

# Heterogeneous Producers Facing Common Shocks: An Overlapping-Generations Example

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DRAFT: 2005.06.07

## 1 Introduction

The extent to which market imperfections or institutional distortions amplify business cycle fluctuations is controversial. Competitive real-business-cycle models (for example, Kydland and Prescott 1982, Hansen 1985) imply that these factors play a secondary role. Imperfectly competitive models ascribe a large role to them. In particular, Bernanke and Gertler (1989) analyze a model according to which collateral constraints on lending to firms have a significant, adverse effect on welfare.

One fact that Bernanke and Gertler adduce to argue that such constraints actually exist is that aggregate shocks affect small firms more dramatically than large firms in the U.S. economy.<sup>1</sup> They remark that small firms do not have access to equity and commercial-paper markets, hypothesize that small firms are excluded from these markets because they have less collateral relative to their scale of operations than large firms have, and infer that the experience of actual small firms corroborates the implication of their model that collateral-constrained firms are particularly sensitive to shocks. This argument is only as strong as the hypothesis that a binding collateral constraint is the basis for exclusion from the market. An alternate hypothesis reverses the causality: small firms are unable to attract capital or to borrow because they are intrinsically risky—particularly so, when there are

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<sup>1</sup>Henceforth ‘shocks’ will denote aggregate shocks. In the simple model to be presented here, heterogeneity will be nonstochastic. There will be no idiosyncratic shocks.

bad shocks—rather than being especially vulnerable to bad shocks because investors and lenders do not treat them on their merits.

If firms are modeled as having respective CRS technologies that are identical (as by Lucas and Prescott 1971, or that differ only by a multiplicative productivity or production-cost factor (as by Jovanovic 1982), then aggregate shocks should affect large and small firms proportionally. Bernanke and Gertler’s suggestion that a market imperfection is required to explain different sensitivity to shocks between large and small firms is valid, conditional on such a background assumption. However, the background assumption also implies that large and small firms should use identical factor proportions even if the small firms face credit constraints, Oi (1983) refutes that implication.

The question remains open, then, whether or not a competitive model can reflect the disproportionate sensitivity to shocks of small firms’ market participation. One aspect of this question concerns how closely a detailed general-equilibrium model can match moments, impulse responses, and other statistical features of the actual economy when it is solved by numerical simulation. Khan and Thomas (2004) and the references cited there have made considerable progress in this regard. This sort of model is a black box, though. If it does entail that small firms are disproportionately sensitive to shocks, it nevertheless may not provide much intuition about what causes that sensitivity. To the extent that it does not provide such intuition, it is uninformative about which moments (among the very large number available when disaggregated data are studied) are the relevant ones to match, to show that the model performs better than one based on financial-market constraints.

The example to be analyzed here is complementary to such quantitative models. It is a schematic model that would be unsuitable for estimation or calibration. Instead, its virtues are that it is analytically tractable and has intuitively intelligible implications.

## 2 An overlapping-generations example with heterogeneous producers

The model has the usual demographic structure of an “initial old” cohort 0 and a sequence of further cohorts  $t = 1, 2, \dots$ . Each cohort is represented as a copy of the interval  $[\theta_L, \theta_H] \subseteq (0, \infty)$  with Lebesgue measure. Each agent in cohort 0 is endowed with  $s_0(\theta) > 0$  units of an investment good that I will call ‘capital’. Output of production at each date is divided into a composite consumption good and capital to be carried over to the next date. Agent  $\theta$  in cohort  $t \geq 1$  takes the following sequence of actions. ( $\alpha_t > 0$ ,  $\phi > 0$ ,  $\gamma \in (0, 1)$ .) The shocks  $\alpha_t$  are generated by a stationary Markov process with finite state space  $A = \{a_1, \dots, a_n\} \subseteq (0, \infty)$ , having transition probabilities

$\Pr(\alpha_{t+1} = a_j | \alpha_t = a_i) = p_{ij} \in (0, 1)$ .

1. Purchase capital, produce output, save and consume at date  $t$ .

- (a) Observe a productivity shock  $\alpha_t \in \mathbb{R}_+$
- (b) Purchase  $\omega \geq 0$  units of capital on short-term credit at competitive purchase price  $q_t$ , to own as an asset.
- (c) Choose an amount  $\kappa > 0$  of capital to be used in production of

$$f(\kappa, \alpha, \theta) = \chi_{\geq \phi}(\kappa) \alpha \theta (\kappa - \phi)^{1/2} \quad (1)$$

units of output, where  $\chi_{\geq \phi}(\kappa) = 1$  if  $\kappa \geq \phi$  and if  $\chi_{\geq \phi}(\kappa) = 0$  if  $\kappa < \phi$ .

- The difference  $\kappa - \omega$  is acquired (or provided, if  $\omega > \kappa$ ) on a rental market at competitive price  $r_t$ .

- (d) After producing, surrender  $q_t \omega + r_t(\kappa - \omega)$  units of output to settle capital purchase and rental obligations. (There is a rental receipt if  $\omega > \kappa$ .)
- (e) Save a quantity  $\sigma$  of output satisfying

$$\sigma \in [(1 - \delta)\omega, f(\kappa, \alpha, \theta) + (1 - \delta)\omega - (q_t \omega + r_t(\kappa - \omega))] \quad (2)$$

as capital for sale at date  $t + 1$ .

- (f) Consume the remainder,  $c = f(\kappa, \alpha, \theta) + (1 - \delta)\omega - (q_t \omega + r_t(\kappa - \omega)) + \sigma \geq 0$

2. Sell  $\sigma$  units of capital and consume  $c' = q_{t+1} \sigma$  units of output at date  $t + 1$

Agents maximize expected consumption with discount factor  $\beta \in (0, 1)$ . That is, if  $\alpha_t = a_i$ , then agent  $\theta$  in cohort  $t$  chooses these actions to maximize

$$c + \beta \sum_{j=1}^n p_{ij} c'_j \quad (3)$$

recognizing that  $q_{t+1}$ , and hence  $c'$ , will depend on  $\alpha_{t+1}$ .<sup>2</sup>

### 3 Equilibrium

For  $t \geq 1$ , define  $w_t(\theta)$ ,  $k_t(\theta)$  and  $s_t(\theta)$  to be the choices of  $\omega$ ,  $\kappa$ , and  $\sigma$  respectively by agent  $\theta$  in cohort  $t$ .

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<sup>2</sup>In the usual fashion of overlapping-generations models, agents in cohort 0 maximize  $c'$  only. The solution of this trivial optimization problem is to sell their entire capital stocks to the agents of cohort 1.

Define  $\mathcal{L}_+^1$  to be the set of integrable nonnegative functions  $g : [\theta_L, \theta_H] \rightarrow \mathbb{R}_+$ . An equilibrium consists of a sequence  $((w_1, k_1, s_1), (w_2, k_2, s_2), \dots) \in (\mathcal{L}_+^1 \times \mathcal{L}_+^1 \times \mathcal{L}_+^1)^{\mathbb{N}_+}$  and a sequence  $((q_1, r_1), (q_2, r_2), \dots) \in (\mathbb{R}_+ \times \mathbb{R}_+)^{\mathbb{N}_+}$  that satisfy the following 3 conditions for each  $t \geq 1$ .

1. For all  $\theta$ ,  $(\omega, \kappa, \sigma) = (w_t(\theta), k_t(\theta), s_t(\theta))$  solves the constrained optimization problem described above.
2. The market-clearing condition for capital purchase holds that

$$\int_{\theta_L}^{\theta_H} w_t(\theta) d\theta = \int_{\theta_L}^{\theta_H} s_{t-1}(\theta) d\theta \quad (4)$$

3. The market-clearing condition for capital rental holds that

$$\int_{\theta_L}^{\theta_H} k_t(\theta) d\theta = \int_{\theta_L}^{\theta_H} w_t(\theta) d\theta \quad (5)$$

The state of the economy at date  $t$  is the pair  $(\alpha_t, \int_{\theta_L}^{\theta_H} s_{t-1}(\theta) d\theta)$ . That is, the state is determined by the aggregate shock and the aggregate quantity of capital carried into the economy by old agents. Define the state space

$$X = A \times \mathbb{R}_+ \quad (6)$$

I adopt the notation that

$$x = (\alpha, \bar{\kappa}) \quad \bar{\kappa}_t = \int_{\theta_L}^{\theta_H} s_{t-1}(\theta) d\theta \quad (7)$$

An equilibrium is recursive if there are functions  $\kappa^* : X \rightarrow \mathbb{R}_+$ ,  $Q : X \rightarrow \mathbb{R}_+$  and  $R : X \rightarrow \mathbb{R}_+$  such that, for all  $t$ ,  $\bar{\kappa}_{t+1} = \kappa^*(x_t)$ ,  $q_t = Q(x_t)$  and  $r_t = R(x_t)$ .

Define a shock-determined equilibrium (SDE) to be a recursive equilibrium such that there is a function  $k^* : A \rightarrow \mathbb{R}_+$  satisfying<sup>3</sup>

$$\kappa^*(x_t) = k^*(\alpha_t) \quad (8)$$

This is a restrictive condition that would not typically be satisfied in equilibrium except with linear utility. However, if it does exist in this case, then an SDE is particularly simple to describe and to analyze.

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<sup>3</sup>A more general condition, allowing for capital-adjustment constraints to bind, would be that

$$|\kappa^*(x_t) - k^*(\alpha_t)| = \min \left\{ |\kappa - k^*(\alpha_t)| \mid \kappa \in \left[ (1 - \delta) \int_{\theta_L}^{\theta_H} \omega_t(\theta) d\theta, \int_{\theta_L}^{\theta_H} f(k_t(\theta), \alpha, \theta) + (1 - \delta)\omega_t(\theta) d\theta \right] \right\}$$

## 4 Competitive production decisions

In this environment with both asset-purchase and rental markets for capital, each young agent's decision problem can be decomposed into an intertemporal and an intratemporal component. The intratemporal problem is to set factor demand for capital to maximize output net of rental cost. Given that the supply of capital is predetermined by the amount carried in by old agents, young agents' demand schedule for capital determines the equilibrium rental price.

Now I derive this demand schedule. Define  $\hat{k}(x, \theta, r)$  to be the value of  $\kappa$  that maximizes  $f(\kappa, \alpha, \theta) - r\kappa$ . This quantity satisfies the first-order condition that

$$\frac{\alpha\theta}{2}(\hat{k}(x, \theta, r) - \phi)^{-1/2} = r \quad (9)$$

which implies that

$$\hat{k}(x, \theta, r) = \left(\frac{\alpha\theta}{2r}\right)^2 + \phi \quad (10)$$

Define  $\tilde{\theta}(x, r)$  to be the marginal firm (that is, the firm for which  $f(\hat{k}(x, \theta, r), \alpha, \theta) - r\hat{k}(x, \theta, r) = 0$ ) or else to be  $\theta_L$  if all firms can earn positive profit at rental price  $r$ , or else to be  $\theta_H$  if no firm can earn positive profit at  $r$ . This firm is defined by

$$\left|\tilde{\theta}(x, r) - \frac{2r\phi^{1/2}}{\alpha}\right| = \min \left\{ \left|\theta - \frac{2r\phi^{1/2}}{\alpha}\right| \mid \theta_L \leq \theta \leq \theta_H \right\} \quad (11)$$

Define  $D(x, r)$  to be aggregate factor demand for capital. That is,

$$D(x, r) = \int_{\tilde{\theta}(x, r)}^{\theta_H} \hat{k}(x, \theta, r) d\theta \quad (12)$$

If

$$\theta_L \leq \frac{2r\phi^{1/2}}{\alpha} \leq \theta_H \quad (13)$$

then

$$D(x, r) = \left(\frac{\alpha}{2r}\right)^2 \left[ \frac{\theta_H^3 - (2r\phi^{1/2}/\alpha)^3}{3} \right] + \phi \left[ \theta_H - \frac{2r\phi^{1/2}}{\alpha} \right] \quad (14)$$

The equilibrium rental price of capital,  $R(x)$ , satisfies

$$\bar{\kappa} = D(x, R(x)) \quad (15)$$

If (13) holds, then by (14) and (15), there exists a function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$R(x) = \alpha\rho(\bar{\kappa}) \quad (16)$$

In particular, substitution of  $\alpha\rho$  for  $r$  on the right side of (14) and substitution of the resulting expression for  $D(x, R(x))$  in (15) yield an equation that simplifies to

$$32\phi^{3/2}\rho^3 + 12(\bar{\kappa} - \phi\theta_H)\rho^2 - \theta_H^3 = 0 \quad (17)$$

Solving this equation defines  $\rho$  as a function of  $\bar{\kappa}$  and parameters of the model. In particular, for  $\bar{\kappa} = 0$ , this procedure yields

$$\rho(0) = \frac{\theta_H}{2\phi^{1/2}} \quad (18)$$

Note that, by (11) and (16),

$$\left| \tilde{\theta}(x, R(x)) - 2\rho\phi^{1/2} \right| = \min \left\{ \left| \theta - 2\rho\phi^{1/2} \right| \mid \theta_L \leq \theta \leq \theta_H \right\} \quad (19)$$

Equations (18) and (19) imply that

$$\bar{\kappa} = 0 \implies \theta_H = \frac{2R(x)\phi^{1/2}}{\alpha} \quad (20)$$

so that  $\tilde{\theta}(x, R(x)) = \theta_H$  but the constraint that  $\tilde{\theta} \leq \theta_H$  does not bind in (19).

It also follows from (19) that

$$\rho(\bar{\kappa}) = \frac{\theta_L}{2\phi^{1/2}} \implies \theta_L = \frac{2R(x)\phi^{1/2}}{\alpha} \quad (21)$$

so that  $\tilde{\theta}(x, R(x)) = \theta_L$  but the constraint that  $\theta_L \leq \tilde{\theta}$  does not bind in (19).

Applying the implicit function theorem to equation (17) yields

$$\rho'(\kappa) = \frac{-\rho(\kappa)}{8\phi^{3/2}\rho(\kappa) - 2\phi\theta_H} \quad (22)$$

so

$$\rho(\kappa) > \frac{\theta_H}{4\phi^{1/2}} \implies \rho'(\kappa) < 0 \quad (23)$$

Therefore, by (20) and (23),

$$\begin{aligned} \theta_L \geq \theta_H/2 \implies \forall \alpha \in A \forall \bar{\kappa} \in \left( 0, \rho^{-1} \left( \frac{\theta_L}{2\phi^{1/2}} \right) \right) \\ [\theta_L < \tilde{\theta}((\alpha, \bar{\kappa}), R(\alpha, \bar{\kappa})) < \theta_H \text{ and } \rho'(\bar{\kappa}) < 0] \end{aligned} \quad (24)$$

## 5 The point of the analysis

My aim here is to explain how entry and exit over the course of the business cycle might be concentrated among small firms, even without institutional

distortions, and then to show that that the explanation is consistent with a fully specified equilibrium. The results of the preceding section enable me largely to accomplish the first part of this aim.

In this model economy, the proper definition of a “small producer” is unambiguous: it is a low- $\theta$  agent. Both factor demand for capital and also output are strictly increasing in  $\theta$ .

Suppose that there is an equilibrium in which the state is always in the range (specified by the range of quantification in the consequent of (24)) where

$$\theta_L < \tilde{\theta}(x, R(x)) < \theta_H \text{ and } \rho'(\kappa) < 0 \quad (25)$$

By (19) and (25),

$$\tilde{\theta}(x, R(x)) = 2\phi^{1/2}\rho(\bar{\kappa}) \quad (26)$$

always in this economy. Now consider 2 possible states at date  $t$  that have identical aggregate capital stocks,  $x = (\alpha_x, \bar{\kappa})$  and  $y = (\alpha_y, \bar{\kappa})$ , such that

$$\alpha_x < \alpha_y \quad (27)$$

By (26), the allocation of capital to producers is identical in these two states. Therefore aggregate output is higher in  $y$  than in  $x$ .

Let  $\bar{\kappa}'_x$  and  $\bar{\kappa}'_y$  be the aggregate amounts of new capital that producers carry into date  $t + 1$  in states  $x$  and  $y$  respectively. Suppose that<sup>4</sup>

$$[\alpha_x < \alpha_y \text{ and } \bar{\kappa}_x \leq \bar{\kappa}_y] \implies \bar{\kappa}'_x < \bar{\kappa}'_y \quad (28)$$

For any value  $\alpha'$  of the productivity shock, consider states  $s'_x = (\alpha', \bar{\kappa}'_x)$  and  $s'_y = (\alpha', \bar{\kappa}'_y)$ . By (25) and (28),  $\rho(\bar{\kappa}'_x) > \rho(\bar{\kappa}'_y)$ . By (26),

$$\tilde{\theta}(\bar{\kappa}'_x) < \tilde{\theta}(\bar{\kappa}'_y) \quad (29)$$

This implication from (27) to (29) shows that small producers' participation in the market is affected by the lagged productivity shock, with small producers' participation (that is,  $\kappa(\theta) > \phi(\theta)$  for low  $\theta$ ) tending to move in the same direction as the productivity shock. If the shock process is stochastically monotone, then I expect that small producers' participation will be correlated with the current productivity shock, but the current shock does not affect small producers' participation causally although it is forecastable. Rather, the entire effect of the lagged productivity shock comes through its effect on the size of the current aggregate capital stock.

These conclusions seem, qualitatively at least, fairly robust. It is easy to imagine embroidering a model like this one with details that would give

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<sup>4</sup>This assumption is plausible if the shock process is stochastically monotone (that is, if a higher value of the shock today implies a stochastically dominating distribution of the shock tomorrow). It will be justified, along with the supposition (25), in the example to be presented in the next section.

the productivity shock some degree of direct effect on the identity of the marginal firm, but it is more difficult to imagine that such an effect would swamp the effect of the capital-stock level. The conclusions of this model are also fairly strong—perhaps strong enough not to be very plausible. It would be worthwhile to know whether or not they qualitatively characterize the actual U.S. economy, and also whether or not they characterize the simulated equilibria of the more elaborate models designed for calibration. If the conclusions characterize equilibria of those calibrated models but not the equilibrium of the actual economy, then that discrepancy would provide grounds for taking models based on market distortions seriously.

## 6 An economy possessing a shock-determined equilibrium

Now I show that, for some economies, an SDE exists. In the equilibrium that I exhibit, each cohort has positive aggregate consumption in both periods of its lifetime. In order for such a consumption pattern to be consistent with optimization of the discounted-consumption objective function (3), prices must satisfy<sup>5</sup>

$$\begin{aligned} \forall t \in \mathbb{N}_+ \quad \beta^{-1} &= \mathbb{E}[q_{t+1}|x_t] \\ \forall x \in X \quad \beta^{-1} &= \mathbb{E}[Q(x_{t+1})|x_t] \end{aligned} \tag{30}$$

The other condition implied by agents' optimization is that

$$\begin{aligned} \forall t \in \mathbb{N}_+ \quad q_t &= r_t + (1 - \delta) \\ \forall x \in X \quad Q(x) &= R(x) + (1 - \delta) \end{aligned} \tag{31}$$

This condition reflects the situation that there is only one difference between purchasing and renting capital: the part of purchased capital that remains after depreciation can be sold (at expected price  $\beta^{-1}$ ) for consumption in the next period, while depreciated rented capital must be returned to the owner.

Combining these equations (for  $x$  and  $x'$  respectively),

$$\forall x \in X \quad \beta^{-1} + \delta - 1 = \mathbb{E}[R(x')|x] \tag{32}$$

Consider a 2-state shock process,

$$A = \{a_1, a_2\} \quad a_1 < a_2 \tag{33}$$

Suppose that  $k^* : A \rightarrow \mathbb{R}_+$  determines an SDE. Define  $\kappa_1 = k^*(a_1)$  and  $\kappa_2 = k^*(a_2)$ . Equations (16), (30) and (31) imply that

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<sup>5</sup>In this pair of equations and in the next pair, the second equation is a sufficient condition for the first, which is an equilibrium condition that must hold a.s.



$$\begin{aligned}
\beta^{-1} + \delta - 1 &= p_{11}a_1\rho(\kappa_1) + p_{12}a_2\rho(\kappa_1) \\
\beta^{-1} + \delta - 1 &= p_{21}a_1\rho(\kappa_2) + p_{22}a_2\rho(\kappa_2)
\end{aligned} \tag{34}$$

This is equivalent to

$$\begin{pmatrix} (\beta^{-1} + \delta - 1)/\rho(\kappa_1) \\ (\beta^{-1} + \delta - 1)/\rho(\kappa_2) \end{pmatrix} = P \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{35}$$

Multiplication by  $P^{-1}$  yields

$$\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= P^{-1} \begin{pmatrix} (\beta^{-1} + \delta - 1)/\rho(\kappa_1) \\ (\beta^{-1} + \delta - 1)/\rho(\kappa_2) \end{pmatrix} \\
&= \frac{\beta^{-1} + \delta - 1}{p_{11} + p_{22} - 1} \begin{pmatrix} p_{22}/\rho(\kappa_1) + (p_{11} - 1)/\rho(\kappa_2) \\ (p_{22} - 1)/\rho(\kappa_1) + p_{11}/\rho(\kappa_2) \end{pmatrix}
\end{aligned} \tag{36}$$

Assume that

$$p_{11} + p_{22} - 1 > 0 \tag{37}$$

which is the condition that the Markov process with transition matrix  $P$  is stochastically monotone. Equations (33) and (37) imply that

$$\kappa_1 < \kappa_2 \tag{38}$$

and

$$a_2 - a_1 = \frac{\beta^{-1} + \delta - 1}{p_{11} + p_{22} - 1} \left[ \frac{1}{\rho(\kappa_2)} - \frac{1}{\rho(\kappa_1)} \right] \tag{39}$$

Now consider  $a_1$  and  $a_2$  to be a function of  $\kappa_1$  and  $\kappa_2$  respectively,  $P$ ,  $\theta_H$ , and  $\phi$ , defined by (17) and (36). An SDE exists in the economy with parameters  $\langle \alpha_t \rangle$  (which determines  $P$ ),  $\theta_H$ ,  $\theta_L \geq \theta_H/2$ , and  $\phi$ , if  $\kappa_1$  and  $\kappa_2$  can be chosen such that, for the values of  $a_1$  and  $a_2$  determined by this function, the following 3 conditions are satisfied.

- If  $\alpha_{t-1} = a_i$  and  $\alpha_t = a_j$ , then aggregate production in the resulting state  $(a_j, k^*(a_i))$  is greater than  $k^*(a_j) - (1 - \delta)a_i$ .
  - It is sufficient that aggregate production in state  $(a_1, k^*(a_1))$  is greater than  $k^*(a_2) - (1 - \delta)a_1$ .
- If  $\alpha_{t-1} = a_i$  and  $\alpha_t = a_j$ , then  $(1 - \delta)k^*(a_i) < k^*(a_j)$  so that the rate of depreciation does not constrain the capital stock from declining to its target in 1 period.
  - It is sufficient that  $(1 - \delta)k^*(a_2) < k^*(a_1)$
- $0 < \kappa_1 < \kappa_2 < \rho^{-1}(\theta_L/[2\phi^{1/2}])$ , as required by (24) and (38).

I prove that this can be done. The key is a change of variable, by which equilibrium quantities are represented as functions of the marginal producer,  $\tilde{\theta}$ , rather than of the capital stock. By (16) and (26), the equilibrium rental price is

$$R^*(\alpha, \tilde{\theta}) = \frac{\alpha \tilde{\theta}}{2\phi^{1/2}} \quad (40)$$

The analogue of  $\hat{k}(x, \theta, R(x))$  is

$$k^*(\theta, \tilde{\theta}) = \hat{k}(x, \theta, R(\alpha, \tilde{\theta})) = \left( \frac{\alpha \theta}{2R^*(\alpha, \tilde{\theta})} \right)^2 + \phi = \phi \left[ \left( \frac{\theta}{\tilde{\theta}} \right)^2 + 1 \right] \quad (41)$$

The analogue of  $D(x, R(x))$  is

$$K(\tilde{\theta}) = \int_{\tilde{\theta}}^{\theta_H} k^*(\theta, \tilde{\theta}) d\theta = \frac{\phi}{3\tilde{\theta}^2} (\theta_H^3 - \tilde{\theta}^3) + \phi(\theta_H - \tilde{\theta}) \quad (42)$$

Aggregate output, when  $K(\tilde{\theta})$  is allocated efficiently in a state having  $\tilde{\theta}$  as marginal producer, is

$$Y(\alpha, \tilde{\theta}) = \int_{\tilde{\theta}}^{\theta_H} f(k^*(\theta, \tilde{\theta}), \theta, \alpha) d\theta = \frac{\alpha \phi^{1/2}}{3\tilde{\theta}} (\theta_H^3 - \tilde{\theta}^3) \quad (43)$$

In order to exhibit an SDE, first consider the case in which aggregate productivity is a nonstochastic constant  $a$ . By (30) and (31),  $\beta^{-1} = R^*(a, \theta) + (1 - \delta)$ . That is, by (40),

$$\beta^{-1} + \delta - 1 = \frac{a\tilde{\theta}}{2\phi^{1/2}} \quad (44)$$

In light of (44), define

$$\alpha(\tilde{\theta}) = \frac{2\phi^{1/2}(\beta^{-1} + \delta - 1)}{\tilde{\theta}} \quad (45)$$

The single-state SDE must reach a stationary allocation at date 2, in which  $Y - \delta K > 0$  so that consumption can be positive. Substituting (45) into the formula for  $Y - \delta K$  yields<sup>6</sup>

$$\begin{aligned} Y(\alpha(\theta), \theta) - \delta K(\theta) &= \frac{2(\beta^{-1} + \delta - 1)\phi}{3\theta^2} (\theta_H^3 - \theta^3) - \\ &\quad \delta \left[ \frac{\phi}{3\theta^2} (\theta_H^3 - \theta^3) + \phi(\theta_H - \theta) \right] \\ &= \left[ \left[ \frac{2(\beta^{-1} + \delta - 1) - \delta}{3\theta^2} \right] (\theta_H^3 - \theta^3) - \right. \\ &\quad \left. \delta(\theta_H - \theta) \right] \phi \end{aligned} \quad (46)$$

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<sup>6</sup>To streamline notation in the following computation, I write ' $\theta$ ' rather than ' $\tilde{\theta}$ '.

In particular,

$$Y(\alpha(\theta_H), \theta_H) - \delta K(\theta_H) = 0 \quad (47)$$

Now I use Taylor's approximation to sign  $Y(\alpha(\theta), \theta) - \delta K(\theta)$  as  $\theta$  approaches  $\theta_H$  from below.

$$\begin{aligned} \frac{\partial}{\partial \theta} [Y(\alpha(\theta), \theta) - \delta K(\theta)] &= \left[ \frac{-2[2(\beta^{-1} + \delta - 1) - \delta]}{3\theta^3} (\theta_H^3 - \theta^3) - \right. \\ &\quad \left. [2(\beta^{-1} + \delta - 1) - \delta] + \delta \right] \phi \\ &= \left[ \frac{-2[2(\beta^{-1} + \delta - 1) - \delta]}{3\theta^3} (\theta_H^3 - \theta^3) - \right. \\ &\quad \left. 2(\beta^{-1} + \delta - 1) \right] \phi \end{aligned} \quad (48)$$

In particular,

$$\frac{\partial}{\partial \theta} [Y(\alpha(\theta), \theta) - \delta K(\theta)]|_{\theta=\theta_H} = -2(\beta^{-1} + \delta - 1)\phi < 0 \quad (49)$$

Therefore, for some  $\theta^* \in (\theta_L, \theta_H)$ ,

$$\forall \theta \in (\theta^*, \theta_H) \quad Y(\alpha(\theta), \theta) > \delta K(\theta) \quad (50)$$

By (50) and by continuity of  $K$  and  $Y$ , the following condition holds. For every  $\theta \in (\theta^*, \theta_H)$ , there exists  $\varepsilon > 0$  such that, for all  $\theta_1$  and  $\theta_2$  satisfying  $\max(\theta^*, \theta - \varepsilon) < \theta_2 < \theta_1 < \min(\theta_H, \theta + \varepsilon)$  and all  $a \in (\alpha(\theta) - \varepsilon, \alpha(\theta) + \varepsilon)$ ,

$$Y(a, \theta_1) > \delta K(\theta_1) + (K(\theta_2) - K(\theta_1)) \quad (51)$$

and

$$K(\theta_1) > (1 - \delta)K(\theta_2) \quad (52)$$

Now a 2-state SDE can be constructed. Let  $P$  be any transition matrix satisfying the stochastic-monotonicity condition (37). Let  $0 < \theta_L < \theta_H/2$  and  $0 < \delta < 1$ . Let  $\theta^*$  satisfy (50), and define  $\theta_1 = (\theta^* + \theta_H)/2$ . Set

$$\kappa_1 = \rho^{-1} \left( \frac{\theta_1}{2\phi^{1/2}} \right) \quad (53)$$

as specified by (40). For  $n \geq 2$ , define

$$\kappa_n^* = \rho^{-1} \left( \frac{\theta_1 + 1/n}{2\phi^{1/2}} \right) \quad (54)$$

Define, according to (36),

$$\begin{pmatrix} a_{1n}^* \\ a_{2n}^* \end{pmatrix} = \frac{\beta^{-1} + \delta - 1}{p_{11} + p_{22} - 1} \begin{pmatrix} p_{22}/\rho(\kappa_1) + (p_{11} - 1)/\rho(\kappa_n^*) \\ (p_{22} - 1)/\rho(\kappa_1) + p_{11}/\rho(\kappa_n^*) \end{pmatrix} \quad (55)$$

Then

$$a_{1n}^* < a_{2n}^* \text{ and } \lim_{n \rightarrow \infty} a_{1n}^* = \lim_{n \rightarrow \infty} a_{2n}^* = \frac{\beta^{-1} + \delta - 1}{\rho(\kappa_1)} = \alpha(\theta_1) \quad (56)$$

Define  $k^*(\alpha_{1n}^*) = \kappa_1$  and  $k^*(\alpha_{2n}^*) = \kappa_n^*$ . It follows from (51), (52), and (56) that, if  $\alpha_1 = \alpha_{1n}^*$  and  $\alpha_2^* = \alpha_{2n}^*$  for sufficiently large  $n$ , then  $k^*$  (together with the capital-rental and -purchase prices specified by (40) and (31)) is an SDE.

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