} \widehat{Q}(F)$ will be close to zero if $n < n_0$ but not too far away from n_0, so that in small samples the penalty term $n.\phi(N)/N$ may dominate $\inf_{F\in\mathcal{F}_n} \widehat{Q}(F)$ too much. Therefore, we propose the following modification of $\widehat{C}_N(n)$:

$$\begin{aligned} \widetilde{C}_N(n) &= \inf_{F\in\mathcal{F}_n} \widehat{Q}(F) + \Phi(n).\frac{\phi(N)}{N}, \\ \phi(N) &= o(N), \quad \lim_{N\to\infty} \phi(N) = \infty. \end{aligned} \tag{55}$$

where $\Phi(n)$ is an increasing but bounded function of n. For example, let for some $\alpha \in (0,1)$,

$$\Phi(n) = 1 - (n+1)^{-\alpha}. \tag{56}$$

Then similar to the Hannan-Quinn and Schwarz information criteria we have:

Theorem 6. *Let* $\widetilde{n}_N = \max_{s.t.\ \widetilde{C}_N(n)\leq\widetilde{C}_N(n-1)} n$ *and* $\widetilde{F} = \arg\min_{F\in\mathcal{F}_{\widetilde{n}_N}(G)} \widehat{Q}(F)$. *Under Assumption 6,* $\lim_{N\to\infty} P\left[\widetilde{n}_N = n_0\right] = 1$*, hence* $||\widetilde{F} - F_0|| \to 0$ *a.s. If Assumption 6 is not true then* $p\lim_{N\to\infty} \widetilde{n}_N = \infty$*, hence* $p\lim_{N\to\infty} ||\widetilde{F} - F_0|| = 0$.

Proof: Appendix.

7 Some Numerical Experiments

In this section we check the performance of the IM test of the validity of the first-price auction model, and the fit of SNP-ISM density estimator with estimated the truncation order $\widetilde{n}_N$, via a few numerical experiments. In all experiments we use the exponential distribution

$$G(v) = 1 - \exp(-v/3),\ g(v) = \frac{1}{3}\exp(-v/3) \tag{57}$$

as the initial guess for the value distribution, and the truncation order $\widetilde{n}_N$ is determined via the approach in Theorem 6, with information criterion

$$\widetilde{C}_N(n) = \inf_{F\in\mathcal{F}_n} \widehat{Q}(F) + \left(1 - (n+1)^{-1/3}\right).\frac{\ln(\ln(N))}{N}. \tag{58}$$

The 5% and 10% bootstrap critical values of the IM test will be based on $K = 500$ bootstrap samples.

7.1 The IM Test

In this subsection we check the performance of the IM test by two numerical examples. The first is the case where then null hypothesis that the observed bids can be rationalized by the first-price sealed bid auction model with independent private values (IPV) is false, and the second case is where this null hypothesis is true. In both cases we have generated bids from 500 identical and independent auctions, each with two sealed bids and no reservation price. In both cases the true value distribution is exponential, which is different from the initial guess (57), namely

$$F_0(v) = 1 - \exp(-v),\ f_0(v) = \exp(-v).$$

In the first case the observed bids come from a second price auction with IPV and two symmetric, risk-neutral bidders. As is well known, in a second price auction, it is a weakly dominant strategy to bid the true value. See Krishna (2002). Therefore, the actual bids are drawn directly from the value distribution $F_0(v)$. As to the results, the estimate of the truncation order is $\widetilde{n}_N = 4$, and the value of the corresponding IM test statistic is $\widetilde{T} = 3.0531$. The 5% and 10% bootstrap critical values are $\widetilde{T}_{0.95K} = 1.1882$ and $\widetilde{T}_{0.90K} = 0.9447$, respectively.[14] Consequently, the null hypothesis is firmly rejected at the 5% significance level, as expected.

The second example is a first-price auction with risk-averse bidders. Again, we have generated bids from 500 auctions, where each auction has two risk-averse bidders, with utility function $U(x) = x^{1/2}$. In this case the equilibrium bid function has a closed form. See example 4.1 in Krishna (2002). But, as is pointed out by Krishna (2002), a first-price auction with two risk-averse bidders and a value distribution $F_0(v)$ is observationally equivalent to a first-price auction with two risk-neutral bidders with the value distribution $F_0(v)^2$.[15] Therefore, in this case we may expect that our IM test will not reject the null hypothesis. Note however that in this case Assumption 6 does not hold. Nevertheless, the IM test statistic now takes the value $\widetilde{T} = 0.0004$, with 5% and 10% bootstrap critical values $\widetilde{T}_{0.95K} = 0.3630$ and $\widetilde{T}_{0.90K} = 0.2661$, respectively, based on the estimated truncation order $\widetilde{n}_N = 2$. Thus, the (true) null hypothesis is not rejected, which is the anticipated result.

[14] Thus, under the null hypothesis, $P(\widetilde{T} \geq 1.1882) = 0.05$ and $P(\widetilde{T} \geq 0.9447) = 0.1$.

[15] This holds when the utility function is a constant relative risk aversion (CRRA) utility function, $U(x) = x^{\alpha}$ with $0 < \alpha < 1$ and $U(0) = 0$. See example 4.1 in Krishna (2002).

7.2 The Fit

In the previous experiments the estimated truncation orders $\widetilde{n}_N = 4$ and $\widetilde{n}_N = 2$ are small, so the question arises whether for such a small truncation order the value density can be adequately approximated. In this section we check this for three cases. In each case we generate independently 200 auctions without a reservation price, where each auction consists of 5 bids whose private values come from a chi-square distribution, so in each case we have a sample of 1000 i.i.d. bids. The three cases only differ with respect to the degrees of freedom r of the chi-square distribution, namely $r = 3, 4, 5$, respectively.

In these cases the true value densities $f_0(v)$ are quite different from the density $g(v)$ of the initial guess (57), in particular the left tails, as shown in Figure 1:

Insert Figure 1 here
Figure 1: $g(v) = \exp(-v/3)/3$ compared with the χ^2_r densities for $r = 3, 4, 5$

Thus, the SNP density $h_n(u)$ needs to convert the exponential density $g(v)$ into an approximation $f_n(v) = h_n\left(G(v)\right) g(v)$ of a χ^2_r density, so that $h_n(u)$ needs to bend down the left tail of $g(v)$ towards zero. This seems challenging. However, it appears that the SNP density $h_n(u)$ has no problem doing that, even for small values of n. First, the estimated truncation orders are small: $\widetilde{n}_N = 4, 2, 4$ for $r = 3, 4, 5$, respectively. To see whether these truncation orders are too small or not, we compare in Figures 2-4 the SNP sieve density estimators $f_{\widetilde{n}_N}(v) = h_{\widetilde{n}_N}\left(G(v)\right) g(v)$ with the true χ^2_r value densities $f_0(v)$ for $r = 3, 4, 5$.

Insert Figures 2, 3 and 4 here
Figure 2: $f_{\widetilde{n}_N}(v)$ (dashed curve) compared with the true χ^2_3 density $f_0(v)$
Figure 3: $f_{\widetilde{n}_N}(v)$ (dashed curve) compared with the true χ^2_4 density $f_0(v)$
Figure 4: $f_{\widetilde{n}_N}(v)$ (dashed curve) compared with the true χ^2_5 density $f_0(v)$

These figures show that our SNP-ISM estimation approach works remarkably well, certainly in view of the bad choice of the initial guess $g(v)$ for $f_0(v)$ (see Figure 1) and the small truncation orders. On the other hand, it seems from Figure 3 that the truncation order $\widetilde{n}_N = 2$ is somewhat too small, as the fit of $f_{\widetilde{n}_N}(v)$ for $\widetilde{n}_N = 4$ in Figures 2 and 4 looks better than in Figure 3.

8 Concluding Remarks

In this paper we have proposed a new semi-nonparametric estimation method for the value distribution of a first-price auction, based on a comparison of the empirical characteristic functions of the actual bids and simulated bids. Our approach differs fundamentally from the nonparametric estimation approaches in the literature in that we estimate the value distribution directly, whereas in the nonparametric auction literature, the value distribution is estimated indirectly via kernel estimation of the inverse bid function. Another novelty of our approach is that it yields as by-product an integrated moment test for the validity of the first-price auction model.

The approach in this paper can be extended to Dutch auctions and auctions with auction-specific heterogeneity. As to the latter, we have already made a start with this. See Bierens and Song (2007).

9 Appendix

9.1 Proof of Lemma 1

Let F be a continuous distribution function with support contained in $(\underline{v}, \overline{v})$, where $\underline{v} = \arg\min_{F(v)>0} v$ and $\overline{v} = \arg\max_{F(v)<1} v$. Suppose first that F is invertible on $(\underline{v}, \overline{v})$, i.e., for each $u \in (0,1)$ there exists a unique $v \in (\underline{v}, \overline{v})$ such that $u = F(v)$. It is a standard textbook exercise to verify that then for a random drawing V from F,

$$F(V) \sim \text{Uniform}[0,1]. \quad (59)$$

If F is not invertible then there exists a $u \in (0,1)$ such that $F(v) = u$ for more than one $v \in (\underline{v}, \overline{v})$. In particular, for such a u let

$$v_1(u) = \inf_{u=F(v)} v, \; v_2(u) = \sup_{u=F(v)} v. \quad (60)$$

Note that by the continuity of $F(v)$, $F(v_1(u)) = F(v_2(u)) = u$, hence $F(v) = u$ for all $v \in [v_1(u), v_2(u)]$. Consequently, $dF(v)/dv = 0$ for all $v \in (v_1(u), v_2(u))$. Moreover, $F(v) < u$ for $v \in (\underline{v}, v_1(u))$ and $F(v) > u$ for $v \in (v_2(u), \overline{v})$. Then for such a u,

$$P[F(V) \leq u] \;=\; E[\mathbf{1}(F(V) \leq u)]$$

$$
\begin{aligned}
&= \int_{\underline{v}}^{v_1(u)} \mathbf{1}\left(F(v) \leq u\right) dF\left(v\right) + \int_{v_1(u)}^{v_2(u)} \mathbf{1}\left(F(v) \leq u\right) dF\left(v\right) \\
&\quad + \int_{v_2(u)}^{\overline{v}} \mathbf{1}\left(F(v) \leq u\right) dF\left(v\right) \\
&= \int_{\underline{v}}^{v_1(u)} dF\left(v\right) + \int_{v_1(u)}^{v_2(u)} dF\left(v\right) = F\left(v_1\left(u\right)\right) = u,
\end{aligned}
$$

where $\mathbf{1}\left(.\right)$ is the indicator function. Since this result also holds if $v_1\left(u\right) = v_2\left(u\right)$, it follows that for all $u \in (0,1)$,

$$
P\left[F\left(V\right) \leq u\right] = u.
$$

Thus, the only requirement for (59) is that F is absolutely continuous.

To prove that, with U a random drawing from the uniform $[0,1]$ distribution, the solution V_U of $U = F(V_U)$ is a.s. unique, it suffices to prove that the set $S = \{u \in (0,1) : v_1\left(u\right) < v_2\left(u\right)\}$ has Lebesgue measure zero. The latter follows from the fact that for any pair $u_1, u_2 \in S$, $u_1 \neq u_2$, the intervals $\left(v_1\left(u_1\right), v_2\left(u_1\right)\right)$ and $\left(v_1\left(u_2\right), v_2\left(u_2\right)\right)$ are disjoint, which implies that S is countable because any collection of disjoint open intervals is countable. In particular, it is easy to verify that S is either empty or takes the form of a finite or countable infinite number of increasingly ordered points u_j in $(0,1)$:

$$
S = \cup_{j \in \mathbb{J}} \left\{u_j\right\}, \; 0 < u_j < u_{j+1} < 1 \text{ for all } j \in \mathbb{J}
$$

where $\mathbb{J}$ is a subset of the space $\mathbb{Z}$ of integers.

Finally, $P[U \in S] = 0$ implies that

$$
\begin{aligned}
P[V_U \leq v] &= P[V_U \leq v \text{ and } U \in S] + P[V_U \leq v \text{ and } U \notin S] \\
&= P[V_U \leq v \text{ and } U \notin S] \\
&= P\left[F\left(V_U\right) \leq F\left(v\right) \text{ and } U \notin S\right] \\
&= P\left[U \leq F\left(v\right) \text{ and } U \notin S\right] \\
&= P\left[U \leq F\left(v\right) \text{ and } U \in S\right] + P\left[U \leq F\left(v\right) \text{ and } U \notin S\right] \\
&= P\left[U \leq F\left(v\right)\right] = F\left(v\right).
\end{aligned}
$$

9.2 Proof of Theorem 1

Originally this uniform strong law was derived by Jennrich (1969, Theorem 2) for the case that Θ is a compact subset of a Euclidean space and

$\mu_j(\theta) = \mu(X_j, \theta)$, where X_j is an i.i.d. sequence of random vectors in a Euclidean space with support $\mathcal{X}$, $\mu(x, \theta)$ is Borel measurable in x for each $\theta \in \Theta$, and $\mu(x, \theta)$ is continuous in θ for each $x \in \mathcal{X}$. However, it is not hard to verify from the more detailed proof in Bierens (2004, Appendix to Chapter 6) of Jennrich's result that this law carries over to a.s. continuous random functions on a compact metric space Θ with metric $\rho(\theta_1, \theta_2)$, provided that for each $\theta_0 \in \Theta$ and arbitrary $\delta > 0$, $\sup_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta)$ and $\inf_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta)$ are measurable, because then by the a.s. continuity condition,

$$\begin{aligned}
\lim_{\delta \downarrow 0} \left(E\left[\sup_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta) \right] - E\left[\inf_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta) \right] \right) &= 0, \\
\lim_{\delta \downarrow 0} \left(\sup_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} E\left[\mu_j(\theta) \right] - \inf_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} E\left[\mu_j(\theta) \right] \right) &= 0,
\end{aligned}$$

where the expectations are well-defined. These results play a key-role in the proof.

To prove the measurability of $\sup_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta)$ and $\inf_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta)$ along the lines of the proof of Lemma 2 of Jennrich(1969), we first establish the existence of an increasing sequence of **finite** subsets Θ_n of Θ which is dense in Θ, i.e., Θ is the closure of $\cup_{n=1}^{\infty} \Theta_n$. These sets Θ_n can be constructed as follows. For each $\theta \in \Theta$ and n, let $U_n(\theta) = \{\theta_* \in \Theta : \rho(\theta, \theta_*) < 1/n\}$. Then $\cup_{\theta \in \Theta} U_n(\theta)$ is a open covering of Θ hence by the definition of compactness there exists a finite set $\Theta_n = \{\theta_{1,n}, ..., \theta_{M_n,n}\}$ such that $\Theta \subset \bigcup_{\theta \in \Theta_n} U_n(\theta)$. To show that $\bigcup_{n=1}^{\infty} \Theta_n$ is dense in Θ, pick an arbitrary $\theta \in \Theta$, and observe that for each n there exists an $\theta_n \in \Theta_n$ such that $\rho(\theta, \theta_n) < 1/n$. Therefore, for each $\theta \in \Theta$ there exists a sequence $\{\theta_n\}$ in $\cup_{n=1}^{\infty} \Theta_n$ such that $\lim_{n \to \infty} \rho(\theta, \theta_n)$, hence $\cup_{n=1}^{\infty} \Theta_n$ is dense in Θ. Consequently

$$\sup_{\theta \in \Theta} \mu_1(\theta) = \lim_{n \to \infty} \sup \sup_{\theta \in \Theta_n} \mu_1(\theta) = \lim_{n \to \infty} \sup \max_{\theta \in \Theta_n} \mu_1(\theta).$$

Since Θ_n is finite, $\max_{\theta \in \Theta_n} \mu_1(\theta)$ is measurable, hence $\sup_{\theta \in \Theta} \mu_1(\theta)$ is measurable. The same holds for the "inf" case, and for $\sup_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta)$ and $\inf_{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta} \mu_j(\theta)$, because the sets $\{\theta \in \Theta, \rho(\theta, \theta_0) \leq \delta\}$ are compact.

9.3 Proof of Theorem 2

The key of the proof of Theorem 2 is the easy inequality

$$\begin{aligned} 0 &\leq \overline{Q}\left(\widehat{\theta}_N\right) - \overline{Q}\left(\theta_0\right) \leq \overline{Q}\left(\widehat{\theta}_N\right) - \widehat{Q}_N\left(\widehat{\theta}_N\right) + \widehat{Q}_N\left(\theta_0\right) - \overline{Q}\left(\theta_0\right) \\ &\leq 2\sup_{\theta\in\Theta}\left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right|, \end{aligned}$$

so that

$$\overline{Q}\left(\widehat{\theta}_N\right) \to \overline{Q}\left(\theta_0\right) \text{ a.s.} \tag{61}$$

The rest of the proof is now similar to the case where Θ is a compact subset of a Euclidean space. See Jennrich (1969) or the proof of Theorem 6.14 in Bierens (2004, p. 174).

9.4 Proof of Theorem 3

It suffices to show that $\overline{Q}(\widetilde{\theta}_N) \to \overline{Q}(\theta_0)$ a.s., because then the rest of the proof is the same as for Theorem 2.

We can choose a sequence $\theta_n \in \Theta_n$ such that

$$\lim_{n\to\infty} \rho(\theta_n, \theta_0). \tag{62}$$

Then

$$\begin{aligned} 0 &\leq \overline{Q}(\widetilde{\theta}_N) - \overline{Q}(\theta_0) = \overline{Q}(\widetilde{\theta}_N) - \widehat{Q}_N(\widetilde{\theta}_N) + \widehat{Q}_N(\widetilde{\theta}_N) - \overline{Q}(\theta_0) \\ &\leq \sup_{\theta\in\Theta_{n_N}}\left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| + \widehat{Q}_N(\theta_{n_N}) - \overline{Q}(\theta_{n_N}) + \overline{Q}(\theta_{n_N}) - \overline{Q}(\theta_0) \\ &\leq 2\sup_{\theta\in\Theta_{n_N}}\left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| + \overline{Q}(\theta_{n_N}) - \overline{Q}(\theta_0) \\ &\leq 2\sup_{\theta\in\Theta}\left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| + \overline{Q}(\theta_{n_N}) - \overline{Q}(\theta_0) \to 0 \text{ a.s.} \end{aligned}$$

because $\sup_{\theta\in\Theta}|\widehat{Q}_N(\theta) - \overline{Q}(\theta)| \to 0$ a.s. by the conditions of Theorem 2, and $\lim_{N\to\infty}\overline{Q}(\theta_{n_N}) = \overline{Q}(\theta_0)$ by (62) and the continuity of $\overline{Q}(\theta)$ on Θ.

9.5 Proof of Lemma 6

We only consider the binding reservation price case. First, we show that

$$\lim_{n\to\infty} \widetilde{B}\left(F_n\right) = 0 \text{ if and only if } \widetilde{B}\left(F\right) = 0 \tag{63}$$

as follows. Suppose that $\widetilde{B}(F) = 0$, which is equivalent to $F(p_0) < U$, and $\limsup_{n\to\infty} \widetilde{B}(F_n) \geq p_0$. The latter implies that there exists a subsequence n_m such that $\widetilde{B}(F_{n_m}) \geq p_0$ for $m = 1, 2, 3, ...$,which by (6) implies that $U \leq F_{n_m}(p_0)$ for $m = 1, 2, 3, ...$ However, this is not possible because then by (19), $||F_{n_m} - F|| > U - F(p_0)$ for $m = 1, 2, 3, ...$, whereas (32) implies that $\lim_{m\to\infty} ||F_{n_m} - F|| = 0$. Thus, $\widetilde{B}(F) = 0$ implies $\lim_{n\to\infty} \widetilde{B}(F_n) = 0$. Similarly, $\lim_{n\to\infty} \widetilde{B}(F_n) = 0$ implies $\widetilde{B}(F) = 0$.

Next we show that (33) is true, by contradiction. Suppose that $\limsup_{n\to\infty} \widetilde{V}(F_n) > \widetilde{V}(F)$, and note that $\widetilde{V}(F)$ is a.s. a continuity point of F. Then there exists a subsequence n_m and an $\varepsilon > 0$ such that for all m, $\widetilde{V}(F_{n_m}) > \widetilde{V}(F) + \varepsilon$. But then $U = F\left(\widetilde{V}(F)\right) < F\left(\widetilde{V}(F) + \varepsilon\right)$ a.s. and

$$U = F_{n_m}\left(\widetilde{V}(F_{n_m})\right) \geq F_{n_m}\left(\widetilde{V}(F) + \varepsilon\right) \to F\left(\widetilde{V}(F) + \varepsilon\right) \text{ a.s.}$$

so that

$$U = F\left(\widetilde{V}(F)\right) < F\left(\widetilde{V}(F) + \varepsilon\right) \leq U \text{ a.s.,}$$

which is impossible. Thus, $\limsup_{n\to\infty} \widetilde{V}(F_n) \leq \widetilde{V}(F)$. Similarly, it follows that $\liminf_{n\to\infty} \widetilde{V}(F_n) \leq \widetilde{V}(F)$. Thus (33) is true.

Finally, it follows straightforwardly from (32) and (33) that

$$\lim_{n\to\infty} \int_0^1 F_n\left(p_0 + u(\widetilde{V}(F_n) - p_0)\right)^{I-1} du = \int_0^1 F\left(p_0 + u(\widetilde{V}(F) - p_0)\right)^{I-1} du. \tag{64}$$

The result (34) now follows from (33), (63) and (64).

9.6 Proof of Lemma 10

Similar to the proof of Lemma 4 in Bierens (1990), we need to show that the following two conditions hold (see also Billingsley 1999):

(i) For each $\delta > 0$ and an arbitrary $t_0 \in [-\kappa, \kappa]$, there exists an ε such that

$$\sup_N P\left(\widehat{W}_N(t_0) > \varepsilon\right) \leq \delta$$

(ii) For each $\delta > 0$ and $\varepsilon > 0$, there exists an $\xi > 0$ such that for $t_1, t_2 \in [-\kappa, \kappa]$,

$$\sup_N P\left(\sup_{|t_1 - t_2| < \xi} |\widehat{W}_N(t_1) - \widehat{W}_N(t_2)| \geq \varepsilon\right) \leq \delta.$$

Condition (i) follows from the fact that for arbitrary $t \in [-\kappa, \kappa]$,

$$\left(\mathrm{Re}\left[\widehat{W}_N(t)\right], \mathrm{Im}\left[\widehat{W}_N(t)\right]\right)'$$

converges in distribution to a bivariate normal distribution. Condition (ii) follows from Chebishev's inequality for first moments:

$$P\left(\sup_{|t_1-t_2|<\xi} |\widehat{W}_N(t_1) - \widehat{W}_N(t_2)| \geq \varepsilon\right) \leq \varepsilon^{-1} E\left[\sup_{|t_1-t_2|<\xi} |\widehat{W}_N(t_1) - \widehat{W}_N(t_2)|\right]$$

and the fact that, with $\widehat{B}_j = \widetilde{B}_j(F_0)$,

$$\begin{aligned} E\left[\sup_{|t_1-t_2|<\xi} |\hat{W}_N(t_1) - \hat{W}_N(t_2)|\right] &\leq E\left[\sup_{|t_1-t_2|<\xi} |\exp(i.t_1 B_1) - \exp(i.t_2 B_1)|\right] \\ &+ E\left[\sup_{|t_1-t_2|<\xi} \left|\exp\left(i.t_1 \widehat{B}_1\right) - \exp\left(i.t_2 \widehat{B}_1\right)\right|\right] \\ &= 2.E\left[\sup_{|t_1-t_2|<\xi} |\exp(i.t_1 B_1) - \exp(i.t_2 B_1)|\right] \leq 2.\xi.\bar{b} \end{aligned}$$

where $\bar{b}$ is the upper bound of the support of B_1. Thus, condition (ii) holds for $\xi = \delta.\varepsilon / \left(2.\bar{b}\right)$.

9.7 Proof of Lemma 11

Part (52) of Lemma 11 follows from

$$\begin{aligned} &\widetilde{\Gamma}(t_1, t_2|\widetilde{F}) = \qquad (65) \\ &E\left[\left(\frac{1}{\sqrt{M}}\sum_{j=1}^{M}\left(\exp\left(i.t_1.\eta\left(\widetilde{U}_j|\widetilde{F}\right)\right) - \exp\left(i.t_1.\eta\left(\widetilde{U}_{j+M}|\widetilde{F}\right)\right)\right)\right)\right. \\ &\left.\times\left(\frac{1}{\sqrt{M}}\sum_{j=1}^{M}\left(\exp\left(-i.t_2.\eta\left(\widetilde{U}_j|\widetilde{F}\right)\right) - \exp\left(-i.t_2.\eta\left(\widetilde{U}_{j+M}|\widetilde{F}\right)\right)\right)\right)\right|\widetilde{F}\Bigg] \\ &\quad = \frac{1}{M}\sum_{j=1}^{M} E\left[\left(\exp\left(i.t_1.\eta\left(\widetilde{U}_j|\widetilde{F}\right)\right) - \exp\left(i.t_1.\eta\left(\widetilde{U}_{j+M}|\widetilde{F}\right)\right)\right)\right. \\ &\quad \left.\times\left(\exp\left(-i.t_2.\eta\left(\widetilde{U}_j|\widetilde{F}\right)\right) - \exp\left(-i.t_2.\eta\left(\widetilde{U}_{j+M}|\widetilde{F}\right)\right)\right)\right|\widetilde{F}\Bigg] \end{aligned}$$

$$= 2 \int_0^1 \exp\left(i.\left(t_1 - t_2\right).\eta\left(u|\widetilde{F}\right)\right) du$$
$$-2 \int_0^1 \exp\left(i.t_1.\eta\left(u|\widetilde{F}\right)\right) du \int_0^1 \exp\left(-i.t_2.\eta\left(u|\widetilde{F}\right)\right) du$$

$$= 2. \int_0^1 \cos\left(\left(t_1 - t_2\right).\eta\left(u|\widetilde{F}\right)\right) du + 2.i. \int_0^1 \sin\left(\left(t_1 - t_2\right).\eta\left(u|\widetilde{F}\right)\right) du$$
$$-2\left(\int_0^1 \cos\left(t_1.\eta\left(u|\widetilde{F}\right)\right) du + i. \int_0^1 \sin\left(t_1.\eta\left(u|\widetilde{F}\right)\right) du\right)$$
$$\times\left(\int_0^1 \cos\left(t_2.\eta\left(u|\widetilde{F}\right)\right) du - i. \int_0^1 \sin\left(t_2.\eta\left(u|\widetilde{F}\right)\right) du\right),$$

the fact that similar to Lemma 6, $\eta\left(u|\widetilde{F}\right) \to \eta\left(u|F_0\right)$ a.s. pointwise in $u \in [0, 1]$, and by the bounded convergence theorem. The results (53) and (54) follow now from the continuous mapping theorem and the fact that zero-mean Gaussian processes are completely determined by their covariance functions.

9.8 Proof of Lemma 12

Similar to (65) it follows that

$$\begin{aligned}
\Gamma(t,t) &= 2 - 2\left(E\left[\cos\left(t.B_1\right)\right] + i.E\left[\sin\left(t.B_1\right)\right]\right) \\
&\quad \times \left(E\left[\cos\left(t.B_1\right)\right] - i.E\left[\sin\left(t.B_1\right)\right]\right) \\
&= 2 - 2\left(E\left[\cos\left(t.B_1\right)\right]\right)^2 - 2\left(E\left[\sin\left(t.B_1\right)\right]\right)^2 \\
&= 2 - 2\left|\varphi\left(t\right)\right|^2
\end{aligned}$$

where $\varphi\left(t\right)$ is the characteristic function of B_1. Therefore, $\widehat{\Gamma}(t,t) = 2 - 2\left|\widehat{\varphi}\left(t\right)\right|^2$ is a consistent estimator of $\Gamma(t,t)$.

9.9 Proof of Theorem 6

The event $\widetilde{n}_N = n_0$ is equivalent to

$$\max_{1 \le n \le n_0} \left(\widetilde{C}_N(n) - \widetilde{C}_N(n-1)\right) \le 0 \text{ and } \widetilde{C}_N(n_0+1) - \widetilde{C}_N(n_0) > 0.$$

so that

$$P\left[\widetilde{n}_N \neq n_0\right] \leq P\left[\max_{1\leq n\leq n_0}\left(\widetilde{C}_N(n)-\widetilde{C}_N(n-1)\right)>0\right] \tag{66}$$
$$+P\left[\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\leq 0\right]$$

For fixed $n \leq n_0$,

$$\widetilde{C}_N(n)-\widetilde{C}_N(n-1) \to \inf_{F\in\mathcal{F}_n(G)} \overline{Q}(F) - \inf_{F\in\mathcal{F}_{n-1}(G)} \overline{Q}(F) \leq 0 \text{ a.s.}$$

hence

$$\lim_{N\to\infty} P\left[\max_{1\leq n\leq n_0}\left(\widetilde{C}_N(n)-\widetilde{C}_N(n-1)\right)>0\right]=0. \tag{67}$$

For $n = n_0+1$,

$$\left|N\left(\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\right)-\phi(N)\left(\Phi(n_0+1)-\Phi(n_0)\right)\right|$$
$$= N\left(\inf_{F\in\mathcal{F}_{n_0}} \widehat{Q}(F) - \inf_{F\in\mathcal{F}_{n_0+1}} \widehat{Q}(F)\right)$$
$$\leq N\left(\inf_{F\in\mathcal{F}_{n_0}} \widehat{Q}(F) + \inf_{F\in\mathcal{F}_{n_0+1}} \widehat{Q}(F)\right) \leq 2.N.\widehat{Q}(F_0)$$

so that with probability 1,

$$\frac{N}{\phi(N)}\left(\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\right) \geq \Phi(n_0+1)-\Phi(n_0)-2.\frac{N.\widehat{Q}(F_0)}{\phi(N)}$$

Therefore,

$$\lim_{N\to\infty} P\left[\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\leq 0\right]$$
$$\leq \lim_{N\to\infty} P\left[\frac{N.\widehat{Q}(F_0)}{\phi(N)} \geq \frac{1}{2}\left(\Phi(n_0+1)-\Phi(n_0)\right)\right]=0 \tag{68}$$

because $N.\widehat{Q}(F_0)/\phi(N) = O_p\left(1/\phi(N)\right) = o_p(1)$. It follows now from (66), (67) and (68) that $\lim_{N\to\infty} P\left[\widetilde{n}_N = n_0\right]=1$.

In the case $n_0 = \infty$ it follows from (67) that for any $\overline{n} \geq 1$,

$$\lim_{N\to\infty} P\left[\widetilde{n}_N \geq \overline{n}\right]=1,$$

which implies that $p\lim_{N\to\infty} \widetilde{n}_N = \infty$. Since for each n we can choose an $F_n \in \mathcal{F}_n$ such that $\lim_{n\to\infty} ||F_n - F_0|| = 0$, it follows that for this sequence F_n, $p\lim_{N\to\infty} ||F_{\widetilde{n}_N} - F_0|| = 0$. Hence

$$p\lim_{N\to\infty} \overline{Q}\left(F_{\widetilde{n}_N}\right) = \overline{Q}\left(F_0\right)$$

The result in Theorem 6 for the case $n_0 = \infty$ now follows from the proof of Theorem 3, adapted to the "plim" case.

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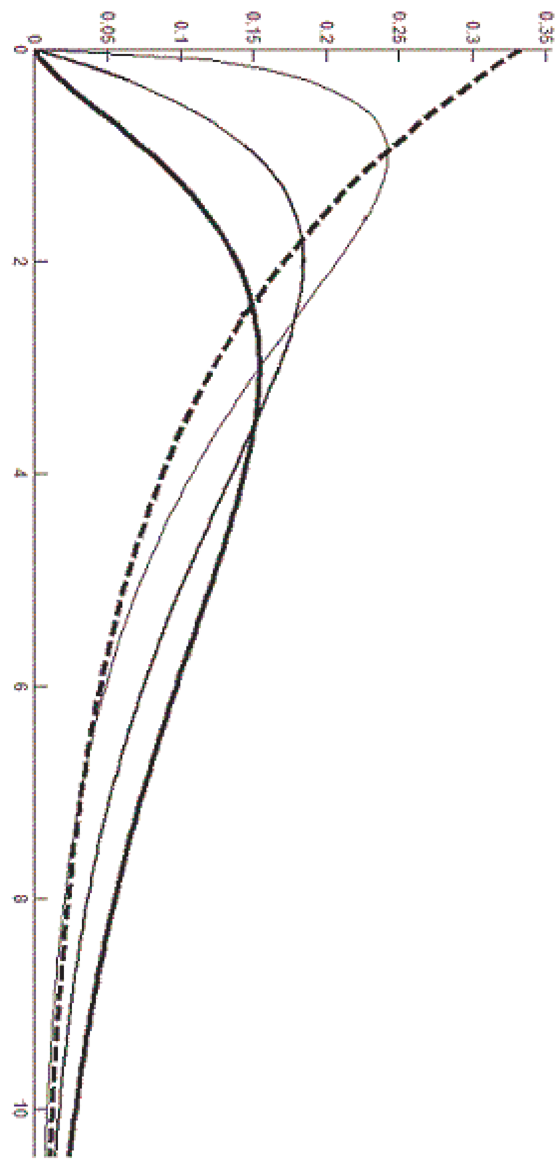
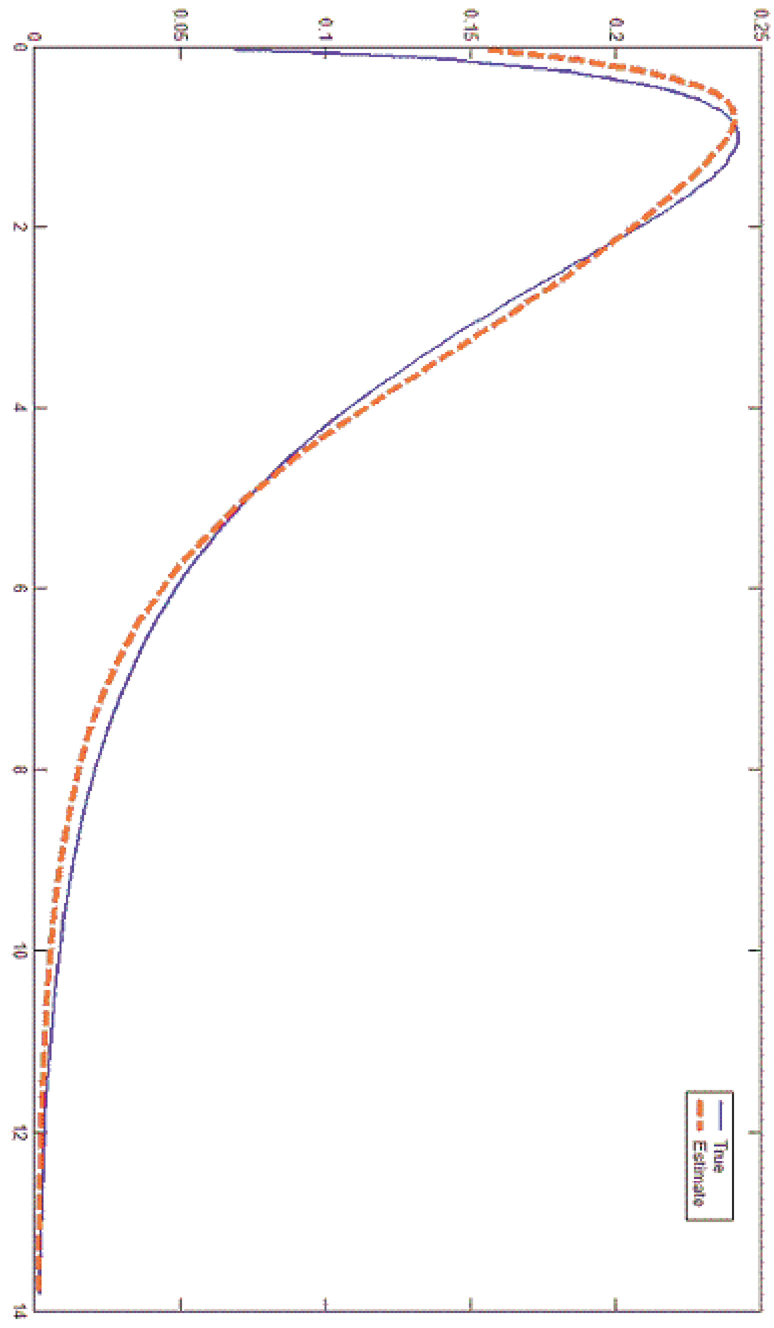


Figure 1:



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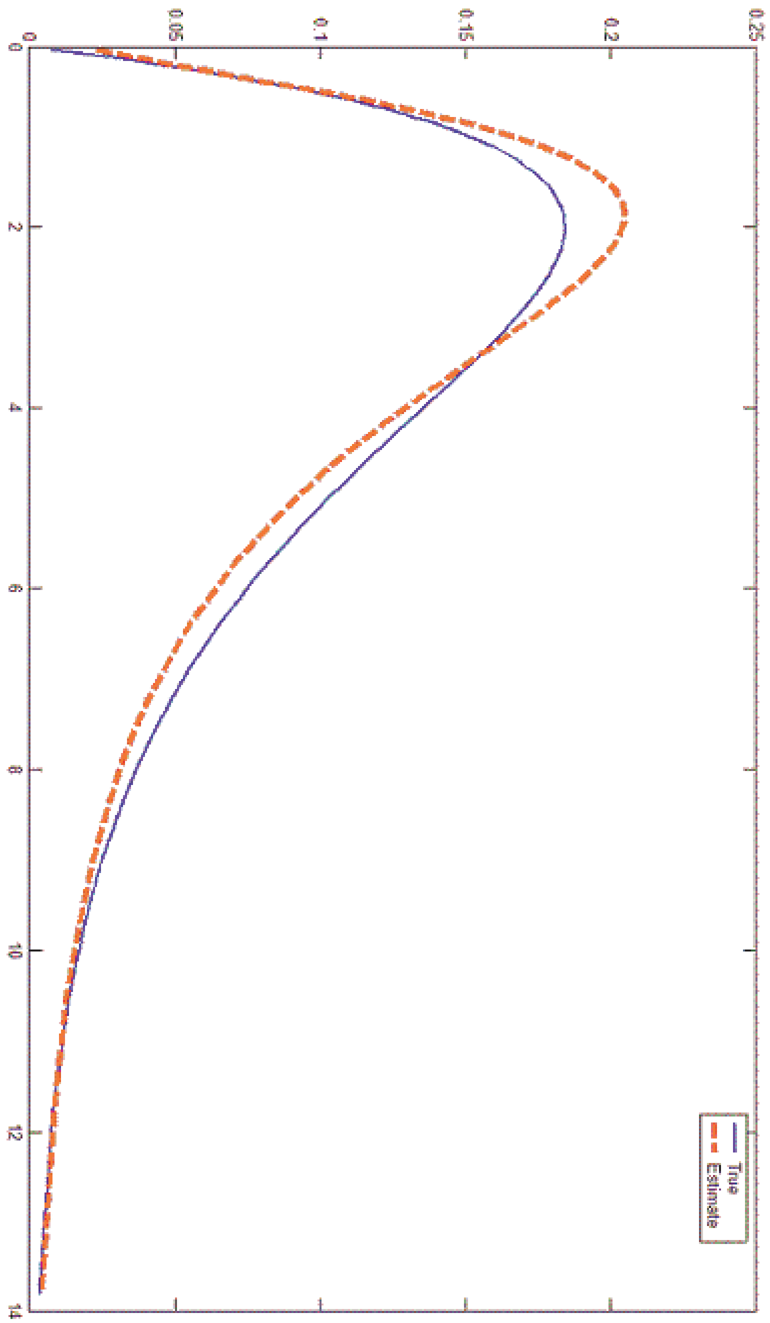

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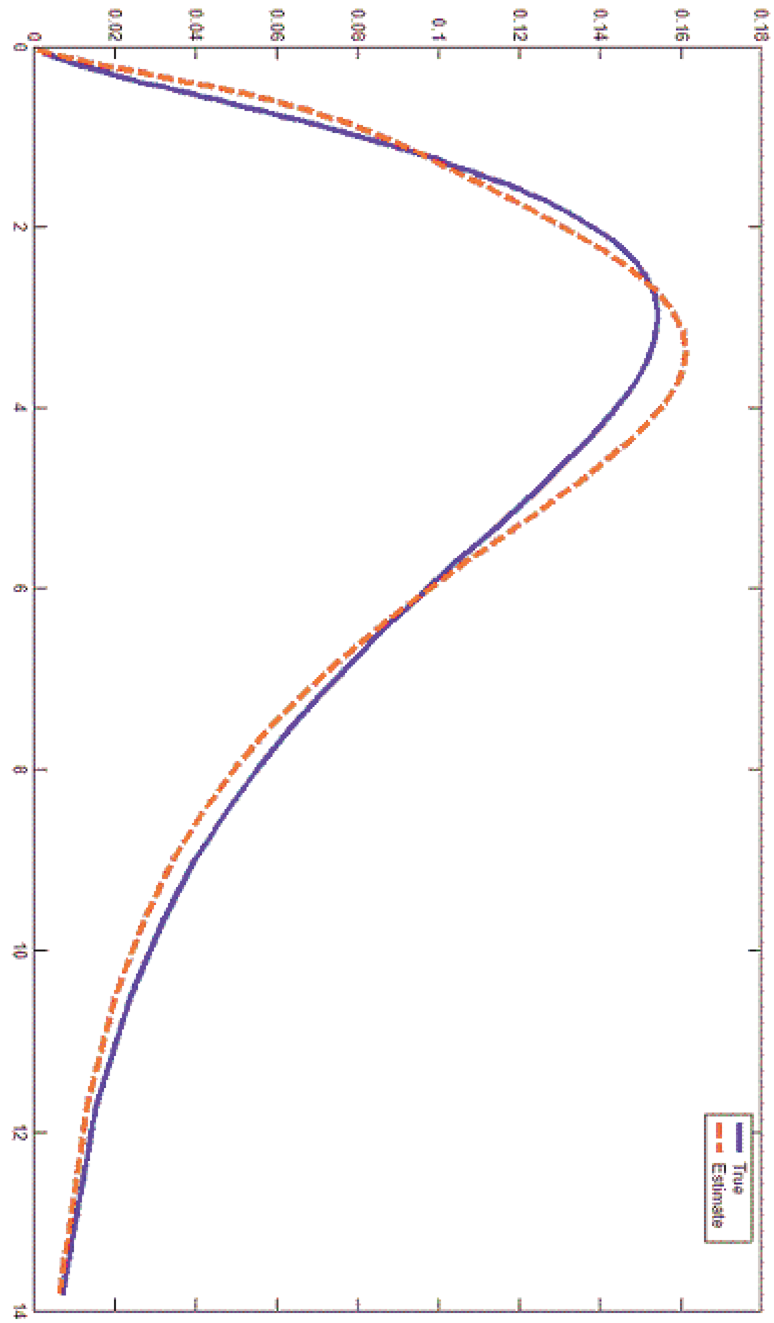

Figure 4: