Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions*

Emmanuel Guerre
Queen Mary University of London

Isabelle Perrigne
Pennsylvania State University

Quang Vuong
Pennsylvania State University

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Correspondence to: I. Perrigne, Department of Economics, Pennsylvania State University, University Park, PA 16802. E-mail address : perrigne@psu.edu.

Abstract

This paper studies the nonparametric identification of the first-price auction model with risk averse bidders within the private value paradigm. We show that the benchmark model is nonindentified in general from observed bids. We derive the restrictions imposed by the model on observables and show that these restrictions are quite weak. We then establish the nonparametric identification of the bidders' utility function under exclusion restrictions. Our primary exclusion restriction takes the form of an exogenous bidders' participation leading to a latent distribution of private values independent of the number of bidders. The key idea is to exploit that the bid distribution varies with the number of bidders while the private value distribution does not. We also characterize all the theoretical restrictions imposed by such an exclusion restriction on observables to rationalize the model. Though derived for a benchmark model, our results extend to more general cases such as a binding reserve price, affiliated private values and asymmetric bidders. Our theoretical results also extend to observed and unobserved heterogeneity. In particular, we consider endogenous bidders' participation with exclusion restrictions and available instruments that do not affect the bidders' private value distribution.

Key words: Risk Aversion, Private Value, Nonparametric Identification, Exclusion Restrictions, Unobserved Heterogeneity

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E. Guerre, I. Perrigne and Q. Vuong

1 Introduction

The empirical and experimental literature suggests that risk aversion is a component of bidders' behavior in auctions. See Baldwin (1995), Athey and Levin (2001), Perrigne (2003) for the former and Cox, Smith and Walker (1988), Goeree, Holt and Palfrey (2002) and Bajari and Hortacsu (2005) for the latter to name a few.[1] With the exception of experimental studies, risk aversion is quite difficult to assess. The main problem lies in the identification of the bidders' utility function. In a companion paper, Campo, Guerre, Perrigne and Vuong (2007) propose minimal parametric restrictions leading to the semiparametric identification of the auction model under a parameterization of the bidders' utility function and a conditional quantile restriction of the private value distribution. They show that both parametric restrictions are needed. Based on this result, they derive an estimator and study its statistical properties.[2] In practice, the choice of a parametric utility function displaying risk aversion may affect the estimated results, yet various concepts of risk aversion have different implications on economic agents' behavior. There is, however, no general agreement on which concept of risk aversion is the most appropriate to explain observed phenomena such as in finance through the diversification of

[1]Using recent structural econometric methods, Bajari and Hortacsu (2005) find that risk aversion provides the best fit to some experimental data among a set of competing models including learning ones.

[2]Since risk aversion does not affect bidding in ascending auctions within the private value paradigm, identification of risk aversion cannot be achieved in ascending auctions. The combination of first-price sealed-bid and ascending auctions, however, allows to identify nonparametrically the bidders' utility function as shown by Lu and Perrigne (2006).

portfolios, insurance when low risk car drivers tend to buy more insurance than needed, or in auctions through overbidding.[3] A typical controversy is whether risk aversion is absolute or relative to economic agent's wealth. As a matter of fact, little is known on the shape of agents' utility functions. Several families of utility functions have been developed to embody some economic properties related to risk aversion. See Gollier (2001) for an extensive survey on risk aversion. Which family is relevant is an empirical question.

Given the importance of risk aversion in auctions and our ignorance about bidders' utility functions, we address the nonparametric identification of the latter in this paper. First, we show that the general model is not identified in general from observed bids. We then derive the theoretical restrictions imposed by the model on observables and show that these restrictions are quite weak. In particular, we show that any smooth bid distribution can be rationalized by a model with general risk aversion. Such a striking result implies that risk aversion does not impose testable restrictions on bids.

Second, we show that the bidders' utility function is nonparametrically identified under some exclusion restrictions. Our primary exclusion restriction takes the form of an exogenous bidders' participation leading to a latent distribution of private values that is independent of the number of bidders. Exclusion restrictions are widely used in econometrics. A typical example is the use of instrumental variables in labor economics to address the endogeneity of education in the estimation of the wage equation. Exclusion restrictions have also been used in the structural auction literature. Athey and Haile (2002) and Haile, Hong and Shum (2003) exploit some exclusion restrictions to test for common values in first-price sealed-bid auctions. Both papers assume exogenous participation to detect the winner's curse. In a different framework, Bajari and Hortacsu (2005) use exogenous participation to estimate an auction model with constant relative risk aversion from experimental data.[4]

Third, we consider observed and unobserved heterogeneity. In particular, we extend our results to a model with endogenous bidders' participation under exclusion restrictions and the availability of instruments that do not affect the bidders' private value distribution. While considering unobserved heterogeneity affecting both bidders' participation

[3] For an empirical analysis of risk aversion in car insurance, see Cohen and Einav (2007).

[4] Exogenous participation is not necessary to estimate the model in their paper. Such a restriction avoids the use of a conditional quantile restriction as in Campo, Guerre, Perrigne and Vuong (2007).

and the latent distribution, Haile, Hong and Shum (2003) also introduce some exogenous variables or instruments independent of the latent distribution but affecting bidders' participation.

Our nonparametric identification result exploits variations of the bid distribution in the number of bidders when the latter does not affect the latent private value distribution. We also characterize all the theoretical restictions on observables implied by such an exclusion restriction. In particular, we show that the rationalization of the observed bid distribution involves additional restrictions that the data must satisfy. Though we consider first a benchmark model with symmetric bidders, independent private values and no reserve price, our results extend to a binding reserve price, affiliated private values and asymmetric bidders, whether asymmetry arises from private values and/or heterogenous preferences. As such, our results apply to a large class of auction models.

The paper is organized as follows. Section 2 presents the benchmark model with independent private values (IPV) and reviews the existence, uniqueness and smoothness of the equilibrium strategy. Section 3 is devoted to the identification of the benchmark model, i.e. whether its structural elements can be uniquely recovered from observed bids. We show that the model is nonidentified from observed bids. In view of this, Section 4 exploits exclusion restrictions in the form of an exogenous number of bidders to achieve nonparametric identification of the bidders' utility function and private value distribution. Under such restrictions, we characterize the theoretical restrictions that observed bids need to satisfy. Section 5 extends our nonidentification and identification results to a binding reserve price, affiliated private values and asymmetric bidders. Section 6 considers observed and unobserved heterogeneity and presents a general approach for dealing with the latter. Section 7 concludes. An appendix collects the proofs.

2 The Benchmark Model

This section presents the IPV first-price sealed-bid auction model with risk averse bidders and properties of its equilibrium strategy. A single and indivisible object is sold through a first-price sealed-bid auction. Within the IPV paradigm, each bidder knows his own private value v_i for the auctioned object but not other bidders' private values. There are I potential bidders with $I \in \mathcal{I}$ a finite subset of $\{2, 3, 4, \ldots\}$. Private values are drawn independently from a distribution $F(\cdot|I)$, which is absolutely continuous with density $f(\cdot|I)$ on a support $[\underline{v}(I), \overline{v}(I)] \subset I\!R_+$. The distribution $F(\cdot|I)$ and the number of potential bidders I are assumed to be common knowledge. Let $U(\cdot)$ be the bidders' von Neuman Morgenstern (vNM) utility function with $U(0) = 0$, $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ thereby allowing for risk aversion. All bidders are thus identical *ex ante* and the game is said to be symmetric. Bidder i then maximizes his expected utility

$$\text{E}\Pi_i = U(v_i - b_i)\Pr(b_i \geq b_j, j \neq i) \tag{1}$$

with respect to his bid b_i, where b_j is the jth player's bid. This corresponds to the most studied case in the auction literature where the quality of the auctioned item is known and has equivalent monetary value. See Case 1 in Maskin and Riley (1984) and Krishna (2002).[5] In addition, because the scale is irrelevant, we impose the normalization $U(1) = 1$. The risk neutral case is obtained when $U(\cdot)$ is the identity function.[6]

From Maskin and Riley (1984), if a symmetric Bayesian Nash equilibrium strategy $s(\cdot) = s(\cdot, U, F, I)$ exists, then it is strictly increasing and continuous on $[\underline{v}(I), \overline{v}(I)]$ and differentiable on $(\underline{v}(I), \overline{v}(I)]$.[7] Thus (1) becomes $\text{E}\Pi_i = U(v_i - b_i)F^{I-1}(s^{-1}(b_i)|I)$, where $s^{-1}(\cdot)$ denotes the inverse of $s(\cdot)$. Hence, imposing bidder i's optimal bid b_i to be $s(v_i)$ gives the following differential equation

$$s'(v_i) = (I-1)\frac{f(v_i|I)}{F(v_i|I)}\lambda(v_i - b_i) \text{ for all } v_i \in (\underline{v}(I), \overline{v}(I)], \tag{2}$$

where $\lambda(\cdot) = U(\cdot)/U'(\cdot)$. As shown by Maskin and Riley (1984), the boundary condition is $U[\underline{v}(I) - s(\underline{v}(I))] = 0$, i.e. $s(\underline{v}(I)) = \underline{v}(I)$ because $U(0) = 0$. Moreover, the second-order conditions are satisfied.

When the reserve price is nonbinding, existence of a pure equilibrium strategy follows from Maskin and Riley (2000) and Athey (2001), while its uniqueness is established by

[5] Maskin and Riley (1984) consider a more general model where the utility of winning is of the form $u(-b_i, v_i)$ and the utility of loosing is equal to $w(\cdot)$. Here, $u(-b_i, v_i) = U(v_i - b_i)$ and $w(0) = U(0) = 0$.

[6] Bidders' wealth w can be introduced in the model. The expected profit becomes $[U(w + v_i - b_i) - U(w)]\Pr(b_i \geq b_j, j \neq i) + U(w)$. Different wealths w_i lead to an asymmetric game if the w_is are common knowledge and to a multisignal game if the w_is are private information. See Che and Gale (1998) for a multisignal auction model with budget constraints.

[7] Moreover, as noted by Maskin and Riley (1984, Remark 2.3), the only equilibria are symmetric when $F(\cdot)$ has bounded support, which is assumed below.

Maskin and Riley (2003) using an argument similar to Lebrun (1999). The main contribution of Theorem 1 below is to derive the smoothness of the equilibrium strategy, which is used in the next section. Determining the smoothness of the equilibrium strategy is difficult when the differential equation (2) does not have an explicit solution, which is the case for general utility functions $U(\cdot)$. This is more so as (2) is known to have a singularity at $\underline{v}(I)$ when the reserve price is nonbinding. To address these difficulties we rewrite (2) as a differential equation in the bid quantile function $b(\alpha, I) = s[v(\alpha, I)]$, where $\alpha \in [0, 1]$ and $v(\alpha, I)$ is the α-quantile of $F(\cdot|I)$. We then view the latter differential equation as a member of a set (also called flow) of differential equations $E(B; t) = 0$ parameterized by $t \in [0, 1]$ in an unknown function $B(\cdot)$, where $E(B; 1) = 0$ corresponds to the general utility function $U(\cdot)$, while $E(B; 0) = 0$ corresponds to an appropriate constant relative risk aversion (CRRA) utility function. See (B.1)–(B.3) in Appendix B. Next, we adopt a functional approach which exploits the existence, uniqueness, and smoothness of the equilibrium strategy in the CRRA case, where the solution of (2) is known explicitly. In particular, our functional approach delivers the existence and uniqueness of the equilibrium strategy for a general utility function $U(\cdot)$ by a Continuation Argument Theorem, thereby providing an alternative proof to those used in the economics literature. Moreover, our framework establishes the smoothness of the equilibrium strategy by an Implicit Functional Theorem.

We assume that $U(\cdot)$ and $F(\cdot|I)$ belong to $\mathcal{U}_R$ and $\mathcal{F}_R$ defined as follows, respectively.

Definition 1: *For $R \geq 1$, let $\mathcal{U}_R$ be the set of utility functions $U(\cdot)$ satisfying*
(i) $U : [0, +\infty) \to [0, +\infty)$, $U(0) = 0$ and $U(1) = 1$,
(ii) $U(\cdot)$ is continuous on $[0, +\infty)$, and admits $R + 2$ continuous derivatives on $(0, +\infty)$ with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $(0, +\infty)$,
(iii) $\lim_{x \downarrow 0} \lambda^{(r)}(x)$ is finite for $1 \leq r \leq R + 1$, where $\lambda^{(r)}(\cdot)$ is the rth derivative of $\lambda(\cdot)$.

Conditions (i) and (ii) have been discussed previously. Note that $\lim_{x \downarrow 0} \lambda(x) = 0$ since $U(0) = 0$ and $U'(\cdot)$ is nonincreasing. Thus, from (ii) and (iii) it follows that $\lambda(\cdot)$ admits $R + 1$ continuous derivatives on $[0, +\infty)$. These regularity assumptions are weak as they are satisfied by many vNM utility functions.

Definition 2: *For $R \geq 1$, let $\mathcal{F}_R$ be the set of distributions $F(\cdot|I)$, $I \in \mathcal{I}$, satisfying*
(i) $F(\cdot|I)$ is a c.d.f. with support of the form $[\underline{v}(I), \overline{v}(I)]$, where $0 \leq \underline{v}(I) < \overline{v}(I) < +\infty$,

(ii) $F(\cdot|I)$ *admits* $R+1$ *continuous derivatives on* $[\underline{v}(I), \overline{v}(I)]$,
(iii) $f(\cdot|I) > 0$ *on* $[\underline{v}(I), \overline{v}(I)]$.

Altogether (i)–(iii) imply that $f(\cdot|I)$ is bounded away from zero on $[\underline{v}(I), \overline{v}(I)]$.

Theorem 1: *Let* $R \geq 1$. *Suppose that* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, *then for each* $I \in \mathcal{I}$, *there exists a unique (symmetric) equilibrium strategy* $s(\cdot)$. *Moreover, this strategy satisfies:*
(i) $\forall v \in (\underline{v}(I), \overline{v}(I)]$, $s(v) < v$ *with* $s(\underline{v}(I)) = \underline{v}(I)$,
(ii) $\forall v \in [\underline{v}(I), \overline{v}(I)]$, $s'(v) > 0$ *with* $s'(\underline{v}) = (I-1)\lambda'(0)/[(I-1)\lambda'(0)+1] < 1$,
(iii) $s(\cdot)$ *admits* $R+1$ *continuous derivatives on* $[\underline{v}(I), \overline{v}(I)]$.

The proof of Theorem 1 can be found in Appendix B.

3 General Nonidentification Results

In this section we study identification of the structure $[U, F]$ from observables. We assume that the number I of bidders is observed, as in a first-price sealed-bid auction with a nonbinding reserve price. We also assume that the distribution $G(\cdot|I)$ of an equilibrium bid is known. Thus the identification problem reduces to whether the structure $[U, F]$ can be recovered uniquely from the knowledge of (I, G). A related issue is whether any bid distribution $G(\cdot|I)$ can be rationalized by a structure $[U, F]$. Such a question relates to the possibility of testing the validity of the auction model under consideration.

Following Guerre, Perrigne and Vuong (2000), we express (2) using the distribution $G(\cdot|I)$ of an equilibrium bid. For every $b \in [\underline{b}(I), \overline{b}(I)] = [\underline{v}(I), s(\overline{v}(I))]$, we have $G(b|I) = F(s^{-1}(b)|I) = F(v|I)$ with density $g(b|I) = f(v|I)/s'(v)$, where $v = s^{-1}(b)$. Thus (2) can be written as

$$1 = (I-1)\frac{g(b_i|I)}{G(b_i|I)}\lambda(v_i - b_i) \text{ for all } b_i \in (\underline{b}(I), \overline{b}(I)]. \tag{3}$$

Since $U(\cdot) \geq 0$ and $U''(\cdot) \leq 0$, we have $\lambda'(\cdot) = 1 - U(\cdot)U''(\cdot)/U'^2(\cdot) \geq 1$. Thus $\lambda(\cdot)$ is strictly increasing. Solving (3) for v_i and using $\underline{b}(I) = \underline{v}(I)$ with $\lambda^{-1}(0) = 0$ give

$$v_i = b_i + \lambda^{-1}\left(\frac{1}{I-1}\frac{G(b_i|I)}{g(b_i|I)}\right) \equiv \xi(b_i, U, G, I) \text{ for all } b_i \in [\underline{b}(I), \overline{b}(I)], \tag{4}$$

where $\lambda^{-1}(\cdot)$ denotes the inverse of $\lambda(\cdot)$. This equation expresses each bidder's private value as a function of his corresponding bid, the bid distribution, the number of bidders and the utility function. Note that $\xi(\cdot)$ is the inverse of the bidding strategy $s(\cdot)$.

The equilibrium bid distribution $G(\cdot|I)$ satisfies some regularity properties implied by the smoothness of $s(\cdot)$ given in Theorem 1 and the regularity assumptions on $[U, F]$.

Definition 3: *For $R \geq 1$, let $\mathcal{G}_R$ be the set of distributions $G(\cdot|I)$, $I \in \mathcal{I}$, satisfying*
(i) $G(\cdot|I)$ is a c.d.f. with support of the form $[\underline{b}(I), \overline{b}(I)]$, where $0 \leq \underline{b}(I) < \overline{b}(I) < +\infty$,
(ii) $G(\cdot|I)$ admits $R+1$ continuous derivatives on $[\underline{b}(I), \overline{b}(I)]$,
(iii) $g(\cdot|I) > 0$ on $[\underline{b}(I), \overline{b}(I)]$,
(iv) $g(\cdot|I)$ admits $R+1$ continuous derivatives on $(\underline{b}(I), \overline{b}(I)]$,
(v) $\lim_{b\downarrow\underline{b}(I)} d^r[G(b|I)/g(b|I)]/db^r$ exists and is finite for $r = 1, \ldots, R+1$.

The regularity properties (i)–(iii) are similar to those of Definition 2 for $F(\cdot|I)$. They imply that $g(\cdot|I)$ is bounded away from zero on $[\underline{b}(I), \overline{b}(I)]$ and $\lim_{b\downarrow\underline{b}} G(b|I)/g(b|I) = 0$ so that $\lim_{b\downarrow\underline{b}(I)} \xi(b, U, G, I) = \underline{b}(I)$. Properties (iv) and (v) are specific to the auction model. In particular, (iv) says that $g(\cdot|I)$ is smoother than $f(\cdot|I)$, extending a similar property noted by Guerre, Perrigne and Vuong (2000) for the risk neutral model. Combined with (iii) and (iv), (v) implies that $G(\cdot|I)/g(\cdot|I)$ is $R+1$ continuously differentiable on $[\underline{b}(I), \overline{b}(I)]$.

The following lemma provides necessary and sufficient conditions for rationalizing a distribution of observed bids by an IPV auction model with risk aversion. Hereafter, we say that a distribution is *rationalized* by an auction model with risk aversion if there exists a structure $[U, F]$ whose equilibrium bid distribution is identical to the given distribution.

Lemma 1: *Let $R \geq 1$, and $\mathbf{G}(\cdot|I)$ be the joint distribution of $(b_1, \ldots, b_I)$ conditional on $I \in \mathcal{I}$. There exists an IPV auction model with risk aversion $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ that rationalizes $\mathbf{G}(\cdot|\cdot)$ if and only if*
(i) $\mathbf{G}(b_1, \ldots, b_I|I) = \prod_{i=1}^{I} G(b_i|I)$, with $G(\cdot|\cdot) \in \mathcal{G}_R$,
(ii) $\exists \lambda : I\!\!R_+ \rightarrow I\!\!R_+$ with $R+1$ continuous derivatives on $[0, +\infty)$, $\lambda(0) = 0$ and $\lambda'(\cdot) \geq 1$ such that $\xi'(\cdot) > 0$ on $[\underline{b}(I), \overline{b}(I)]$, where $\xi(b, U, G, I) = b + \lambda^{-1}\left[G(b|I)/((I-1)g(b|I))\right]$.

Condition (i) is related to the IPV paradigm and requires that bids be i.i.d., where $G(\cdot|\cdot)$ satisfies the regularity properties of Definition 3. Condition (ii) arises from $\xi(\cdot, U, G, I)$ being the inverse of the equilibrium strategy, which is strictly increasing for each $I \in \mathcal{I}$.[8]

[8]As shown in the proof of Lemma 1, if condition (ii) is satisfied, then $G(\cdot|I)$ is rationalized by the

The next proposition shows that any distribution $G(\cdot|\cdot) \in \mathcal{G}_R$ can be rationalized by an IPV auction model with a utility function displaying risk aversion.

Proposition 1: *Let $R \geq 1$. A bid distribution $G(\cdot|\cdot)$ can be rationalized by a risk averse structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ if and only if $G(\cdot|\cdot) \in \mathcal{G}_R$.*

Proposition 1 is striking. It implies that the restriction (ii) in Lemma 1 for rationalizing a bid distribution with risk averse bidders is redundant. Specifically, our proof indicates that we can always find a function $\lambda(\cdot)$ corresponding to a utility function $U(\cdot) \in \mathcal{U}_R$ so that condition (ii) in Lemma 1 is satisfied whenever $G(\cdot|\cdot) \in \mathcal{G}_R$. Alternatively, the IPV auction model with general risk aversion does not impose any restriction on observed bids beyond their independence and the weak regularity conditions embodied in $\mathcal{G}_R$. Thus, by allowing for risk aversion, one does enlarge the set of rationalizable bid distributions relative to the risk neutral case studied in Guerre, Perrigne and Vuong (2000).[9]

A model is a set of structures $[U, F]$. Hereafter, a structure $[U, F]$ is *nonidentified* if there exists another structure $[\tilde{U}, \tilde{F}]$ within the model that leads to the same equilibrium bid distribution. If no such structure $[\tilde{U}, \tilde{F}]$ exists for any $[U, F]$, the model is (globally) identified. Given the weakness of the restrictions imposed by the model, it is not surprising that the model with general risk aversion is not identified.

Proposition 2: *Let $R \geq 1$. Any structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ is not identified.*

Proposition 2 implies that the auction model with risk averse bidders is nonparametrically nonidentified. This contrasts with Guerre, Perrigne and Vuong (2000) who show that the auction model with risk neutral bidders is nonparametrically identified. Thus the nonidentification of the general risk aversion model $\mathcal{U}_R \times \mathcal{F}_R$ arises from the unknown utility function $U(\cdot)$, which is restricted to the identity function under risk neutrality.

In view of this result, one can entertain several strategies to identify the model. A first natural strategy is to require more data. For instance, the availability of ascending

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structure $[U, F]$, where $U(x) = \exp \int_1^x (1/\lambda(t))dt$ and $F(\cdot|I)$ is the distribution of $\xi(b, U, G, I)$ with $b \sim G(\cdot|I)$. Because $\lambda(x) \sim \lambda'(0)x$ in the neighborhood of 0, $\int_1^0 (1/\lambda(t))dt = -\infty$ so that $U(0) = 0$, as required.

[9] Campo, Guerre, Perrigne and Vuong (2007) show another interesting result: Any distribution $G(\cdot|\cdot) \in \mathcal{G}_R$ can be rationalized by some constant relative or absolute risk aversion model. Thus, allowing for constant relative or absolute risk aversion can explain any smooth bid distribution.

auction data allows to identifiy nonparametrically $[U, F]$ as shown in Lu and Perrigne (2006). A second strategy is to impose more restrictions on the structure $[U, F]$ through a parameterization of $U(\cdot)$ and/or $F(\cdot|\cdot)$. This is pursued in Campo, Guerre, Perrigne and Vuong (2007) where $U(\cdot)$ and one quantile of $F(\cdot|\cdot)$ are parameterized, which are minimal parametric assumptions to identify semiparametrically the model. Hereafter, we exploit another type of restrictions, namely exclusion restrictions, which have been used extensively in econometrics for identifying demand and supply as well as sample selection models in labor economics.

4 Nonparametric Identification and Restrictions

We now assume that $F(\cdot|\cdot)$ does not depend on the number I of bidders, which corresponds to the restriction $F(\cdot|I) = F(\cdot)$. As such, bidders' participation is exogenous. The functions $U(\cdot)$ and $F(\cdot)$ satisfy the regularity conditions of Definitions 1 and 2 with $R \geq 1$. The previous definitions and equations defining the model need to be revised accordingly with $F(\cdot)$ and $f(\cdot)$ defined on support $[\underline{v}, \underline{v}]$, while the bid distribution $G(\cdot|I)$ still depends on I through $s(\cdot, U, F, I)$ but its support is now $[\underline{v}, s(\overline{v})]$, where the upper bound depends implicitly on I. The key idea of our nonparametric identification result is to exploit variations in the quantiles of the bid distribution with the number of bidders, while the corresponding quantiles of the private value distribution remain the same. Our result relies on a property of the equilibrium strategy, namely that $s(\cdot)$ is increasing in bidders' participation. In simple terms, increased competition renders bidding more aggressive.[10]

Let $I_2 > I_1 \geq 2$ be two different numbers of bidders. We use the index $j = 1, 2$ to refer to the level of competition. Because the equilibrium strategy defined in (2) varies with the number of bidders, the bid distribution will also vary with the number of bidders giving $s_j(\cdot)$ and $G_j(\cdot) \equiv G(\cdot|I_j)$. Though the lower bound of the bid distribution remains the same because of the boundary condition, the upper bound $\overline{b}_j$ varies with competition. The next lemma gives some lower and upper bounds for each equilibrium strategy in terms of the other equilibrium strategy. In particular, it establishes that the equilibrium strategy strictly increases in the number of bidders. As far as we know, the latter property was

[10] Identification of the bidders' utility function when the equilibrium strategies are nonincreasing in competition is discussed in Section 5.4.

obtained for the risk neutral case and the CRRA case, but not when risk aversion takes the general form $U(\cdot)$.[11]

Lemma 2: *Under the previous assumptions, we have*

$$\frac{I_1-1}{I_2-1}s_2(v)+\frac{I_2-I_1}{I_2-1}\underline{v}<s_1(v)<s_2(v)<\frac{I_2-1}{I_1-1}s_1(v)+\frac{I_1-I_2}{I_1-1}\underline{v}$$

for any $v\in(\underline{v},\overline{v}]$.

The preceding lemma provides some testable implications in terms of stochastic dominance between the observed equilibrium bid distributions as well as their quantiles. Let $G_1(\cdot)\prec_{\underline{b}}G_2(\cdot)$ denote that the distribution $G_1(\cdot)$ is strictly (first-order) stochastically dominated by $G_2(\cdot)$ except at the common lower bound $\underline{b}$ of their supports. That is, $G_1(b)>G_2(b)$ for any $b\in(\underline{b},\overline{b}_1]$, where the support of $G_j(\cdot)$ is $[\underline{b},\overline{b}_j]$, which is a compact subset with nonempty interior of $[0,\infty)$. For $j=1,2$, let $b_j(\alpha)$ denote the α-quantile of the equilibrium bid distribution $G_j(\cdot)$, i.e. $G_j[b_j(\alpha)]=\alpha$ for $\alpha\in[0,1]$. Because $b_j=s_j(v)$ where $s_j(\cdot)$ is strictly increasing on $[\underline{v},\overline{v}]$, $b_j(\alpha)=s_j[v(\alpha)]$, where $v(\alpha)$ is the α-quantile of $F(\cdot)$. Hence, from Lemma 2 the quantiles of $G_1(\cdot)$ and $G_2(\cdot)$ satisfy

$$\frac{I_1-1}{I_2-1}b_2(\alpha)+\frac{I_2-I_1}{I_2-1}\underline{b}<b_1(\alpha)<b_2(\alpha)<\frac{I_2-1}{I_1-1}b_1(\alpha)+\frac{I_1-I_2}{I_1-1}\underline{b} \tag{5}$$

for any $\alpha\in[0,1]$. Equivalently, let $G_{jk}(\cdot)$ denote the distribution of $[(I_k-1)b_j+(I_j-I_k)\underline{b}]/[I_j-1]$, where $j,k=1,2$, and $b_j=s_j(v)$.[12] Thus, the lower bound of the support of $G_{jk}(\cdot)$ is $\underline{b}$ and we have $G_{21}(\cdot)\prec_{\underline{b}}G_1(\cdot)\prec_{\underline{b}}G_2(\cdot)\prec_{\underline{b}}G_{12}(\cdot)$.

When the number I of bidders can take more than two values, the previous results imply several testable stochastic dominance relations among the observed bid distributions associated with the different values for I. Several of them are actually redundant. For instance, suppose that $I\in[\underline{I},\overline{I}]$ with $2\le\underline{I}<\overline{I}<\infty$. The above implies that there are $4[1+2+\ldots+(\overline{I}-\underline{I})]=2(\overline{I}-\underline{I})(\overline{I}-\underline{I}+1)$ stochastic dominance relations. The next corollary shows that there are at most $2(\overline{I}-\underline{I}+1)$ relevant relations.[13]

[11] Only the middle inequality will be used for establishing the nonparametric identification of $[U(\cdot),F(\cdot)]$.

[12] When $j=k$, $G_{jk}=G_j(\cdot)$.

[13] See Barrett and Donald (2003) for consistent tests of stochastic dominance.

Corollary 1: *Suppose that* $I \in \mathcal{I} \equiv [\underline{I}, \overline{I}]$ *with* $2 \le \underline{I} < \overline{I}$. *Under the previous assumptions, the quantiles* $b(\alpha, I)$ *of the equilibrium bid distribution* $G(\cdot|I)$ *satisfy*

$$\max\left\{b(\alpha, I-1), \frac{I-1}{I} b(\alpha, I+1) + \frac{1}{I}\underline{b}\right\} < b(\alpha, I) < \min\left\{b(\alpha, I+1), \frac{I-1}{I-2} b(\alpha, I-1) - \frac{1}{I-2}\underline{b}\right\}$$

for any $\alpha \in (0, 1]$ *and any* $I \in [\underline{I}, \overline{I}]$.[14] *Equivalently, let* $b(I)$ *denote the equilibrium bid with* I *bidders. Let* $\underline{G}_I(\cdot)$ *denote the distribution of the maximum of* $b(I-1)$ *and* $[(I-1)b(I+1) + \underline{b}]/I$ *and* $\overline{G}_I(\cdot)$ *denote the distribution of the minimum of* $b(I+1)$ *and* $[(I-1)b(I-1) - \underline{b}]/(I-2)$. *Hence,* $\underline{G}_I(\cdot) \prec_{\underline{b}} G(\cdot|I) \prec_{\underline{b}} \overline{G}_I(\cdot)$, *for any* $I \in [\underline{I}, \overline{I}]$.

Given two different numbers of bidders $I_2 > I_1$, we now turn to the nonparametric identification of $[U(\cdot), F(\cdot)]$ or equivalently $[\lambda(\cdot), F(\cdot)]$ as $U(x) = \exp \int_1^x 1/\lambda(t)dt$ using the normalization $U(1) = 1$. Specifically, our proof is constructive and shows that the inverse function $\lambda^{-1}(\cdot)$, which exists because $\lambda(\cdot)$ is strictly increasing on $[0, +\infty)$, is nonparametrically identified on the range $\mathcal{R}_1$ of the function $R_1(\alpha)$, where $\alpha \in [0, 1]$ and

$$R_j(\alpha) = \frac{1}{I_j - 1} \frac{\alpha}{g_j[b_j(\alpha)]} \tag{6}$$

for $j = 1, 2$. Note that the range $\mathcal{R}_j$ of $R_j(\cdot)$ is of the form $[0, \overline{r}_j]$ with $0 < \overline{r}_j < \infty$ because $g_j(\cdot)$ is bounded away from zero and continuous on $[0, \overline{b}_j]$ by Definition 3 and Lemma 1. Moreover, note that $R_j(\alpha) = \lambda[v_j(\alpha) - s_j(v(\alpha))]$ from (4). Thus, identifying nonparametrically $\lambda^{-1}(\cdot)$ on $\mathcal{R}_j$ is equivalent to identifying nonparametrically $\lambda(\cdot)$ on the range of the markdown/rent $v - s_j(v)$, where $v \in [\underline{v}, \overline{v}]$. Because $s_1(\cdot) < s_2(\cdot)$ on $(\underline{v}, \overline{v}]$ by Lemma 2, we have $R_1(\cdot) > R_2(\cdot)$ on $(0, 1]$. Thus, $\overline{r}_2 < \overline{r}_1$ so that $\mathcal{R}_2$ is strictly included in $\mathcal{R}_1$. Hence, the risk aversion function $\lambda(\cdot)$ is identified nonparametrically on the largest set of possible markdowns $[0, \max_{v \in [\underline{v}, \overline{v}]} v - s_1(v)]$.[15] The next proposition provides explicit expressions for $\lambda(\cdot)$ and $F(\cdot)$.

Proposition 3: *Under the previous assumptions,* $\lambda^{-1}(\cdot)$ *is identified nonparametrically on* $\mathcal{R}_1$. *Specifically,* $\lambda^{-1}(0) = 0$ *and for any* $u_0 \in \mathcal{R}_1 \setminus \{0\}$, $\lambda^{-1}(\cdot)$ *is given by*

$$\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} \Delta b(\alpha_t),$$

[14] Obviously, $b(\cdot, I-1)$ is dropped when $I = \underline{I}$, while $b(\cdot, I+1)$ is dropped when $I = \overline{I}$.

[15] In general, $\max_{v \in [\underline{v}, \overline{v}]} v - s_j(v) \neq \overline{v} - s_j(\overline{v})$. On the other hand, if the markdown $v - s_j(v)$ is increasing in v, then $\max_{v \in [\underline{v}, \overline{v}]} v - s_j(v) = \overline{v} - s_j(\overline{v})$. Moreover, $R_j(\cdot)$ would be increasing in α and $\overline{r}_j = R_j(1)$.

where $\Delta b(\alpha) = b_2(\alpha) - b_1(\alpha)$, *and the sequence* $\{\alpha_t\}$ *is strictly decreasing with* $0 < \alpha_t \leq 1$ *satisfying the nonlinear recursive relation* $R_1(\alpha_t) = R_2(\alpha_{t-1})$ *with initial condition* $R_1(\alpha_0) = u_0$. *Moreover,* $F(\cdot)$ *is identified nonparametrically on* $[\underline{v}, \overline{v}]$ *with* $F(\cdot) = G_j[\xi_j^{-1}(\cdot)]$ *for* $j = 1, 2$.

The sequence $\{\alpha_t\}$ is not necessarly unique. Proposition 3 explains how to construct such a sequence recursively. The key idea is to use the invariance of the quantile of the private value distribution $v(\alpha)$ for two different numbers of bidders I_1 and I_2. Specifically, using (4) leads to the *compatibility condition* $\lambda^{-1}[R_1(\alpha)] = \lambda^{-1}[R_2(\alpha)] + \Delta b(\alpha)$, where $\Delta b(\alpha) > 0$ by Lemma 2. Because $R_1(0) = 0$ and $R_1(\cdot) > R_2(\cdot)$ on $(0, 1]$, the continuity of $R_1(\cdot)$ implies that there exists a value $\tilde{\alpha}$ such that $\tilde{\alpha} < \alpha$ and $R_1(\tilde{\alpha}) = R_2(\alpha)$, which can be used to rewrite the preceding compatibility condition. But the latter also holds at $\tilde{\alpha}$. Continuing the same exercise gives the sequence of values α_t. We show that there exists at least one such sequence $\{\alpha_t\}$. When $R_1(\cdot)$ is strictly increasing, i.e. when the markdown or bidders' rent with I_1 bidders is strictly increasing in v, such a sequence is unique. When $R_1(\cdot)$ is not strictly increasing, the sequence $\{\alpha_t\}$ may not be unique, but all such sequences must lead to the same value for $\sum_{t=0}^{\infty} \Delta b(\alpha_t)$, which then defines $\lambda^{-1}(u_0)$ uniquely.

The construction of such a sequence is illustrated in Figure 1. Figure 1 displays the equilibrium strategies $s_1(\cdot) < s_2(\cdot)$. For $\alpha_0 \in (0, 1]$, consider the α_0-quantile $v(\alpha_0)$ of $F(\cdot)$. The markdown $v(\alpha_0) - b_1(\alpha_0)$ is the sum of $\Delta b(\alpha_0)$, which is known and $\lambda^{-1}[R_2(\alpha_0)]$, which is unknown. The latter is equal to the markdown $v(\alpha_1) - b_1(\alpha_1)$, which is also the sum of $\Delta b(\alpha_1)$ and $\lambda^{-1}[R_2(\alpha_1)]$. Continuing this construction gives the sequence $\{\alpha_t\}$ and establishes the unknown component $\lambda^{-1}[R_2(\alpha_0)]$ as the infinite series of known differences in bid quantiles $\Delta(\alpha_t)$.

An important related question is to characterize all the restrictions on the observed equilibrium bid distributions that arise from the independence of the private value distribution $F(\cdot)$ on the number I of bidders. In particular, it is useful to assess whether the observed bid distributions, which typically vary with the number I of bidders, can be rationalized by a structure $[U(\cdot), F(\cdot)]$ that is independent of I. In other words, these restrictions allow to test the validity of the model and its assumptions. Violation of one of these restrictions leads to reject the model for explaining the observed bids. In particular, it could mean that the exogeneity of bidders' participation is not justified. Lemma 3 provides such restrictions when I takes two different values $I_2 > I_1$.

Lemma 3: *Let $\mathcal{I} = \{I_1, I_2\}$ with $I_1 < I_2$. Let $\mathbf{G}_j(\cdot, \ldots, \cdot)$ be the joint distribution of $(b_1, \ldots, b_{I_j})$, $j = 1, 2$. The equilibrium bid distributions $\mathbf{G}_j(\cdot)$, $j = 1, 2$, are rationalized by a structure $[U(\cdot), F(\cdot)]$ independent of I if and only if*

(i) For each $j = 1, 2$, $\mathbf{G}_j(b_1, \ldots, b_{I_j}) = \prod_{i=1}^{I_j} G_j(b_i)$, where $G_j(\cdot) = G(\cdot|I_j)$ with support of the form $[\underline{b}, \overline{b}_j]$ and $\{G(\cdot|I); I \in \mathcal{I}\} \in \mathcal{G}_R$,

(ii) The α-quantiles of $G_1(\cdot)$ and $G_2(\cdot)$ satisfy $b_1(\alpha) < b_2(\alpha)$ for $\alpha \in (0, 1]$, i.e. $G_1(\cdot) \prec_{\underline{b}} G_2(\cdot)$,

(iii) $\exists \lambda(\cdot) : I\!R_+ \to I\!R_+$ with $R + 1$ continuous derivatives on $[0, +\infty)$, $\lambda(0) = 0$ and $\lambda'(\cdot) \geq 1$ such that

(a) the compatibility condition is satisfied for any $\alpha \in [0, 1]$, namely,

$$b_2(\alpha) + \lambda^{-1}\left(\frac{1}{I_2 - 1}\frac{\alpha}{g_2(b_2(\alpha))}\right) = b_1(\alpha) + \lambda^{-1}\left(\frac{1}{I_1 - 1}\frac{\alpha}{g_1(b_1(\alpha))}\right), \quad (7)$$

(b) for $I_j \in \mathcal{I}$, $\xi_j'(\cdot) > 0$ on $[\underline{b}, \overline{b}_j]$, where $\xi_j(b) = b + \lambda^{-1}[G_j(b)/((I_j - 1)g_j(b))]$.

Unlike Lemma 2, which only provides some (testable) implications, Lemma 3 characterizes all the theoretical restrictions imposed by the model with an exogenous bidders' participation. Relative to the general case of Section 3 in which $F(\cdot)$ can vary with I, the set of bid distributions that can be rationalized is much reduced because of the restrictions (ii) and (iii)(a). Indeed, Lemma 1 implies that any distribution $G_j(\cdot) \in \mathcal{G}_R$ can be rationalized by a structure $[U(\cdot), F(\cdot|I_j)]$, which is not identified. Thus, these additional restrictions help in identifying nonparametrically the structure $[U, F]$.[16] As in Corollary 1, Lemma 3 can be extended straightforwardly to the case where $I \in \mathcal{I} \equiv [\underline{I}, \overline{I}]$. Specifically, (i) and (iii)-(b) hold, while (ii) and (iii)-(a) still hold for all pairs $(I_j, I_k) \in \mathcal{I}, k \neq j$.

5 Extensions

This section extends our results to a binding reserve price, affiliated private values and asymmetric biddders in private values and/or preferences. Except for the first part of

[16]The above compatibility conditions are similar in spirit to the ones used to identify the model with risk aversion and heterogenous preferences. See Campo, Guerre, Perrigne and Vuong (2007).

Proposition 7, we do not provide formal proofs of Propositions 4–7 though we indicate how they can be established in the text.

5.1 Binding Reserve Price

A binding reserve price, i.e. $p_0 > \underline{v}$, introduces a truncation in the observed bid distribution as only the I^* bidders who have a value above p_0 will bid at the auction. Let $G^*(\cdot|I)$ be the truncated bid distribution on $[p_0, \overline{b}(I)]$. We observe I^* the number of active bidders, $I^* \leq I, I \in \mathcal{I}$. Because $G^*(b^*|I) = [F(v|I) - F(p_0|I)]/[1 - F(p_0|I)]$ for $b^* \in [p_0, \overline{b}(I)]$, elementary algebra gives the following inverse equilibrium strategy

$$\begin{aligned} v &= s^{-1}(b^*) = b^* + \lambda^{-1}\left(\frac{1}{I-1}\frac{G^*(b^*|I)}{g_j^*(b|I)} + \frac{1}{I-1}\frac{1}{g^*(b^*|I)}\frac{F(p_0|I)}{1-F(p_0|I)}\right) \\ &\equiv \xi(b^*, U, G^*, I, F(p_0|I)). \end{aligned} \quad (8)$$

Definitions 1, 2 and 3 remain the same with the exception that p_0 replaces $\underline{b}(I)$ in Definition 3. Moreover, because $\lim_{b \downarrow p_0} g^*(b|I) = +\infty$ as $s'(p_0) = 0$ from (2), we allow derivatives and limits to be infinite at p_0 in Definition 3.[17] Given that I^* is Binomial distributed with parameters $[I, 1 - F(p_0|I)]$, I and $F(p_0|I)$ are identified. This identification applies on subsets of auctions.

Proposition 4: *Any structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ with a binding reserve price is not identified. On the other hand, the structure $[U, F]$ with the exclusion restriction $F(\cdot|I) = F(\cdot)$ is identified. Namely, $U(\cdot)$ is identified on $[0, \max_{v\in[\underline{v},\overline{v}]} v - s_1(v)]$, while $F(\cdot)$ is identified on $[p_0, \overline{v}]$.*

The observed bid distribution $\mathbf{G}^*(\cdot, \ldots, \cdot)$ is rationalized if only if Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi(\cdot)$ as defined above. From this rationalization result, any $G^*(\cdot|I) \in \mathcal{G}_R, I \in \mathcal{I}$ can be rationalized by a risk averse structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. It is then straightforward to show that the structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ is not identified.

Under the exogeneity of the number of bidders, we assume that we identify at least two levels of potential bidders $I_1 < I_2$. Let $G_j^*(\cdot)$ be the truncated bid distribution on

[17] To avoid infinite derivatives/limits at p_0, we can consider the bid transformation used in Guerre, Perrigne and Vuong (2000, Section 4), in which case rationalization and identification are based on the density of the transformed bids.

$[p_0, \bar{b}_j]$. Note that Lemma 2 still holds with a binding reserve price as the latter simply reduces the shading relative to the case with no reserve price. In view of (8), the function $R_j(\alpha)$ becomes

$$R_j(\alpha) = \frac{1}{I_j - 1} \frac{1}{g_j^*[b_j^*(\alpha)]} \left(\alpha + \frac{F(p_0)}{1 - F(p_0)} \right),$$

for $j = 1, 2$. Note that $R_j(\alpha)$ differs from (6) by the additional term $F(p_0)/(1 - F(p_0))$. As before, the number of potential bidders I_j and $F(p_0)$ are identified from the distribution of the number of actual bidders. The problem reduces to identifying $\lambda^{-1}(\cdot)$ and $F(\cdot)$ on $[0, \bar{r}_1]$ and $[p_0, \bar{v}]$, respectively. A simple extension of Proposition 3 shows that $[\lambda^{-1}(\cdot), F(\cdot)]$ is nonparametrically identified on these intervals using the quantiles $b_j^*(\alpha)$ of $G_j^*(\cdot)$. Similarly, Lemma 3 can be readily adapted.

5.2 Affiliated Private Values

The vector $(v_1, \ldots, v_I)$ is distributed as $F(\cdot, \ldots, \cdot|I)$, which is exchangeable in its I arguments, affiliated and defined on $[\underline{v}(I), \bar{v}(I)]^I$. We follow the notations of Li, Perrigne and Vuong (2002). Let $G_{B_i|b_i}(b_i|b_i, I)$ be the probability that i has a bid larger than all his opponents conditional on his bid b_i with $B_i = \max_{k \neq i} b_k$ and $b_i = s(v_i)$. Without loss of generality, we can consider $G_{B_1|b_1}(\cdot|\cdot, I)$ as all bidders are symmetric. To simplify the notations, we omit the index 1. The inverse equilibrium strategy becomes

$$v = s^{-1}(b) = b + \lambda^{-1} \left(\frac{G_{B|b}(b|b, I)}{g_{B|b}(b|b, I)} \right) \equiv \xi(b, U, \mathbf{G}, I) \text{ for all } b \in [\underline{b}(I), \bar{b}(I)] \qquad (9)$$

with the joint bid distribution $\mathbf{G}(\cdot, \ldots, \cdot|I)$. Definitions 1 and 2 remain the same except that $F(\cdot, \ldots, \cdot|I)$ is $R + I$ continuously differentiable following Li, Perrigne and Vuong (2000, 2002). Note that $G_{B|b}(\cdot| \cdot |I)/g_{B|b}(\cdot| \cdot |I) = G_{B \times b}(\cdot, \cdot|I)/g_{Bb}(\cdot, \cdot|I)$, where $G_{B \times b}(\cdot, \cdot|I) \equiv \partial G_{Bb}(\cdot, \cdot|I)/\partial b$ and $g_{Bb}(\cdot, \cdot|I)$ are the b-derivative of the joint c.d.f. and the joint density of (B, b), respectively. Let $\mathcal{G}_R$ be the set of exchangeable and affiliated distributions $\{\mathbf{G}(\cdot, \ldots, \cdot|I), I \in \mathcal{I}\}$ with R continuously differentiable densities such that $G_{B \times b}(b, b|I)/g_{Bb}(b, b|I)$ is $R + 1$ continuously differentiable in $b \in [\underline{b}(I), \bar{b}(I)]$ and strictly positive on $(\underline{b}(I), \bar{b}(I)]$.

Proposition 5: *Any structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ with affiliated values is not identified. On the other hand, the structure $[U, F]$ with the exclusion restriction $F(\cdot, \ldots, \cdot|I) = F(\cdot, \ldots, \cdot)$*

is identified. Namely, $U(\cdot)$ *is identified on* $[0, \max_{v\in[\underline{v},\overline{v}]} v - s_1(v)]$*, while* $F(\cdot,\ldots,\cdot)$ *is identified on* $[\underline{v},\overline{v}]^I$*.*

The observed bid distribution $\mathbf{G}(\cdot,\ldots,\cdot|\cdot)$ is rationalized if and only Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi(\cdot)$ as defined above. From this rationalization result, any $\mathbf{G}(\cdot,\ldots,\cdot|\cdot)\in\mathcal{G}_R$ can be rationalized by some risk averse structure $[U,F]\in\mathcal{U}_R\times\mathcal{F}_R$. We can then show that the structure $[U,F]\in\mathcal{U}_R\times\mathcal{F}_R$ is not identified using a similar argument as in Proposition 2, where $G(\cdot|I)/[(I-1)g(\cdot|I)]$ is replaced by $G_{B\times b}(\cdot,\cdot|I)/g_{Bb}(\cdot,\cdot|I)$ in view of (4) and (9).

Under two competitive environments $I_1 < I_2$ and exogenous bidders' participation, the vector $(v_1,\ldots,v_{I_j})$ is distributed as $F_j(\cdot,\ldots,\cdot)$, which is exchangeable in its I_j arguments and affiliated. Moreover, as participation is exogenous, $F_1(\cdot,\ldots,\cdot)$ and $F_2(\cdot,\ldots,\cdot)$ are related by $F_1(v_1,\ldots,v_{I_1}) = \int_{\underline{v}}^{\overline{v}}\ldots\int_{\underline{v}}^{\overline{v}} F_2(v_1,\ldots,v_{I_1},\ v_{I_1+1},\ldots,v_{I_2})dv_{I_1+1}\ldots dv_{I_2}$, i.e. $F_1(\cdot,\ldots,\cdot)$ is the marginal of $F_2(\cdot,\ldots,\cdot)$. Hence, $F_j(\cdot,\ldots,\cdot)$ has support $[\underline{v},\overline{v}]^j$. Assume that the structures $[U,F_j], j=1,2$ satisfy $s_1(v) < s_2(v)$, i.e. competition renders bidding more aggressive.[18] Here again, the exogeneity of the number of bidders allows us to identify $\lambda^{-1}(\cdot)$ on $[0,\overline{r}_1]$ by exploiting variations in the number of bidders, where $R_j(\cdot) = G^j_{B\times b}(b_j(\alpha),b_j(\alpha))/g^j_{B,b}(b_j(\alpha),b_j(\alpha))$ with $b_j(\alpha)$ the α-quantile of the marginal bid density $g_j(\cdot)$ associated with I_j bidders. Specifically, Proposition 3 and Lemma 3 similarly extend to this case.

5.3 Asymmetric Bidders

Asymmetry among bidders, which is known ex ante to all participants, can arise from two different sources, namely from (i) different distributions of private values and/or (ii) different utility functions. We consider these cases separately.

Asymmetry in Private Values

Given $I\in\mathcal{I}$, the joint private value distribution is $\mathbf{F}(\cdot,\ldots,\cdot|I)=\prod_i F_i(\cdot|I)$ with each $F_i(\cdot|I)$ satisfying Definition 2 on the support $[\underline{v}(I),\overline{v}(I)]$. To simplify, we assume that all $F_i(\cdot|I)$ have the same support. Let $\mathcal{F}_R$ be the set of such distributions $\mathbf{F}(\cdot,\ldots,\cdot|I)$ when $I\in\mathcal{I}$. Because of the boundary conditions $s_i(\underline{v}(I))=\underline{v}(I)$ and $s_i(\overline{v}(I))=s_j(\overline{v}(I))$,

[18]Section 5.4 relaxes this assumption. The competition effect is unclear with affiliated private values as some distributions $F(\cdot,\ldots,\cdot)$ may lead to bidding strategies decreasing in the number of bidders as shown by Pinkse and Tan (2005).

$j \neq i$, bidder i's distribution $G_i(\cdot|I)$ is defined on $[\underline{b}(I), \overline{b}(I)]$ for all $i = 1, \ldots, I$. Following Campo, Perrigne and Vuong (2003), we have

$$v_i = b_i + \lambda^{-1}\left(\frac{1}{H_i(b_i|I)}\right) \equiv \xi_i(b_i, U, \mathbf{G}, I), \text{ where } H_i(\cdot|I) = \sum_{j \neq i} \frac{g_j(\cdot|I)}{G_j(\cdot|I)}, \tag{10}$$

for $i = 1, \ldots, I$. Let $\mathcal{G}_R$ be the set of distributions $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$ such that each marginal distribution $G_i(\cdot|\cdot)$ satisfies Definition 3 with $G(b|I)/g(b|I)$ replaced by $1/H_i(b|I)$ in (v).

Proposition 6: *Any structure $[U, \mathbf{F}] \in \mathcal{U}_R \times \mathcal{F}_R$ with asymmetry in private values is not identified. On the other hand, the structure $[U, \mathbf{F}]$ with the exclusion restriction $F_i(\cdot|I) = F_i(\cdot)$, $i \in \mathcal{I}$ is partially identified. Namely, $U(\cdot)$ is identified on $[0, \max_{v \in [\underline{v}, \overline{v}], i=1,\ldots,I_0} v - s_i(v)]$, while $F_i(\cdot), i = 1, \ldots, I_0$ are identified on $[\underline{v}, \overline{v}]$, where I_0 is the number of bidders participating to both auctions. For the remaining bidders, $F_i(\cdot)$ is identified for the quantiles satisfying $R_i(\alpha) \in [0, \max_{j=1,\ldots,I_0} \overline{r}_{j1}]$, where $R_i(\alpha)$ is defined below.*

The bid distribution $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$ is rationalized if and only if Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi_i(\cdot), i = 1, \ldots, I$ as defined above. Hence, any $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$ can be rationalized by a structure with $[U, \mathbf{F}] \in \mathcal{U}_R \times \mathcal{F}_R$. It is then straigthforward to show that any structure $[U, \mathbf{F}] \in \mathcal{U}_R \times \mathcal{F}_R$ is not identified.

Under two competitive environments $I_2 > I_1 \geq 2$ and exogenous bidders' participation, the bidder of type i has the same private value distribution irrespective of the number of bidders participating to the auction. Thus, all $F_i(\cdot|I)$s are defined on the same support $[\underline{v}, \overline{v}]$. Since our results under exclusion restrictions exploit the difference in bidding behavior under two competitive environments, it is crucial that at least one bidder participates in both auctions.[19] For instance, when $I_1 = 2$ and $I_2 = 3$, at least one of the bidder in the two bidders auction must participate in the auction with three bidders. In the case of asymmetry, because of the complexity of the system of differential equations defining the equilibrium strategies, it is difficult if not impossible to prove that equilibirum strategies are increasing with competition. Nevertheless, because of the independence of

[19] More generally, it is important that we observe at least one bidder's "type" in both auctions. This is useful in practice as a few "types" are often entertained in empirical studies involving asymmetric bidders. See for instance Campo, Perrigne and Vuong (2003), Athey, Levin and Seira (2004) and Flambard and Perrigne (2006).

private values, it is reasonable to postulate that equilibrium strategies are increasing in the number of bidders due to the competition effect.

Let $s_{ij}(\cdot)$ denote the equilibrium strategy for bidder $i = 1, \dots, I_j$, when the number of bidders is I_j, $j = 1, 2$. The boundary conditions are $s_{i1}(\underline{v}) = s_{i'2}(\underline{v}) = \underline{v}$ for $i = 1, \dots, I_1$ and $i' = 1, \dots, I_2$, and $s_{ij}(\overline{v}) = s_{i'j}(\overline{v})$ for $i, i' = 1, \dots, I_j, j = 1, 2$. We assume that bidders $1, \dots, I_0$ participate to both auctions, where $1 \leq I_0 \leq I_1$. Let $v_i(\alpha)$ and $b_{ij}(\alpha)$ be the α-quantiles of $F_i(\cdot)$ and $G_{ij}(\cdot) = G_i(\cdot|I_j)$, respectively. Instead of (4), we now have at the α-quantile

$$v_i(\alpha) = b_{ij}(\alpha) + \lambda^{-1}\left(R_{ij}(\alpha)\right), i = 1, \dots, I_j, j = 1, 2 \tag{11}$$

for $\alpha \in [0, 1]$, where $R_{ij}(\alpha) = 1/H_{ij}(b_{ij}(\alpha))$ takes values in the range $\mathcal{R}_{ij} = [0, \overline{r}_{ij}]$ with $H_{ij}(\cdot) = \sum_{k \neq i} g_{kj}(\cdot)/G_{kj}(\cdot)$. A straightforward extension of Proposition 3 shows that $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{i=1,\dots,I_0} \overline{r}_{i1}]$, while $[F_1, \cdots, F_{I_0}]$ are identified on $[\underline{v}, \overline{v}]$. Because $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{i=1,\dots,I_0} \overline{r}_{i1}]$, which may be a strict subset of $[0, \overline{r}_{i1}]$, where i refers to a remaining bidder, his private value distribution may not be identified everywhere justifying the partial identification result of Proposition 6. Lemma 3 also extends where the compatibility conditions (7) now hold for each of the I_0 bidders.

Asymmetry in Preferences

We consider structures of the form $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ with $\mathbf{U} = \{(U_1, \dots, U_I) \in \mathcal{U}_R^I \equiv \bigotimes_{i=1}^I \mathcal{U}_R, I \in \mathcal{I}\} \in \mathcal{U}_R^{\mathcal{I}}$. Given $I \in \mathcal{I}$ and dropping the superscript to simplify, we obtain for $i = 1, \dots, I$

$$v_i = b_i + \lambda_i^{-1}\left(\frac{1}{H_i(b_i|I)}\right) \equiv \xi_i(b_i, U_i, \mathbf{G}, I), \tag{12}$$

where $\lambda_i(\cdot) = U_i(\cdot)/U_i'(\cdot)$ and $H_i(\cdot|I) = \sum_{j \neq i} g_j(\cdot|I)/G_j(\cdot|I)$. For each I, the boundary conditions $s_1(\underline{v}) = \ldots = s_I(\underline{v}) = \underline{v}$ and $s_1(\overline{v}) = \ldots = s_I(\overline{v})$ give a common support $[\underline{b}(I), \overline{b}(I)]$ for the bid distributions across bidders. Let $\mathcal{G}_R$ be the set of distributions $\mathbf{G}(\cdot, \dots, \cdot|\cdot)$ such that each marginal distribution $G_i(\cdot|\cdot)$ satisfies Definition 3 with $G(b|I)/g(b|I)$ replaced by $1/H_i(b|I)$ in (v).

Proposition 7: *Any structure $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ with asymmetry in preferences satisfying $H_i'(\cdot|I) < 0, i = 1, \dots, I, I \in \mathcal{I}$ is not identified.*[20] *On the other hand, the structure $[\mathbf{U}, F]$*

[20]The assumption $H_i'(\cdot|I) < 0$ corresponds to an increasing markup $v_{i\alpha} - b_{i\alpha}$ in α from (12). If all the

with the exclusion restriction $F(\cdot|I) = F(\cdot)$ *is identified. Namely,* U_i *is identified on* $[0, \max_{v\in[\underline{v},\overline{v}]} v - s_i(v)]$ *for* $i = 1, \dots, I$*, while* $F(\cdot)$ *is identified on* $[\underline{v}, \overline{v}]$.

Because the α-quantiles $(b_{1\alpha}, \dots, b_{I\alpha})$ all correspond to the same α-quantile v_α, (12) evaluated at an α-quantile for an arbitrary pair (i, j) of bidders gives the compatibility condition

$$b_{j\alpha} + \lambda_j^{-1}\left(\frac{1}{H_j(b_{j\alpha}|I)}\right) = b_{i\alpha} + \lambda_i^{-1}\left(\frac{1}{H_i(b_{i\alpha}|I)}\right). \tag{13}$$

The bid distribution $\mathbf{G}(\cdot, \dots, \cdot|\cdot)$ is then rationalized if and only if (i) Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi_i(\cdot), i = 1, \dots, I$ as defined above and (ii) the compatibility condition (13) is satisfied for any pair (i, j) and $\alpha \in [0, 1]$.[21] The latter condition reduces the set of bid distributions that can be rationalized relative to the symmetric case and may help in identification. Despite this condition, the nonparametric model is still not identified. Since the proof is more involved than in previous cases, we provide a proof of such a result in the appendix.[22]

Under exogenous bidders' participation, we assume again that I_0 bidders participate to both auctions, where $I_0 \geq 1$ and that equilibrium strategies are increasing in competition. Equation (11) takes a similar form with $v(\alpha)$ and $\lambda_i^{-1}(\cdot)$ replacing $v_i(\alpha)$ and $\lambda^{-1}(\cdot)$. A similar argument as in Proposition 3 applies for identifying nonparametrically $\lambda_i^{-1}(\cdot)$ on $[0, \overline{r}_{i1}]$ for $i = 1, \dots, I_0$ from which we can identify $F(\cdot)$ on $[\underline{v}, \overline{v}]$. Since $F(\cdot)$ is identified everywhere, following a similar argument as in Lu and Perrigne (2006), the $\lambda_j^{-1}(\cdot)$s for the remaining bidders are identified from (12). Specifically, because the $v(\alpha)$s are identified, we can recover the remaining $\lambda_j^{-1}(\cdot)$ on $[0, \overline{r}_j]$. Again Lemma 3 extends with the compatibility conditions (13) holding for each of the I_0 bidders and $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$.

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bid distributions $G_1, \dots, G_I$ are log-concave, this assumption is automatically satisfied. Our requirement is weaker as some bid distributions may not be log-concave. Log-concavity is usually verified on data.

[21] Though similar in spirit, the compatibility condition (13) applies within each auction, while the compatibility condition (7) applies across auctions.

[22] On the other hand, if (say) bidder 1 participates to all auctions and his utility $U_1(\cdot)$ is known, the nonparametric model $\mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ becomes identified as (12) for $i = 1$ allows to identify $F(\cdot|\cdot)$. Thus, evaluated at the α-quantile, (12) for $i \neq 1$ allows to identify $\lambda_i(\cdot)$ on $[0, \max_\alpha(v_\alpha - b_{i\alpha})]$. This result is useful when bidders differ by their sizes and "large" ones, wich participate to all auctions, can be assumed to be risk neutral.

Asymmetry in Both Preferences and Private Values

This third case involves asymmetry in both private value distributions and preferences. Given the above results, the model is not identified in general. We then consider exogenous bidders' participation. Thus, (11) takes a similar form with $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$. Despite the complexity of this case, which has not been considered to our knowledge, it can be shown that the structure $[\lambda_i, F_i]$ is nonparametrically identified for the $I_0 \geq 1$ bidders who participate to both auctions. Specifically, we can apply Proposition 3 to each of these participating bidders to identify nonparametrically $\lambda_i^{-1}(\cdot)$ and $F_i(\cdot)$ on $[0, \overline{r}_{i1}]$ and $[\underline{v}, \overline{v}]$, respectively. On the other hand, we cannot identify the pair $[\lambda_i^{-1}(\cdot), F_i(\cdot)]$ for the other bidders. Again Lemma 3 extends with the compatibility condition (7) holding for each of the common bidders where $\lambda_i^{-1}(\cdot)$ replaces $\lambda^{-1}(\cdot)$.

5.4 Bidding Strategies Nonincreasing in Competition

In the previous extensions, we have assumed that equilibrium strategies are increasing in the number of bidders to simplify the exposition. This may not be always the case. For instance, as indicated previously, affiliated private values may lead to equilibirum strategies that are decreasing in competition for some $\mathbf{F}(\cdot, \ldots, \cdot|\cdot)$. In this section, we discuss how our results extend when the equilibrium strategies are nonincreasing in competition. As before, let $I_1 < I_2$. We assume that for a bidder participating to both auctions his equilibrium strategies $s_1(\cdot)$ and $s_2(\cdot)$ intersect a finite number of times at most. This excludes the case where these strategies are identical on some open interval of private values. The nonparametric identification of the model is then established through the following steps:

- Step 1: From the knowledge of $G_1(\cdot)$ and $G_2(\cdot)$, we can identify the positive values $0 \geq \alpha_1^* < \ldots < \alpha_K^* \geq 1$ at which the equilibrium strategies $s_1(\cdot)$ and $s_2(\cdot)$ intersect, i.e. such that $b_1(\alpha_k^*) = b_2(\alpha_k^*)$.

- Step 2: Let $s_j(\cdot) < s_{j'}(\cdot)$ on $(\underline{v}, v(\alpha_1^*)), j, j' = 1, 2$. By Proposition 3, for any $\alpha_0 \in (0, \alpha_1^*)$, we can identify $\lambda^{-1}(u_0)$ as $\sum_{t=0}^{+\infty} |\Delta b(\alpha_t)|$, where $u_0 = R_j(\alpha_0)$. By continuity of $\lambda^{-1}(\cdot)$, we can also identify $\lambda^{-1}(R_j(\alpha_1^*))$ which is also equal to $\lambda^{-1}(R_{j'}(\alpha_1^*))$ by the compatibility condition (7). Hence, $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{\alpha \in [0, \alpha_1^*]} R_j(\alpha)] =$

$[0, \max\{\max_{\alpha\in[0,\alpha_1^*]} R_j(\alpha), \max_{\alpha\in[0,\alpha_1^*]} R_{j'}(\alpha)\}]$.

- Step 3: We have $s_{j'}(\cdot) < s_j(\cdot)$ on $(v(\alpha_1^*), v(\alpha_2^*))$. For any $\alpha_0 \in (\alpha_1^*, \alpha_2^*)$, we let $u_0 = R_{j'}(\alpha_0)$ and by Proposition 3 we construct recursively α_{t+1} from the equation $R_{j'}(\alpha_{t+1}) = R_j(\alpha_t)$ subject to $\alpha_{t+1} < \alpha_t$. There are two possibilities:

  (i) If $\alpha_{t+1} \in [0, \alpha_1^*]$, we stop this sequence as we switch to values for which $s_j(\cdot) \leq s_{j'}(\cdot)$. We have $\lambda^{-1}(u_0) = \lambda^{-1}(R_{j'}(\alpha_{t+1})) + \sum_{s=0}^{t} |\Delta b(\alpha_s)|$. But $\lambda^{-1}(R_j(\alpha_{t+1})) = \lambda^{-1}(R_{j'}(\alpha_{t+1})) + |\Delta b(\alpha_{t+1})|$, where $\lambda^{-1}(R_j(\alpha_{t+1}))$ is identified from Step 1 and hence equal to $\sum_{r=0}^{+\infty} |\Delta b(\alpha'_r)|$ for some decreasing sequence $\{\alpha'_r\}$ with $\alpha'_0 = \alpha_{t+1}$. Thus $\lambda^{-1}(u_0) = \sum_{r=1}^{+\infty} |\Delta b(\alpha'_r)| + \sum_{s=0}^{t} |\Delta b(\alpha_s)|$, which gives $\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} |\Delta b(\alpha_t)|$ by letting $\alpha'_r \equiv \alpha_{t+r}$. Figure 2 illustrates this case, where $j = 1$, $j' = 2$ and $t + 1 = 2$.

  (ii) If α_{t+1} remains in (α_1^*, α_2^*) for all t, the sequence $\{\alpha_t\}$ will converge to α_1^*. Taking the limit gives $\lambda^{-1}(u_0) = \lambda^{-1}(R_{j'}(\alpha_1^*)) + \sum_{s=0}^{+\infty} |\Delta b(\alpha_s)|$. But $\lambda^{-1}(R_{j'}(\alpha_1^*)) = \lambda^{-1}(R_j(\alpha_1^*))$, where the latter is identified from Step 2. Figure 3 illustrates this case with $j = 1$ and $j' = 2$.

  By continuity of $\lambda^{-1}(\cdot)$, we can also identify $\lambda^{-1}(R_{j'}(\alpha_2^*))$ which is also equal to $\lambda^{-1}(R_j(\alpha_2^*))$ by the compatibility condition (7). Hence, at the end of Step 3, $\lambda^{-1}(\cdot)$ is identified on $[\ \min\{\min_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_j(\alpha), \min_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_{j'}(\alpha)\}, \max\{\max_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_j(\alpha),$ $\max_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_{j'}(\alpha)\}\ ]$. By combining Step 2 and Step 3, $\lambda^{-1}(\cdot)$ is identified on $[0, \max\{\max_{\alpha\in[0,\alpha_2^*]} R_j(\alpha), \max_{\alpha\in[0,\alpha_2^*]} R_{j'}(\alpha)\}]$.

- Step 4: For any $k \geq 2$ and $\alpha_0 \in (\alpha_k^*, \alpha_{k+1}^*)$, we repeat Step 3 and its two possibilities. If $\alpha_{t+1} \in [0, \alpha_k^*]$, we stop the sequence and we have $\lambda^{-1}(u_0) = \lambda^{-1}(R_{j'}(\alpha_{t+1})) + \sum_{s=0}^{t} |\Delta b(\alpha_s)|$, where $\lambda^{-1}(R_{j'}(\alpha_{t+1}))$ is identified from previous steps. Thus $\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} |\Delta b_{\alpha_t}|$ as in Step 3-(i). If α_{t+1} remains in $(\alpha_k^*, \alpha_{k+1}^*)$, Step 3-(ii) applies and $\lambda^{-1}(u_0)$ is identified. As before, by continuity, $\lambda^{-1}[R_j(\alpha_{k+1}^*)] = \lambda^{-1}[R_{j'}(\alpha_{k+1}^*)]$ is identified. Applying a similar argument, the combination of the various steps allows us to identify $\lambda^{-1}(\cdot)$ on $[0, \max\{\max_{\alpha\in[0,\alpha_{k+1}^*]} R_j(\alpha), \max_{\alpha\in[0,\alpha_{k+1}^*]} R_{j'}(\alpha)\}]$ and hence on $[0, \max\{\max_{\alpha\in[0,1]} R_j(\alpha), \max_{\alpha\in[0,1]} R_{j'}(\alpha)\}]$ when $k + 1 = K$.

The identification of the bidder's private value distribution $F(\cdot)$ follows as in Proposition 3. The above procedure shows that assuming bidding strategies increasing in competition is not necessary to identify nonparametrically the model. The construction of the sequence $\{\alpha_t\}$ is however more involved.

6 Observed and Unobserved Heterogeneity

The previous sections have shown that exogenous variations in the number of bidders can be exploited to identify nonparametrically the bidders' utility functions and their private value distributions with a binding or nonbinding reserve price, affiliated private values and asymmetric bidders. In practice, some additional variables and possible unobserved heterogeneity can explain auctioned objects heterogeneity and bidders' participation. This section discusses how our previous results extend to this context. In particular, bidders' participation may be endogenous and some of these additional variables can play the role of instrumental variables through exclusion restrictions.

We first consider the case of unobserved heterogeneity in bidder's participation. This leads to a model of exogenous participation. Let W be a vector of observed variables characterizing heterogeneity across auctioned objects. These variables are assumed to affect both bidders' private value distribution and bidders' participation.[23] Bidders' participation is modeled as $I = I(W, \epsilon)$, where ϵ can be interpreted as a term of unobserved heterogeneity or as a traditional error term. We assume $v \perp \epsilon | W$, namely bidders' private values are independent of ϵ given the auction characteristics W. This assumption translates into the exclusion restriction $F(v|W, \epsilon) = F(v|W)$, while the observed bid distribution is $G(b|I, W)$ since $b = s(v, U, F, I)$. This model is similar to the one in Section 4 since the latent private value distribution does not depend on the number of bidders or equivalently bidders' private values are independent of I given W leading to the exogeneity of I. The only difference between the two models is the introduction of the vector of conditioning variables W. This exclusion restriction allows us to exploit variations in bidding behavior under two competitive environments, i.e. $I_2 > I_1$, at W given. Proposition 3 applies and the pair $[U(\cdot), F(\cdot|\cdot)]$ is nonparametrically identified, while the quantile

[23]See Athey, Levin and Seira (2004) for example of such variables.

becomes $b_j(\alpha, W)$. Regarding Lemma 3, the bids are now conditionally independent given W in (i), while the rest extends straightforwardly.

The term of unobserved heterogeneity ϵ may, however, affect the private value distribution as ϵ can capture some unobserved characteristics affecting both private values and bidders' participation. This leads to a model of endogenous participation. The introduction of additional variables or instruments combined with appropriate exclusion restrictions solves this problem. Specifically, bidders' participation is modeled as $I = I(W, Z, \epsilon)$, while we assume $v \perp Z|(W, \epsilon)$, namely the bidders' private values are independent of Z given (W, ϵ). This translates into the exclusion restriction $F(v|W, Z, \epsilon) = F(v|W, \epsilon)$. Hence, the variables Z can be viewed as instruments. This model corresponds to an endogenous number of bidders as the unobserved heterogeneity ϵ affects both bidders' private values and bidders' participation. This model is reminiscent of Bajari and Hortacsu (2003) modeling of bidders' entry in online coin auctions within a common value framework. Their empirical results show that the variables explaining bidder's entry are the appraisal value of the auctioned object (W), the reserve price for the auctioned object (Z_1) and the seller's reputation (Z_2), while the bidder's private signal distribution depends on the appraisal value W only. Haile, Hong and Shum (2003) also adopt a similar framework to test for common value in first-price sealed-bid auctions when I is endogenous.

Proposition 8: *The structure $[U, F]$ with endogenous participation and unobserved heterogeneity is identified under the exclusion restriction $F(\cdot|W, Z, \epsilon) = F(\cdot|W, \epsilon)$, additive separability of $I(W, Z, \epsilon)$ in ϵ and $\mathrm{E}[\epsilon|W, Z] = 0$.*

The argument is as follows. The observed bid distribution satisfies $G(b|W, Z, \epsilon) = G(b|I, W, \epsilon)$ as $s(\cdot) = s(\cdot, I, W, \epsilon)$, where $v \sim F(v|W, Z, \epsilon) = F(v|W, \epsilon)$ and $I = I(W, Z, \epsilon)$. The parallel with Proposition 3 appears as we can exploit variations in bidding behavior under two competitive environments while the latent distribution remains the same at (W, ϵ) given. The term of heterogeneity ϵ is, however, unobserved. Under additive separability of ϵ, we have $I(W, Z, \epsilon) = I(W, Z) + \epsilon$, where ϵ takes a finite number of values. Under the assumption $\mathrm{E}(\epsilon|W, Z) = 0$, $I(W, Z) = \mathrm{E}(I|W, Z)$. Because $\mathrm{E}(I|\cdot, \cdot)$ is the regression of I on (W, Z), $\mathrm{E}(I|\cdot, \cdot)$ is nonparametrically identified so that ϵ can be recovered as $\epsilon = I - \mathrm{E}(I|\cdot, \cdot)$. Proposition 3 applies and $[U, F]$ is nonparametrically identified, while the quantile becomes $b_j(\alpha, W, \epsilon)$. Regarding Lemma 3, the bids are now conditionally independent given (W, ϵ) in (i), while the rest extends straightforwardly.

It should be noted that endogenous entry with an additive error term is a general method that can be used for solving the problem of unobserved heterogeneity. The exclusion restriction or existence of instruments Z is not needed in general and is used here because the model is not identified otherwise. For instance, consider the risk neutral model with endogenous entry $I = I(W) + \epsilon$, where the bidders' private value distribution is $F(v|W, \epsilon)$. Hence, some unobserved heterogeneity affects both bidders' participation and private values. As above, ϵ can be recovered as $\epsilon = I - \mathrm{E}(I|W, \epsilon)$. Thus, $F(\cdot|W, \epsilon)$ is nonparametrically identified following Guerre, Perrigne and Vuong (2000, Theorem 1). This method differs from Krasnokutskaya (2004) who identifies the term of unobserved heterogeneity in a private value model $F(v|W, \epsilon)$ with exogenous participation $I = I(W)$ using a multiplicative decomposition of private values. Her result then relies on a measurement error model with multiple indicators studied in Li and Vuong (1998).

7 Conclusion

This paper addresses the problem of nonparametric identification of bidders' utility function(s). We show that the auction model with risk aversion is not identified in general and that it imposes weak restrictions on observables. This implies that the auction model with risk averse biddders is not testable in view of bids only. In view of these results, we exploit exclusion restrictions under the form of an exogenous bidders' participation to identify nonparametrically the bidders' utility function and their private value distribution. The results are general as they extend to a binding reserve price, affiliated private values and asymmetric bidders. We also provide some conditions that must be verified by the data and can be used for model testing. More generally, identifiying risk aversion is an important problem in the analysis of microeconomic data. In this respect, it would be interesting to investigate how similar ideas can be exploited to identify nonparametrically agents' utility function in other economic contexts such as in insurance.

A nonparametric estimation method clearly needs to be developed. Recent results on the nonparametric estimation of quantiles can be used. The difficulty relies, however, in the infinite series of differences in quantiles that identify $\lambda^{-1}(\cdot)$ in Proposition 3 as the sequence of estimated $\{\hat{\alpha}_t\}$ and hence the estimated $\Delta\hat{b}(\hat{\alpha}_t)$ are serially correlated. This greatly complicates the derivation of the asymptotic properties of the resulting estimator and its rate of convergence. Another possibility could be to exploit directly the compatibility conditions (7) to estimate nonparamerically $\lambda^{-1}(\cdot)$. On the other hand, if one is willing to impose some parameterization on $U(\cdot)$ and a quantile restriction on $F(\cdot)$, one can use the semiparametric estimator proposed in Campo, Guerre, Perrigne and Vuong (2007) as the model is semiparametrically identified under the exclusion restriction considered here. If data permit, an alternative method is to combine bidding data from ascending and sealed-bid auctions as proposed by Perrigne and Lu (2006).

Appendix A

This appendix contains the proofs of Lemmas 1–3, Propositions 1–3, the first part of Proposition 7 and Corollary 1.

Proof of Lemma 1: First, we prove that (i) and (ii) are necessary. Because $b_i = s(v_i, U, F, I)$ and the v_is are i.i.d., the b_is are also i.i.d. The fact that $G(\cdot|\cdot) \in \mathcal{G}_R$ follows from Theorem 1, Definitions 1-2 and (3). To prove that (ii) is also necessary, consider (4), where $\lambda(\cdot) \equiv U(\cdot)/U'(\cdot)$. Thus $\lambda(\cdot)$ is defined from $I\!R_+$ to $I\!R_+$ because $\lambda(0) = \lim_{x\downarrow 0} \lambda(x) = 0$, as noted after Definition 1. As $U(\cdot)$ admits $R+2$ continuous derivatives on $(0, +\infty)$ with $U'(\cdot) > 0$, and $\lim_{x\downarrow 0} \lambda^{(r)}$ is finite for $r = 1, \ldots, R+1$, then $\lambda(\cdot)$ has $R+1$ continuous derivatives on $[0, +\infty)$. As $\lambda'(\cdot) = 1 - \lambda(\cdot)U''(\cdot)/U'(\cdot)$, we have $\lambda'(\cdot) \geq 1$ because $\lambda(\cdot) \geq 0$, $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$. It remains to show that $\xi'(\cdot) > 0$. The equilibrium strategy must solve the differential equation (2). As (4) follows from (2), $s(\cdot)$ must satisfy $\xi[s(v), U, G, I] = v$ for all $v \in [\underline{v}, \overline{v}]$. We then obtain $\xi(b, U, G, I) = s^{-1}(b, U, F, I)$. This implies $\xi'(\cdot) = [s^{-1}(\cdot)]' > 0$ using Theorem 1.

Second, we show that (i) and (ii) are together sufficient. First, we construct a pair $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. Let $U(\cdot)$ be such that $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ or $U'(\cdot)/U(\cdot) = 1/\lambda(\cdot)$. Integrating with the normalization $U(1) = 1$ gives $U(x) = \exp \int_1^x 1/\lambda(t)dt$. We verify that $U(\cdot) \in \mathcal{U}_R$. Because $\lambda(\cdot)$ admits $R+1$ continuous derivatives on $[0, +\infty)$, then Definition 1-(iii) is clearly satisfied. Moreover, in the neighborhood of zero, $\lambda(t) \sim \lambda'(0)t$ with $1 \leq \lambda'(0) < \infty$. Thus $\int_x^1 1/\lambda(t)dt$ diverges to infinity, which implies that $U(x)$ tends to zero as $x \downarrow 0$. Define $U(0) = 0$ so that $U(\cdot)$ is continuous on $[0, +\infty)$. Because $U'(x) = \exp \int_1^x 1/\lambda(t)dt/\lambda(x)$, where $\lambda(\cdot) > 0$ on $(0, +\infty)$, we have $U'(\cdot) > 0$ on $(0, +\infty)$. The second-order derivative gives $U''(x) = [-\lambda'(x) + 1] \exp \int_1^x 1/\lambda(t)dt/\lambda^2(x)$. Since $\lambda'(x) \geq 1$, $U''(\cdot) \leq 0$ on $(0, +\infty)$. It remains to show that $U(\cdot)$ admits $R+2$ continuous derivatives on $(0, +\infty)$. By assumption, $\lambda(\cdot)$ has $R+1$ continuous derivatives on $[0, +\infty)$. It follows that $U(\cdot)$ admits $R+2$ continuous derivatives on $(0, +\infty)$.

Let $F(\cdot|I)$ be the distribution of $X = b + \lambda^{-1}[G(b|I)/(I-1)g(b|I)]$ conditional on I, where $b \sim G(\cdot|I)$. We verify that $F(\cdot|\cdot) \in \mathcal{F}_R$. We have $F(x|I) = \Pr(X \leq x|I) = \Pr[\xi(b) \leq x|I] = \Pr[b \leq \xi^{-1}(x)|I] = G[\xi^{-1}(x)|I]$, because $\xi'(\cdot) > 0$ by assumption. This implies $F(\cdot|I) = G[\xi^{-1}(\cdot)|I]$ on $[\underline{v}(I), \overline{v}(I)]$, where $\underline{v}(I) \equiv \xi(\underline{b}(I)) = \underline{b}(I)$ and $\overline{v}(I) \equiv \xi(\overline{b}(I)) < \infty$ by continuity of $\xi(\cdot)$. Because $\xi(\cdot)$ and $G(\cdot|I)$ are strictly increasing, $F(\cdot|I)$ is strictly increasing on its support $[\underline{v}(I), \overline{v}(I)]$. Moreover, $\xi(\cdot)$ is $R+1$ continuously differentiable on $[\underline{b}(I), \overline{b}(I)]$. This follows from the definition of $\xi(\cdot)$, the $R+1$ continuous differentiability of $\lambda^{-1}(\cdot)$ on $[0, +\infty)$, and the $R+1$ continuous differentiability of $G(\cdot|I)/g(\cdot|I)$ on $[\underline{b}(I), \overline{b}(I)]$, which follows from Definition 3-(iv,v). Thus $F(\cdot|I) = G[\xi^{-1}(\cdot)|I]$ admits $R+1$ continuous derivatives on $[\underline{v}(I), \overline{v}(I)]$ because $G(\cdot|I)$ has

$R+1$ continuous derivatives on $[\underline{b}(I),\overline{b}(I)]$. It remains to show that $f(\cdot|I)>0$ on $[\underline{v}(I),\overline{v}(I)]$. This follows from $f(\cdot|I)=g[\xi^{-1}(\cdot)|I]/\xi'[\xi^{-1}(\cdot)]$, where $g(\cdot|I)>0$ from Definition 3 and $\xi'(\cdot)$ is finite on $[\underline{b}(I),\overline{b}(I)]$.

Lastly, we show that the pair $[U,F]$ rationalizes $G(\cdot|\cdot)$, i.e. that $G(\cdot|I)=F[s^{-1}(\cdot,U,F,I)|I]$ on $[\underline{b}(I),\overline{b}(I)]$, where $s(\cdot,U,F,I)$ solves (2) with the boundary condition $s(\underline{v}(I),U,F,I)=\underline{v}(I)$. By construction of $F(\cdot|I)$, $G(\cdot|I)=F[\xi(\cdot)|I]$. Thus, it suffices to show that $\xi^{-1}(\cdot)$ solves (2) with the boundary condition $\xi^{-1}(\underline{v}(I))=\underline{v}(I)$. The boundary condition is straightforward as $\xi(\underline{b}(I))=\underline{b}(I)=\underline{v}(I)$. By construction of $F(\cdot|I)$, $f(\cdot|I)/F(\cdot|I)=[\xi^{-1}(\cdot)]'g[\xi^{-1}(\cdot)|I]/G[\xi^{-1}(\cdot)|I]$. Thus $\xi^{-1}(\cdot)$ solves (2) if $1=\{(I-1)g[\xi^{-1}(v)|I]\lambda[v-\xi^{-1}(v)]\}/G[\xi^{-1}(v)|I]$ for all $v\in[\underline{v}(I),\overline{v}(I)]$. Making the change of variable $v=\xi(b)$ and noting that $\xi(b)-b=\lambda^{-1}[G(b|I)/(I-1)g(b|I)]$ from the definition of $\xi(\cdot)$, it follows that $\xi^{-1}(\cdot)$ solves (2) with boundary condition $\xi^{-1}(\underline{v}(I))=\underline{v}(I)$. $\square$

Proof of Proposition 1: In view of Lemma 1, it suffices to prove sufficiency. Specifically, it suffices to find a function $\lambda(\cdot)$ satisfying Lemma 1-(ii), where $G(\cdot|\cdot)\in\mathcal{G}_R$. Denote $G(b|I)/[(I-1)g(b|I)]$ by $\psi(b)$. Let $\min_{b\in[\underline{b}(I),\overline{b}(I)]}\psi'(b)=\underline{\psi}'$, which is finite from Definition 3. If $\underline{\psi}'>0$, any strictly increasing function $\lambda(\cdot)$ will satisfy $\xi'(\cdot)\geq 0$, where $\xi(b)=b+\lambda^{-1}(\psi(b))$. It then suffices to take a $\lambda(\cdot)$ function that admits $R+1$ continuous derivatives on $[0,+\infty)$ with $\lambda(0)=0$ and $\lambda'(\cdot)\geq 1$. If $\underline{\psi}'<0$, we must find a strictly increasing and differentiable function $\lambda(\cdot)$ such that $\min_{b\in[\underline{b}(I),\overline{b}(I)]}\{(1/\lambda'[\lambda^{-1}(\psi(b))])\times\psi'(b)\}>-1$ to satisfy $\xi'(\cdot)>0$ on $[\underline{b}(I),\overline{b}(I)]$. To satisfy the latter inequality, it suffices that the function $\lambda(\cdot)$ satisfies $\lambda'(\cdot)\geq 1$ and $\underline{\psi}'\max_{b\in[\underline{b}(I),\overline{b}(I)]}1/\lambda'[\lambda^{-1}(\psi(b))]>-1$, where the latter inequality is equivalent to $\underline{\psi}'>-1/[\max_{b\in[\underline{b}(I),\overline{b}(I)]}1/\lambda'[\lambda^{-1}(\psi(b))]$. But $\max_{b\in[\underline{b}(I),\overline{b}(I)]}1/\lambda'[\lambda^{-1}(\psi(b))]=\max_{x\in[0,\overline{x}]}1/\lambda'(x)$ because $x\equiv\lambda^{-1}(\psi(b))$ takes its value between $\lambda^{-1}[\psi(\underline{b}(I))]=0$ and $\overline{x}=\equiv\lambda^{-1}[\max_{b\in[\underline{b}(I),\overline{b}(I)]}\psi(b(I))]<+\infty$. Moreover, $\max_{x\in[0,\overline{x}]}1/\lambda'(x)=1/\min_{x\in[0,\overline{x}]}\lambda'(x)\equiv 1/\underline{\lambda}'$, where $\underline{\lambda}'\geq 1$. Hence $\underline{\psi}'>-\underline{\lambda}'$, i.e. $0<-\underline{\psi}'<\underline{\lambda}'$. Thus, $\lambda(\cdot)$ must have a sifficiently steep slope. To complete the proof, it suffices to take a $\lambda(\cdot)$ function that admits $R+1$ continuous derivatives on $[0,+\infty)$ with $\lambda(0)=0$.$\square$

Proof of Proposition 2: Let $[U,F]\in\mathcal{U}_R\times\mathcal{F}_R$ with bid distribution $G(\cdot|\cdot)\in\mathcal{G}_R$ by Lemma 1. Let $[\tilde{U},\tilde{F}]$ be such that $\tilde{U}(\cdot)=[U(\cdot/\delta)/U(1/\delta)]^{\delta}$, with $\delta\in(0,1)$ and $\tilde{F}(\cdot|I)$ be the conditional distribution given I of

$$\tilde{\xi}(b,\tilde{U},G,I)=b+\tilde{\lambda}^{-1}\left(\frac{1}{I-1}\frac{G(b|I)}{g(b|I)}\right)=b+\delta\lambda^{-1}\left(\frac{1}{I-1}\frac{G(b|I)}{g(b|I)}\right)=(1-\delta)b+\delta\xi(b,U,G,I),$$

where $b\sim G(\cdot|I)$. It is easy to check that $[\tilde{U},\tilde{F}]\in\mathcal{U}_R\times\mathcal{F}_R$. Because $\tilde{\xi}(\cdot)$ is the weighted sum of two strictly increasing functions in b, then $\tilde{\xi}(\cdot)$ is strictly increasing. Hence, from Lemma 1 the structures $[U, F]$ and $[\tilde{U}, \tilde{F}]$ are observationally equivalent, and $[U, F]$ is not identified. □

Proof of Lemma 2: (i) We first prove that $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$. We have $s_2(\underline{v}) = s_1(\underline{v}) = \underline{v}$. Moreover, from Theorem 1-(ii) we have

$$0 < s_j'(\underline{v}) = \frac{(I_j - 1)\lambda'(0)}{(I_j - 1)\lambda'(0) + 1} = 1 - \frac{1}{(I_j - 1)\lambda'(0) + 1} < 1 \tag{A.1}$$

where $\lambda'(0) \geq 1$. Thus $0 < s_1'(\underline{v}) < s_2'(\underline{v}) < 1$. In particular, by continuity of $s_j(\cdot)$ it follows that $\underline{v} < s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \epsilon)$ for some ϵ satisfying $\underline{v} < \epsilon \leq \overline{v}$. The proof is now by contradiction. Suppose that $s_1(v) \geq s_2(v)$ for some $v \in [\epsilon, \overline{v}]$. Because $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \epsilon)$, the continuity of $s_j(\cdot)$ would imply the existence of some $v_0 \in [\epsilon, \overline{v}]$ such that $s_1(v_0) = s_2(v_0)$. Moreover, for (at least) one of such v_0 denoted v_0^*, the strategy $s_1(\cdot)$ must intersect the strategy $s_2(\cdot)$ from below, i.e. $s_1'(v_o^*) \geq s_2'(v_o^*)$. From (2), we have

$$s_j'(v_0^*) = (I_j - 1)\frac{f(v_0^*)}{F(v_0^*)}\lambda\left(v_0^* - s_j(v_0^*)\right)$$

for $j = 1, 2$, where $f(v_0^*) > 0$ and $F(v_0^*) > 0$ since $v_0^* \in (\underline{v}, \overline{v}]$, while $\lambda(v_0^* - s_j(v_0^*)) > 0$ since $v_0^* > s_j(v_0^*)$ by Theorem 1-(i) and $\lambda(\cdot) > 0$ on $(0, +\infty)$. By construction $s_1(v_0^*) = s_2(v_0^*)$. The previous equation then implies $s_1'(v_0^*) < s_2'(v_0^*)$, contradicting $s_1'(v_0^*) \geq s_2'(v_0^*)$.

(ii) Next, we prove the first inequality, which implies the third inequality after immediate algebra. For each $j = 1, 2$, (2) gives

$$\frac{s_j'(v)}{I_j - 1} = \frac{f(v)}{F(v)}\lambda\left(v - s_j(v)\right) \tag{A.2}$$

for any $v \in [\underline{v}, \overline{v}]$. From (i), $v - s_2(v) < v - s_1(v)$ for any $v \in (\underline{v}, \overline{v}]$. Because $0 < v - s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$ by Theorem 1-(i), and $\lambda(\cdot)$ is strictly increasing with $\lambda(\cdot) > 0$ on $(0, +\infty)$, then $0 < \lambda(v - s_2(v)) < \lambda(v - s_1(v))$ for any $v \in (\underline{v}, \overline{v}]$. Hence, because $f(\cdot) > 0$ and $F(\cdot) > 0$ on $(\underline{v}, \overline{v}]$, it follows from (A.2) that

$$s_2'(v)/(I_2 - 1) < s_1'(v)/(I_1 - 1) \tag{A.3}$$

for any $v \in (\underline{v}, \overline{v}]$.[24] Integrating (A.3) from $\underline{v}$ to $v > \underline{v}$ and using $s_j(\underline{v}) = \underline{b}$ give

$$\frac{s_2(v) - \underline{b}}{I_2 - 1} < \frac{s_1(v) - \underline{b}}{I_1 - 1} \tag{A.4}$$

for any $v \in (\underline{v}, \overline{v}]$. The desired result follows after immediate algebra. □

[24]Equation (A.1) shows that $s_2'(v)/(I_2 - 1) < s_1'(v)/(I_1 - 1)$ also holds at $v = \underline{v}$.

Proof of Corollary 1: The desired result holds when $I = \underline{I}$ and $I = \overline{I}$ by Lemma 2. Let $I \in (\underline{I}, \overline{I})$. With $I = I_1 < I_2$ in (5), the first inequality in (5) gives

$$(I-1)\frac{b_2(\alpha) - \underline{b}}{I_2 - 1} + \underline{b} < b_I(\alpha),$$

for any $\alpha \in (0,1]$ and any $I_2 > I$. Equation (A.4) applied to an arbitrary pair (I_2, I_2') with $I_2 < I_2'$ shows that the LHS in the above inequality is strictly decreasing in I_2. Hence, the most stringent inequality is obtained when I_2 is the smallest, i.e. when $I_2 = I + 1$. Similarly, with $I = I_2 > I_1$ in (5), the third inequality in (5) gives

$$b_I(\alpha) < (I-1)\frac{b_1(\alpha) - \underline{b}}{I_1 - 1} + \underline{b}$$

for any $\alpha \in (0,1]$ and any $I_1 < I$. The RHS in the above inequality is strictly decreasing in I_1 from (A.4). Hence, the most stringent inequality is obtained when I_1 is the largest, i.e. when $I_1 = I - 1$. Combining these two results gives

$$\frac{I-1}{I} b_{I+1}(\alpha) + \frac{1}{I}\underline{b} < b_I(\alpha) < \frac{I-1}{I-2} b_{I-1}(\alpha) - \frac{1}{I-2}\underline{b}, \tag{A.5}$$

for any $\alpha \in (0,1]$. On the other hand, the middle inequality of (5) gives

$$b_{I-1}(\alpha) < b_I(\alpha) < b_{I+1}(\alpha), \tag{A.6}$$

for any $\alpha \in (0,1]$ and any $I \in [\underline{I}, \overline{I}]$. The desired result follows by combining (A.5) and (A.6).

The second part of the corollary follows by noting that b_I is a strictly increasing function of v, namely $b_I = s_I(v)$ fo each I. Hence, the random variables $\max\{b_{I-1}, [(I-1)b_{I+1} + \underline{b}]/I\}$ and $\min\{b_{I+1}, [(I-1)b_{I-1} - \underline{b}]/(I-2)\}$ are also strictly increasing functions of v. It follows that the α-quantiles of their corresponding distributions $\underline{G}_I(\cdot)$ and $\overline{G}_I(\cdot)$ are equal to these functions evaluated at $v(\alpha)$. Thus, they are equal to the first term and third term of the two inequalities displayed in Corollary 1, respectively since $b_I(\alpha) = s_I[v(\alpha)]$. The stochastic dominance assertion then follows from these two inequalities. $\square$

Proof of Proposition 3: From $b_j(\alpha) = s_j[v(\alpha)]$ and (4) evaluated at $v = v(\alpha)$, we obtain the crucial relation

$$v(\alpha) = b_j(\alpha) + \lambda^{-1}\left(\frac{1}{I_j - 1}\frac{\alpha}{g_j[b_j(\alpha)]}\right) \tag{A.7}$$

for $j = 1, 2$ and any $\alpha \in [0,1]$. Hence, using (6) we obtain the nonlinear relation

$$\lambda^{-1}[R_1(\alpha)] = \lambda^{-1}[R_2(\alpha)] + \Delta b(\alpha) \tag{A.8}$$

for any $\alpha \in [0,1]$. For future use, we note that $\Delta b(0) = 0$ as $s_1(\underline{v}) = s_2(\underline{v}) = \underline{b}$. Moreover, from Lemma 2, we know that $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$, which implies $b_1(\alpha) < b_2(\alpha)$ for any $\alpha \in (0,1]$. Hence, $\Delta b(\alpha) > 0$ for any $\alpha \in (0,1]$. Because $\lambda^{-1}(\cdot)$ is strictly increasing and $R_j(\alpha) > 0$ for any $\alpha \in (0,1]$, it follows from (A.8) that $R_1(\alpha) > R_2(\alpha) > 0$ for any $\alpha \in (0,1]$. In particular, because $R_j(\cdot)$ is continuous on $[0,1]$ and $R_j(0) = 0$ for $j = 1,2$, the range $\mathcal{R}_j$ of $R_j(\cdot)$ must be of the form $[0, \overline{r}_j]$ with $0 < \overline{r}_j < \infty$ and $\overline{r}_1 > \overline{r}_2$, as claimed in the text.

Now, by assumption u_0 belongs to $\mathcal{R}_1 = [0, \overline{r}_1]$. If $u_0 = 0$, then $\lambda^{-1}(0) = 0$. Next, consider the general case $u_0 \in (0, \overline{r}_1]$. Thus, there exists some $\alpha_0 \in (0,1]$ such that $u_0 = R_1(\alpha_0)$. In particular, we have $u_0 = R_1(\alpha_0) > R_2(\alpha_0) > 0 = R_1(0)$ because $R_1(\cdot) > R_2(\cdot) > 0$ on $(0,1]$. Moreover, because $R_1(\cdot)$ is continuous on $[0,1]$, there exists some α_1 satisfying $\alpha_0 > \alpha_1 > 0$ and $R_1(\alpha_1) = R_2(\alpha_0)$. Continuing such a construction, we have $R_1(\alpha_1) > R_2(\alpha_1) > 0 = R_1(0)$, which implies that there exists some α_2 satisfying $\alpha_1 > \alpha_2 > 0$ and $R_1(\alpha_2) = R_2(\alpha_1)$. Thus, we have constructed a sequence, which is not necessarily unique such that $1 \geq \alpha_0 > \alpha_1 > \ldots > \alpha_t > \ldots > 0$ with $u_0 = R_1(\alpha_0) > R_2(\alpha_0) = R_1(\alpha_1) > R_2(\alpha_1) = R_1(\alpha_2) > \ldots > R_2(\alpha_{t-1}) = R_1(\alpha_t) > \ldots > 0$, as indicated in the text.[25] Because the sequence $\{\alpha_t\}$ is strictly decreasing and is in $(0,1]$, it must converge to some finite limit $\alpha_\infty \in [0,1]$. Because $R_j(\cdot)$ is continuous on $[0,1]$, then $\lim_{t\to+\infty} R_j(\alpha_t) = R_j(\alpha_\infty)$ for $j = 1,2$. But $R_2(\alpha_{t-1}) = R_1(\alpha_t)$ by construction, implying that $R_2(\alpha_\infty) = R_1(\alpha_\infty)$. Because $R_2(\alpha) = R_1(\alpha)$ only for $\alpha = 0$, this implies that $\alpha_\infty = 0$, and consequently $\lim_{t\to+\infty} R_j(\alpha_t) = 0$ for $j = 1,2$.

We now iterate (A.8). Specifically, for any $u_0 \in \mathcal{R}_1 \setminus \{0\}$ and any corresponding sequence $\{\alpha_t\}$ as constructed above, we must have the nonlinear dynamic relation

$$\begin{aligned}
\lambda^{-1}(u_0) &= \lambda^{-1}[R_2(\alpha_0)] + \Delta b(\alpha_0) \\
&= \lambda^{-1}[R_1(\alpha_1)] + \Delta b(\alpha_0) \\
&= \lambda^{-1}[R_2(\alpha_1)] + \Delta b(\alpha_0) + \Delta b(\alpha_1) \\
&\vdots \\
&= \lambda^{-1}[R_2(\alpha_t)] + \Delta b(\alpha_0) + \ldots \Delta b(\alpha_t).
\end{aligned}$$

See Figure 1 for an illustration. Because $\lambda^{-1}(\cdot)$ is continuous on $[0, +\infty)$ with $\lambda^{-1}(0) = 0$ and $\lim_{t\to+\infty} R_2(\alpha_t) = 0$, as shown above, then $\lim_{t\to+\infty} \lambda^{-1}[R_2(\alpha_t)] = 0$. Because $\lambda^{-1}(u_0)$ is finite, it follows from the above equation that $\lim_{t\to+\infty} \sum_{\tau=0}^{t} \Delta b(\alpha_\tau)$ must exist and that it is equal

[25]When $R_1(\cdot)$ is strictly increasing, or equivalently by (4) when the bidder's rent is strictly increasing in v, then $\alpha_0 = R_1^{-1}(u_0)$ and $\alpha_t = [R_1^{-1} \circ R_2]^t(\alpha_0)$, for $t = 1, 2, \ldots$, where $\circ$ denotes the composition of two functions and $[R_1^{-1} \circ R_2]^t$ denotes the t-composition of $R_1^{-1} \circ R_2$. Thus the sequence $\{\alpha_t\}$ is unique.

to $\lambda^{-1}(u_0)$, i.e. $\lambda^{-1}(u_0) = \sum_{\tau=0}^{+\infty} \Delta b(\alpha_\tau)$ as desired. Note that this must be so irrespective of the sequence $\{\alpha_t\}$, whether such a sequence is unique. Moreover, because $\Delta b(\alpha_\tau)$ depends only on $b_j(\cdot)$ and $R_j(\cdot)$, which depend only on the distributions $G_j(\cdot)$, the latter equality shows that $\lambda^{-1}(\cdot)$ is identified nonparametrically on $\mathcal{R}_1$ from observed equilibrium bids.

The nonparametric identification of $F(\cdot)$ follows immediately from $F(\cdot) = G_j[\xi_j^{-1}(\cdot)]$. For, the nonparametric identification of $\lambda^{-1}(\cdot)$ on $\mathcal{R}_1$ and hence on $\mathcal{R}_2 \subset \mathcal{R}_1$ implies the nonparametric identification of $\xi_j(\cdot)$ on $[\underline{b}, \overline{b}_j]$ for $j = 1, 2$ by (4) and (6). The latter implies the nonparametric identification of $\xi_j^{-1}(\cdot) = s_j(\cdot)$ on $[\underline{v}, \overline{v}]$. Alternatively, pick an arbitrary $\alpha_0 \in [0, 1]$. From (6) and (A.7) for (say) $j = 1$, we have $v(\alpha_0) = b_1(\alpha_0) + \lambda^{-1}[R_1(\alpha_0)]$. Thus, the above explicit expression for $\lambda^{-1}(u_0)$ with $u_0 = R_1(\alpha_0)$ gives

$$v(\alpha_0) = b_1(\alpha_0) + \sum_{t=0}^{+\infty} \Delta b(\alpha_t) \tag{A.9}$$

showing that the α_0-quantile of $F(\cdot)$ is identified. Because α_0 is arbitrary in $[0, 1]$, it follows that $F(\cdot)$ is identified on $[\underline{v}, \overline{v}]$. □

Proof of Lemma 3: First, we prove that (i), (ii) and (iii) are necessary. We use a double index (i, j) with i indexing bidder i among the I_j bidders and $j = 1, 2$ indicating the level of competition. Because $b_{ij} = s_j(v_i, U, F, I_j)$ and the v_{ij}s are i.i.d., the b_{ij}s are also i.i.d. given $I_j, j = 1, 2$. The fact that $G_j(\cdot) \in \mathcal{G}_R, j = 1, 2$ follows from Lemma 1. This establishes (i). Because $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$ from Lemma 2 and noting that $b_j = s_j(v)$ with $s_j(\cdot)$ strictly increasing, it follows that $b_j(\alpha) = s_j[v(\alpha)]$. Hence, $b_1(\alpha) < b_2(\alpha)$ for any $\alpha \in (0, 1]$ or equivalently $G_1(\cdot) \prec_{\underline{b}} G_2(\cdot)$. This establishes (ii). Lastly, because $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ and $U(\cdot)$ satisfies Definition 1, then $\lambda(\cdot)$ is defined from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\lambda(0) = 0$, $\lambda'(\cdot) \geq 1$ and $\lambda(\cdot)$ is continuously differentiable on $[0, \infty)$. Because $F(\cdot)$ is invariant in I, its quantiles are also invariant in I. Thus, considering (4) for two values I_1 and I_2 at any α-quantile with $\alpha \in [0, 1]$ and $I_1 \neq I_2$ leads to (7). It remains to show that $\xi_j'(\cdot) > 0, j = 1, 2$. The equilibrium strategy $s_j(\cdot)$ must satisfy $\xi_j[s_j(v), U, G, I_j] = v$ for any $v \in [\underline{v}, \overline{v}]$ and $j = 1, 2$. We then obtain $\xi_j(b, U, G, I_j) = s_j^{-1}(b, U, F, I)$. This implies $\xi_j'(\cdot) = [s^{-1}(\cdot)]' > 0$. This establishes (iii).

Conversely, we show that (i), (ii) and (iii) are together sufficient. We construct a pair $[U, F]$ that satisfies Definitions 1 and 2 and is independent of I. Let $U(\cdot)$ be such that $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ or $1/\lambda(\cdot) = U'(\cdot)/U(\cdot)$. Integration of the latter with the normalization $U(1) = 1$ gives $U(x) = \exp \int_1^x 1/\lambda(t)dt$. We need to verify that $U(\cdot)$ satisfies Definition 1. This follows from the proof of Lemma 1. Let $F_j(\cdot)$ be the distribution of $X_j = b + \lambda^{-1}[G_j(b)/((I_j - 1)g_j(b))]$ given I_j, where $b \sim G_j(\cdot), j = 1, 2$. Note that $F_j(\cdot)$ satisfies Definition 2 by the proof of Lemma

1. Moreover, because the compatibility condition is satisfied for any $\alpha \in [0,1]$, it implies that the corresponding α-quantile of $F_j(\cdot)$ does not depend on I_j. Hence $F_1(\cdot) = F_2(\cdot) \equiv F(\cdot)$, which thereby satisfies Definition 2. Lastly, we show that the pair $[U, F]$ can be rationalized by $G_1(\cdot)$ and $G_2(\cdot)$ with $I_2 > I_1$, i.e. that $G_j(\cdot) = F[s_j^{-1}(\cdot, U, F, I_j)], j = 1, 2$, where $s_j(\cdot, U, F, I_j)$ solves the first-order differential equation defining the equilibrium strategy with the boundary condition $s_j(\underline{v}, U, F, I_j) = \underline{v}$. By construction, $G_j(\cdot) = F[\xi_j(\cdot)]$. Thus it suffices to show that $\xi_j^{-1}(\cdot), j = 1, 2$ solves the differential equation (2). This proof can be found in Lemma 1, which shows that $\xi_j^{-1}(\cdot), j =, 1, 2$ solves the differential equation with $I = I_j$ under the boundary condition $\xi_j^{-1}(\underline{v}) = \underline{v}$. □

Proof of Proposition 7: We prove the first part only. Let $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$. This structure generates $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$ whose marginal distributions satisfy Definition 3 and the compatibility condition (13). We show that there exists another structure $[\tilde{\mathbf{U}}, \tilde{F}] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ rationalizing $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$. The proof is in four steps and is done for every fixed $I \in \mathcal{I}$.

Step 1: *Construction of* $[\tilde{U}_1, \ldots, \tilde{U}_I, \tilde{F}(\cdot|I)]$. Let $\tilde{U}_1(\cdot) = [U_1(\cdot/\delta)/U_1(1/\delta)]^\delta$ with $\delta \in (0,1)$. Thus, $\tilde{\lambda}_1(\cdot) = \lambda_1(\cdot/\delta)$ and $\tilde{\lambda}_1^{-1}(\cdot) = \delta\lambda_1^{-1}(\cdot)$. For $i = 2, \ldots, I$, let $\tilde{U}_i(x) = \exp\left[\int_1^x 1/\tilde{\lambda}_i(t)dt\right]$ so that $\tilde{\lambda}_i(\cdot) = \tilde{U}_i(\cdot)/\tilde{U}_i'(\cdot)$, where $\tilde{\lambda}_i(\cdot)$ is such that $\tilde{\lambda}_i^{-1}\left[1/H_i(b_{i\alpha}|I)\right] = \tilde{\lambda}_1^{-1}\left[1/H_1(b_{1\alpha}|I)\right] + b_{1\alpha} - b_{i\alpha}$, for all $\alpha \in [0,1]$. The latter well-defines $\tilde{\lambda}_i^{-1}(\cdot)$ because $1/H_i(b_{i\alpha}|I)$ strictly increases as α increases given $H_i'(\cdot|I) < 0$ by assumption. Moreover, $\tilde{\lambda}_i(\cdot)$ is strictly increasing as shown in Step 3. Note that the compatibility condition (13) is satisfied by construction. We then let $\tilde{F}(\cdot|I)$ be the conditional distribution given I of $\tilde{v}_i \equiv b_i + \tilde{\lambda}_i^{-1}[1/H_i(b_i|I)] \equiv \tilde{\xi}_i(b_i)$ for an arbitrary i, where $b_i \sim G_i(\cdot|I)$. Using $\tilde{\lambda}_1^{-1}(\cdot) = \delta\lambda_1^{-1}(\cdot)$, we obtain $\tilde{\lambda}_i^{-1}\left[1/H_i(b_{i\alpha}|I)\right] = \delta\lambda_1^{-1}\left[1/H_1(b_{1\alpha}|I)\right] + b_{1\alpha} - b_{i\alpha}$. Thus, (13) with $j = 1$ gives

$$\tilde{\lambda}_i^{-1}\left(\frac{1}{H_i(b_{i\alpha}|I)}\right) = \delta\lambda_i^{-1}\left(\frac{1}{H_i(b_{i\alpha}|I)}\right) + (1-\delta)(b_{1\alpha} - b_{i\alpha}). \tag{A.10}$$

Equivalently, $\tilde{\lambda}_i^{-1}[1/H_i(b_{i\alpha}|I)] = \lambda_i^{-1}[1/H_i(b_{i\alpha}|I)] - (1-\delta)\lambda_1^{-1}[1/H_1(b_{1\alpha}|I)]$. In particular, since $\lambda_i^{-1}(\cdot)$ is bidder's i shading, the shading under $[\tilde{U}_1, \ldots, \tilde{U}_I, \tilde{F}]$ is smaller than under $[U_1, \ldots, U_I, F]$, i.e. bidders bid more aggressively under the former than under the latter.

Step 2: $\tilde{\lambda}_i(0) = 0$ *and* $\tilde{\xi}_i'(\cdot) > 0$ *on* $[\underline{b}, \overline{b}]$. Because $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ so that $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$, we have $\lambda_i^{-1}(0) = 0$ and $\lim_{b \downarrow \underline{b}} 1/H_i(b|I) = 0$ for $I \in \mathcal{I}$. Thus, (A.10) with the boundary conditions $\underline{b}_1 = \ldots = \underline{b}_I \equiv \underline{b} = \underline{v}$ imply $\tilde{\lambda}_i^{-1}(0) = 0$ and hence $\tilde{\lambda}_i(0) = 0$. Regarding $\tilde{\xi}_i'(\cdot) > 0$, we note that $\tilde{\xi}_i(b_{i\alpha}) = (1-\delta)b_{1\alpha} + \delta\xi_i(b_{i\alpha})$ from (A.10) and (12). Noting that $b_{1\alpha} = G_1^{-1}[G_i(b_{i\alpha})] \equiv B_i(b_{i\alpha})$ and letting $b_{i\alpha} = b$, we obtain $\tilde{\xi}_i'(b) = (1-\delta)B_i'(b) + \delta\xi_i'(b)$, where $B_i'(b) = g_i(b)/g_1[B(b)]$. Hence, $\tilde{\xi}_i'(b) > 0$ since $B_i'(b) > 0$ and $\xi_i'(b) > 0$.

STEP 3: $\tilde{\lambda}_i'(\cdot) \geq 1$. From (A.10) and (12), $\tilde{\lambda}_i^{-1}[1/H_i(b_{i\alpha}|I)] = \delta\xi_i(b_{i\alpha}) + (1-\delta)b_{1\alpha} - b_{i\alpha}$, i.e. $1/H_i(b_{i\alpha}|I) = \tilde{\lambda}_i[\delta\xi_i(b_{i\alpha}) + (1-\delta)b_{1\alpha} - b_{i\alpha}]$. From the structure $[\mathbf{U}, F]$, we have $1/H_i(b_{i\alpha}|I) = \lambda_i[\xi_i(b_{i\alpha}) - b_{i\alpha}]$. Thus, $\lambda_i[\xi_i(b_{i\alpha}) - b_{i\alpha}] = \tilde{\lambda}_i[\delta\xi_i(b_{i\alpha}) + (1-\delta)b_{1\alpha} - b_{i\alpha}]$. Differentiating with respect to $b = b_{i\alpha}$ and noting that $b_{1\alpha} = G_1^{-1}[G_i(b_{i\alpha})] \equiv B_i(b_{i\alpha})$ gives

$$\tilde{\lambda}_i'(**) = \frac{\xi_i'(b) - 1}{\delta\xi_i'(b) + (1-\delta)B_i'(b) - 1}\lambda_i'(*) \equiv R_i(b)\lambda_i'(*), \tag{A.11}$$

where the different arguments of $\lambda_i'(\cdot)$ and $\tilde{\lambda}_i'(\cdot)$ are indicated by * and **, respectively. Thus, it suffices to show that $R_i(\cdot) \geq 1$ since $\lambda_i'(\cdot) \geq 1$. We note that $\xi_1(b_{1\alpha}) = \xi_i(b_{i\alpha}) = v_\alpha$ for all $\alpha \in [0,1]$ from the compatibility condition. Using $b_{1\alpha} = B_i(b_{i\alpha})$, this gives $\xi_1[B_i(b)] = \xi_i(b)$ for all $b \in [\underline{b}, \overline{b}]$. Differentiating gives $\xi_1'[B_i(b)]B_i'(b) = \xi_i'(b)$, i.e. $B_i'(b) = \xi_i'(b)/\xi_1'[B_i(b)]$. Hence,

$$R_i(b) = \frac{\xi_i'(b) - 1}{\delta\xi_i'(b) - 1 + (1-\delta)\frac{\xi_i'(b)}{\xi_1'[B_i(b)]}} = 1 + \frac{(1-\delta)\xi_i'(b)\{\xi_1'[B_i(b)] - 1\}}{\delta\xi_i'(b)\{\xi_1'[B_i(b)] - 1\} - \{\xi_1'[B_i(b)] - \xi_i'(b)\}}, \tag{A.12}$$

for $b \in [\underline{b}, \overline{b}]$. Note that $\xi_i'(\cdot) > 1$ on $(\underline{b}, \overline{b}]$ for every $i = 1, \ldots, I$ since differentiating (12) gives $\xi_i'(b) = 1 - \lambda_i^{-1'}[1/H_i(b|I)][H_i'(b|I)/H_i^2(b|I)]$, where $\lambda_i^{-1'}(\cdot) > 0$ and $H_i'(\cdot|I) < 0$ by assumption. Hence, $\xi_i'(\cdot) \geq 1$ on $[\underline{b}, \overline{b}]$ by continuity. Since $1 - \delta > 0$ and $\xi_i'(\cdot) > 0$, it suffices to show that the denominator $D_i(b)$ (say) in the RHS is strictly positive for all $b \in [\underline{b}, \overline{b}]$ and some $\delta \in [\delta^*, 1]$.

To study the sign of $D_i(\cdot)$ on $[\underline{b}, \overline{b}]$, we note that

$$\frac{g_j(b)}{G_j(b)} = \frac{g_j(\underline{b}) + o(1)}{g_j(\underline{b})(b - \underline{b}) + o(b - \underline{b})} = \frac{1}{b - \underline{b}}\frac{g_j(\underline{b}) + o(1)}{g_j(\underline{b}) + o(1)} = \frac{1}{b - \underline{b}}(1 + o(1)).$$

Thus, a Taylor expansion of $1 = \lambda_i[\xi_i(b) - b]\sum_{j \neq i}[g_j(b)/G_j(b)]$ from (12) gives

$$1 = \{\lambda_i'(0)[\xi_i'(\underline{b}) - 1](b - \underline{b}) + o(b - \underline{b})\}\frac{I-1}{b - \underline{b}}(1 + o(1)) = \{\lambda_i'(0)[\xi_i'(\underline{b}) - 1](I-1)\} + o(1).$$

Hence, $\xi_i'(\underline{b}) = 1 + \{1/[(I-1)\lambda_i'(0)]\} > 1$ as $\lambda_i'(\cdot) \geq 1$. Thus, because $\xi_i'(\cdot) > 1$ on $[\underline{b}, \overline{b}]$, then $D_i(\cdot) > 0$ on $[\underline{b}, \overline{b}]$ if and only if $\delta > \delta^* \equiv \max_{b \in [\underline{b}, \overline{b}]} \underline{\delta}(b)$, where $\underline{\delta}(\cdot)$ is continuous on $[\underline{b}, \overline{b}]$ with

$$\underline{\delta}(b) \equiv \frac{\xi_1'[B_i(b)] - \xi_i'(b)}{\xi_i'(b)\{\xi_1'[B_i(b)] - 1\}} = \frac{1}{\xi_i'(b)}\left[1 - \frac{\xi_i'(b) - 1}{\xi_1'[B_i(b)] - 1}\right].$$

It remains to show that $\underline{\delta}(\cdot) < 1$ on $[\underline{b}, \overline{b}]$ so that $\delta^* < 1$. Clearly, $\underline{\delta}(\cdot) < 1$ on $(\underline{b}, \overline{b}]$ as $\xi_i'(\cdot) > 1$ on $(\underline{b}, \overline{b}]$. Moreover, at $b = \underline{b}$, we have $\underline{\delta}(\underline{b}) = [1/\xi_i'(\underline{b})]\{1 - [\lambda_i'(0)/\lambda_1'(0)]\} < 1$.

STEP 4: $[\tilde{\mathbf{U}}, \tilde{F}] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$. From the previous steps and the rationalization result given after (13), it follows that $[\tilde{\mathbf{U}}, F]$ rationalizes $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$. It remains to show that $[\tilde{\mathbf{U}}, \tilde{F}] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$. From the proof of Lemma 1, it sufffices to show that $\tilde{\lambda}_i(\cdot)$ is $R+1$ continuously differentiable on $[0, \infty)$ for $i = 1, \ldots, I$. This follows from (A.11)–(A.12) and the $R+1$ continuous differentiablity of $\lambda_i(\cdot)$ and $\xi_i(\cdot)$ as $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$. $\square$

Appendix B

This appendix contains the proof of Theorem 1. As our argument is done for every $I \in \mathcal{I}$, hereafter we omit the dependence on I. Theorem 1 follows from Theorems B1 and B2, where $b(\alpha) = s[v(\alpha)]$ is the α-bid quantile function with $\alpha \in [0,1]$, $v(\alpha)$ is the α-quantile of $F(\cdot)$ and $s(\cdot)$ is a solution of (2) which must be strictly increasing by Lemma B1 below. Since $F[v(\alpha)] = \alpha$ implies $v'(\cdot) = 1/f[v(\cdot)]$, we have $b'(\alpha) = s'(v(\alpha))/f(v(\alpha))$. Hence, from (2), the bid quantile function $b(\cdot)$ must solve

$$b'(\alpha) = \frac{I-1}{\alpha}\lambda\left[v(\alpha) - b(\alpha)\right] \text{ for } \alpha \in (0,1] \text{ with } b(0) = v(0), \tag{B.1}$$

where $\lambda(\cdot)$ is $R+1$ continuously differentiable with $\lambda'(\cdot) \geq 1$ on $[0,\infty)$ and $v(\cdot)$ is $R+1$ continuously differentiable with $v'(\cdot) > 0$ on $[0,1]$ as $U(\cdot) \in \mathcal{U}_R$ and $F(\cdot) \in \mathcal{F}_R$. Note that (B.1) is ill-conditioned at $\alpha = 0$. As for (2), the solutions of (B.1) are not explicit except for simple utility functions such as CRRA. Specifically, when $U(x) = x^{1-c}/[1-c]$ for $0 \leq c < 1$, it is well-known that the equilibrium strategy exists and is unique so that the solution of (B.1) exists and is unique, namely,

$$b(\alpha) = \frac{I-1}{(1-c)\alpha^{\frac{I-1}{1-c}}} \int_0^\alpha r^{\frac{I-1}{1-c}-1} v(r) dr\ . \tag{B.2}$$

Moreover, following the proof of Lemma A2 in Guerre, Perrigne and Vuong (2000), the equilibrium strategy in the CRRA case and hence the bid quantile function $b(\cdot) = s[v(\cdot)]$ are $R+1$ continuously differentiable on $[v(0), v(1)] = [\underline{v}, \overline{v}]$ and $[0,1]$, respectively.

We now define our flow of differential equations $\{E(B;t) = 0; t \in [0,1]\}$. For $t \in (0,1]$, let

$$\begin{aligned} \Lambda(x;t) &= \frac{\lambda(tx)}{t} \text{ for } x \in I\!R_+, \quad \Lambda(x;t) = \lambda'(0)x \text{ for } x \in I\!R_-, \\ V(\alpha;t) &= v(0) + \frac{v(\alpha t) - v(0)}{t} \text{ for } \alpha \in [0,1]. \end{aligned}$$

These two functions are extended at $t = 0$ by considering their limits as $t \downarrow 0$, namely, $\Lambda(x;0) = \lambda'(0)x$ for $x \in I\!R$, and $V(\alpha;0) = v(0) + v'(0)\alpha$ for $\alpha \in [0,1]$. For every $t \in [0,1]$, note that $\Lambda(\cdot;t)$ and $V(\cdot;t)$ correspond to a utility function $U(x;t) = \exp(\int_0^x [1/\Lambda(u;t)]du) \in \mathcal{U}_R$ and a private value distribution $F(\cdot|\cdot;t) \in \mathcal{F}_R$, respectively. The flow of differential equations $\{E(B;t) = 0; t \in [0,1]\}$ is then defined by

$$B'(\alpha;t) = \frac{I-1}{\alpha}\Lambda(V(\alpha;t) - B(\alpha;t);t) \text{ for } \alpha \in (0,1] \text{ with } B(0;t) = v(0), \tag{B.3}$$

which is analogous to (B.1). Note that $E(B;0) = 0$ is

$$B'(\alpha;0) = \frac{(I-1)\lambda'(0)}{\alpha}[v(0) + v'(0)\alpha - B(\alpha;0)] \text{ for } \alpha \in (0,1] \text{ with } B(0;0) = v(0),$$

which corresponds to (B.1) for a CRRA utility function with parameter $0 \leq c = 1 - 1/\lambda'(0) < 1$ as $\lambda'(\cdot) \geq 1$, and a uniform private value distribution on $[v(0), v(0) + v'(0)]$. In particular, a key property is that $E(B; 0) = 0$ is known to admit a unique solution, namely

$$B(\alpha; 0) = v(0) + \frac{(I-1)\lambda'(0)}{(I-1)\lambda'(0) + 1} v'(0)\alpha$$

from (B.2). On the other hand, solving $E(B; 1) = 0$ is equivalent to solving (B.1) since $\Lambda(x; 1) = \lambda(x)$ and $V(\alpha; 1) = v(\alpha)$. Thus, the flow of differential equations $\{E_I(B; t) = 0; t \in [0, 1]\}$ is a path between $E(B; 0) = 0$ and $E(B; 1) = 0$.

The existence and uniqueness of the solution to $E(B; 1) = 0$ can be inferred from the existence and uniqueness of the solution to $E(B; 0)$ by a Continuation Argument given by Proposition 6.10 in Zeidler (1985) and reproduced below as Theorem Z1. Roughly this argument says that $E(B; 1) = 0$ admits a unique solution if $E(B; 0) = 0$ does under some regularity conditions on the functional operator associated with the differential equation $E(B; t) = 0$ and a socalled a priori condition defining the set of functions containing the potential solutions of $E(B; t) = 0$. This gives us

Theorem B1: *If $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, then for every $I \in \mathcal{I}$*
(i) the differential equation (B.1) has a unique solution $b(\cdot)$, which is strictly increasing and continuously differentiable over $[0, 1]$ with $b(\alpha) < v(\alpha)$ for all $\alpha \in (0, 1]$,
(ii) $s(\cdot) = b(F(\cdot))$ is the unique solution of the differential equation (2) with initial condition $s(v(0)) = v(0)$. Moreover, this solution is strictly increasing and continuously differentiable on $[v(0), v(1)]$, with $s(v) < v$ for all $v \in (v(0), v(1)]$, $s'(v) > 0$ for all $v \in [v(0), v(1)]$ and $s'(v(0)) = (I-1)\lambda'(0)/[(I-1)\lambda'(0) + 1] < 1$.

A main advantage of our functional approach is that it also delivers the smoothness of the equilibrium strategy. As above, we first study the differentiability of the bid quantile function $b(\cdot)$ on $[0, 1]$ building on an Implicit Functional Theorem 4.B in Zeidler (1985) and reproduced below as Theorem Z2. This theorem is applied to the flow of differential equations $\{E(B; t) = 0; t \in [0, 1]\}$.

Theorem B2: *If $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, then for every $I \in \mathcal{I}$*
(i) the unique solution $b(\cdot)$ of (B.1) admits $R + 1$ continuous partial derivatives on $[0, 1]$, while $b'(\alpha)$ has $R + 1$ continuous partial derivatives on $(0, 1]$,
(ii) the unique solution $s(\cdot)$ of the differential equation (2) with initial condition $s(v(0)) = v(0)$ admits $R + 1$ continuous partial derivatives on $[v(0), v(1)]$.

To prove Theorems B1–B2 requires to establish some properties so as to satisfy the conditions of the Continuation Argument Theorem and the Implicit Functional Theorem. These properties follow from the next series of lemmas and corollaries, most of which are used to check the conditions of either theorem. In what follows, $\pi^{(k)}(\alpha;t)$, $V^{(k)}(\alpha;t)$ and $\Lambda^{(k)}(x;t)$ denote the kth derivatives of $\pi(\alpha;t)$, $V(\alpha;t)$ and $\Lambda(x;t)$ with respect to α, α and x, respectively.

We first establish some properties that potential solutions to (2) must satisfy.

Lemma B1: *If $[U,F] \in \mathcal{U}_R \times \mathcal{F}_R$, then for every $I \in \mathcal{I}$ solutions $s(\cdot)$ of (2) with boundary condition $s(v(0)) = v(0)$, if any, are such that*
(i) $s(\cdot)$ is continuously differentiable on $[v(0), v(1)]$,
(ii) $s(v) < v$ for all $v \in (v(0), v(1)]$ and $s'(v) > 0$ for all $v \in [v(0), v(1)]$ with $s'(v(0)) = (I-1)\lambda'(0)/[(I-1)\lambda'(0)+1] < 1$.

Proof of Lemma B1: Fix $I \in \mathcal{I}$. Let $\tilde{\lambda}(x) = \lambda(x)$ for $x \geq 0$, and $\tilde{\lambda}(x) = \lambda'(0)x$ for $x < 0$. Note that $\tilde{\lambda}(\cdot)$ is strictly increasing and continuously differentiable over $I\!R$ because $\lambda'(\cdot) \geq 1$ on $I\!R_+$. We establish (i) and (ii) for the potential solutions of the "extended" differential equation

$$s'(v) = (I-1)\frac{f(v)}{F(v)}\tilde{\lambda}\left(v - s(v)\right) \text{ for } v \in (v(0), v(1)] \text{ with } s(v(0)) = v(0). \tag{B.4}$$

Since $\lambda(\cdot)$ and $\tilde{\lambda}(\cdot)$ coincide over $I\!R_+$, a solution of (2) with $s(v(0)) = v(0)$ is also a solution of (B.4). Conversely, a solution of (B.4) satisfying $s(v) < v$ for all $v \in (v(0), v(1)]$ is a solution of (2) with $s(v(0)) = v(0)$. In Step 2 we show that potential solutions of (B.4) must satisfy $s(v) < v$ for all $v \in (v(0), v(1)]$. Hence, $s(\cdot)$ is a solution of (2) with $s(v(0)) = v(0)$ if and only if it is a solution of (B.4). The desired result then follows.

STEP 1: *Proof of (i) for solutions of (B.4).* Solutions $s(\cdot)$ of (B.4) are continuous on $[v(0), v(1)]$ and continuously differentiable on $(v(0), v(1)]$. Thus, it suffices to show the existence of $s'(v(0))$ with $\lim_{v\downarrow v(0)} s'(v) = s'(v(0))$. For $v \in (v(0), v(1)]$, let

$$\Psi(v) = (I-1)\frac{f(v)(v-v(0))}{F(v)}\frac{\tilde{\lambda}(v-s(v))}{v-s(v)}, \qquad r(v) = \exp\left(-\int_v^{v(1)} \frac{\Psi(u)}{u-v(0)}du\right).$$

Also, let $\Psi(v(0)) = (I-1)\lambda'(0)$ and $r(v(0)) = 0$. Thus, $\Psi(\cdot)$ is continuous and strictly positive on $[v(0), v(1)]$ since $[U,F] \in \mathcal{U}_R \times \mathcal{F}_R$. Hence, $0 < r(\cdot) < 1$ on $(v(0), v(1)]$. Moreover, $\frac{\Psi(v)}{v-v(0)} = \frac{\Psi(0)+o(1)}{v-v(0)}$ when $v \downarrow v(0)$. Thus, $\lim_{v\downarrow v(0)} r(v) = 0$ and $r(\cdot)$ is continuous on $[v(0), v(1)]$.

Now, (B.4) can be written as $s'(v) = \Psi(v)\frac{v-s(v)}{v-v(0)}$ for $v \in (v(0), v(1)]$ with $s(v(0)) = v(0)$, i.e.

$$(v-v(0))s'(v) + \Psi(v)(s(v)-v(0)) = \Psi(v)(v-v(0)) \text{ for } v \in (v(0), v(1)] \text{ with } s(v(0)) = v(0). \tag{B.5}$$

Letting $C(v) = r(v)[s(v) - v(0)]$ yields, for $v \in (v(0), v(1)]$, $C'(v) = r(v)\Psi(v)\frac{s(v)-v(0)}{v-v(0)} + r(v)s'(v)$, so that (B.5) gives $C'(v) = r(v)\Psi(v)$. Thus, $C(v) = C_0 + \int_{v(0)}^{v} r(u)\Psi(u)du$, where $C_0 = 0$ because $C(v(0)) = 0$. Hence, the potential solutions of (B.4) satisfy

$$s(v) = v(0) + \int_{v(0)}^{v} \frac{r(u)}{r(v)} \Psi(u) du \; .$$

But for $v(0) < u \leq v \leq v(1)$,

$$\begin{aligned} \frac{r(u)}{r(v)} &= \exp\left(-\int_{u}^{v} \frac{\Psi(v(0)) + o(1)}{x - v(0)} dx\right) = \exp\left(-[\Psi(v(0)) + o(1)] \ln \frac{v - v(0)}{u - v(0)}\right) \\ &= \left(\frac{u - v(0)}{v - v(0)}\right)^{[\Psi(v(0)) + o(1)]} . \end{aligned}$$

It follows that

$$\begin{aligned} s(v) - v(0) &= \int_{v(0)}^{v} \left(\frac{u - v(0)}{v - v(0)}\right)^{(\Psi(v(0)) + o(1))} [\Psi(v(0)) + o(1)] \, du \\ &= \frac{\Psi(v(0))}{\Psi(v(0)) + 1} (v - v(0)) (1 + o(1)) \; , \end{aligned}$$

showing that $s(v(0)) = v(0)$ as desired. Moreover, $s(\cdot)$ is differentiable at $v(0)$ with $s'(v(0)) = (I-1)\lambda'(0)/[(I-1)\lambda'(0)+1]$ using $\Psi(v(0)) = (I-1)\lambda'(0)$. On the other hand, $s'(v) = \Psi(v)\frac{v-s(v)}{v-v(0)}$ for $v > v(0)$ gives

$$\lim_{v \downarrow v(0)} s'(v) = \lim_{v \downarrow v(0)} \Psi(v) \left(1 - \frac{s(v) - v(0)}{v - v(0)}\right) = \Psi(v(0)) \left(1 - s'(v(0))\right) = s'(v(0))$$

as desired.

Step 2: *Proof of (ii) for solutions of (B.4).* We first prove that $s(v) < v$ for $v \in (v(0), v(1)]$ by contradiction. Observe that $0 < s'(v(0)) < 1$. It follows that $s(v) < v$ for $v > v(0)$ close enough to $v(0)$. Suppose that there is a v^* in $(v(0), v(1)]$ such that $s(v) < v$ for $v \in (v(0), v^*)$ and $s(v^*) = v^*$, so that $s'(v^*) = 0$ by (B.4). Since $R \geq 1$, differentiating (B.4) at v^* yields

$$\frac{s''(v^*)}{I-1} = \frac{\partial}{\partial v}\left(\frac{f(v^*)}{F(v^*)}\right) \lambda(v^* - s(v^*)) + \frac{f(v^*)}{F(v^*)} \lambda'(v^* - s(v*)) \left(1 - s'(v^*)\right) = \frac{f(v^*)}{F(v^*)} \lambda'(0) > 0.$$

Hence, a second-order Taylor expansion for $\epsilon > 0$ small enough yields $s(v^* - \epsilon) = v^* + [s''(v^*) + o(1)]\epsilon^2/2$. Thus, $s(v^* - \epsilon) > v^* > v^* - \epsilon$ for $\epsilon > 0$ small enough, contradicting $s(v^* - \epsilon) < v^* - \epsilon$.

We next show that $s'(v) > 0$ for $v \in [v(0), v(1)]$. This follows immediately from $s'(v(0)) = (I-1)\lambda'(0)/[(I-1)\lambda'(0)+1]$ and (B.4) using $s(v) < v$ for $v \in (v(0), v(1)]$. □

The next result, which follows from Lemma B1, relates the potential solutions of (2) to those of (B.1). It also provides some properties of the bid quantile function $b(\cdot)$.

Corollary B1: *If* $[U,F] \in \mathcal{U}_R \times \mathcal{F}_R$*, then for every* $I \in \mathcal{I}$
(i) $b(\cdot)$ *solves (B.1) if and only if* $b(\alpha) = s(v(\alpha))$*, where* $s(\cdot)$ *is a solution of (2) with* $s(v(0)) = v(0)$*. Equivalently,* $s(\cdot)$ *solves (2) with* $s(v(0)) = v(0)$ *if and only if* $s(v) = b(F(v))$ *where* $b(\cdot)$ *is a solution of (B.1).*
(ii) Solutions $b(\cdot)$ *of (B.1), if any, are continuously differentiable on* $[0,1]$*, with* $b'(v) > 0$ *for all* $\alpha \in [0,1]$ *and* $b(\alpha) < v(\alpha)$ *for all* $\alpha \in (0,1]$*.*

Proof of Corollary B1: Note that $v(\alpha)$ is continuously differentiable on $[0,1]$ with $v'(\alpha) = 1/f(v(\alpha)) > 0$ as $v(\alpha) = F^{-1}(\alpha)$. For part (i), setting $b(\alpha) = s(v(\alpha))$ yields $b'(\alpha) = s'(v(\alpha))/f(v(\alpha))$. So, if $b(\cdot)$ solves (B.1), then the change of variable $\alpha = F(v)$ yields that $s(\cdot)$ solves (2) with the desired initial condition. Conversely, if $s(\cdot)$ solves (2) with $s(v(0)) = v(0)$, then elementary algebra yields that $b(\cdot)$ solves (B.1). The second assertion of (i) follows similarly. Part (ii) follows from Lemma B1 with $b(\alpha) = s(v(\alpha))$. □

Instead of working with $B(\cdot;t)$, it is more convenient to make the change of variable $\pi(\cdot;t) = V(\cdot;t) - B(\cdot;t)$, where $V(\cdot;t)$ is continuously differentiable on $[0,1]$. This gives the companion flow of differential equations $\{\tilde{E}(\pi;t) = 0; t \in [0,1]\}$ defined by

$$\pi'(\alpha;t) = V'(\alpha;t) - \frac{I-1}{\alpha}\Lambda(\pi(\alpha;t);t) \text{ for } \alpha \in (0,1] \text{ with } \pi(0;t) = 0. \tag{B.6}$$

The next result, which also follows from Lemma B1, provides a set Σ in which the potential solutions of (B.6) lies. Hereafter, we let $\mathbf{C}_1^0$ be the set of functions $\pi(\cdot)$ from $[0,1]$ to $I\!R$ that are continuously differentiable on $[0,1]$ and satisfy $\pi(0) = 0$.

Corollary B2: *Let* $[U,F] \in \mathcal{U}_R \times \mathcal{F}_R$*. For every* $I \in \mathcal{I}$ *define* $\bar{v}' = \max_{\alpha\in[0,1]} v'(\alpha)$*, where* $0 < \bar{v}' < \infty$*, and let* $\mathbf{\Sigma} = \{\pi(\cdot) \in \mathbf{C}_1^0; 0 < \pi(\alpha) < \bar{v}' \text{ for } \alpha \in (0,1], \pi'(0) > 0\}$*. Then, for any* t *in* $[0,1]$*, solutions* $\pi(\cdot;t)$ *of the differential equation* $\tilde{E}(\pi;t) = 0$*, if any, are in* $\mathbf{\Sigma}$*.*

Proof of Corollary B2: Fix $t \in [0,1]$. For $\alpha \in [0,1]$, note that $V'(\alpha;t) = v'(\alpha t)$ and $V(0,t) = v(0)$. Hence, $0 \le V(\alpha;t) = v(0) + \int_0^\alpha v'(ut)du \le v(0) + \sup_{x\in[0,1]} v'(x) = v(0) + \bar{v}'$. Moreover, $V(\cdot;t)$ is $R+1$ continuously differentiable on $[0,1]$, while $\Lambda(\cdot;t)$ has the same properties as $\lambda(\cdot)$. Thus, (B.3) is similar to (B.1), thereby yielding that $B(\cdot;t)$ is continuously differentiable on $[0,1]$ with $v(0) < B(\alpha;t) < V(\alpha;t)$ for all $\alpha \in (0,1]$ by Corollary B1-(ii). Now, $\pi(\cdot;t) = V(\cdot;t) - B(\cdot;t)$ solves (B.6) if and only if $B(\cdot;t)$ solves (B.3). Thus, $\pi(\cdot;t) \in C_1^0$ and $0 < \pi(\alpha;t) = V(\alpha;t) - B(\alpha;t) < V(\alpha;t) - v(0) \le \bar{v}'$ for $\alpha \in (0,1]$. Moreover, $\pi'(0;t) = V'(0;t) - B'(0;t) = v'(0) - s'(v(0);t)v'(0) > 0$ since $v'(0) > 0$ and $s'(v(0);t) = (I-1)\Lambda'(0;t)/[(I-1)\Lambda'(0;t)+1] < 1$ by Lemma B1-(ii), where $\Lambda'(0;t) = \lambda'(0) > 0$. □

Next, we establish the smoothness of the auxiliary functions $\Lambda(x;t)$ and $V'(\alpha;t)$.

Lemma B2: *If $[U,F] \in \mathcal{U}_R \times \mathcal{F}_R$, then for every $I \in \mathcal{I}$*
(i) $\Lambda(x;t)$ is R continuously differentiable in $(x,t) \in I\!R_+ \times [0,1]$, Moreover, $(1/x)\partial^r \Lambda(x;t)/\partial t^r$ is continuous in $(x,t) \in I\!R_+ \times [0,1]$ for $r = 0,\ldots,R$,
(ii) $V'(\alpha;t)$ is R continuously differentiable in $(\alpha,t) \in [0,1]^2$.

Proof of Lemma B2: Let $0 < t \leq 1$. For $x > 0$ the Liebnitz-Newton formula yields

$$\frac{\partial^r \Lambda(x;t)}{\partial t^r} = \frac{\partial^r}{\partial t^r}\left(\frac{\lambda(tx)}{t}\right) = \sum_{j=0}^{r} \frac{r!}{j!(r-j)!} \frac{\partial^j \lambda(tx)}{\partial t^j} \frac{\partial^{r-j}}{\partial t^{r-j}}\left(\frac{1}{t}\right) = \frac{(-1)^r r!}{t^{r+1}} \sum_{j=0}^{r} \frac{\lambda^{(j)}(tx)}{j!}(-tx)^j$$

for $0 \leq r \leq R$. On the other hand, a Taylor expansion of $\lambda(0) = \lambda(tx - tx) = 0$ around tx with integral remainder (see e.g Zeidler (1985, p.77)) shows that

$$0 = \sum_{j=0}^{r} \frac{\lambda^{(j)}(tx)}{j!}(-tx)^j + \frac{(-tx)^{r+1}}{r!} \int_0^1 (1-u)^r \lambda^{(r+1)}(tx - utx)\, du.$$

Hence, using the change of variable $\nu = 1 - u$, we obtain for $(x,t) \in (\infty) \times (0,1]$

$$\begin{aligned}
\frac{1}{x}\frac{\partial^r \Lambda(x;t)}{\partial t^r} &= x^r \int_0^1 \nu^r \lambda^{(r+1)}(\nu t x)\, d\nu, \\
\frac{\partial^{r_1+r_2} \Lambda(x;t)}{\partial x^{r_1} \partial t^{r_2}} &= \frac{\partial^{r_1}}{\partial x^{r_1}} \left(x^{r_2+1} \int_0^1 \nu^{r_2} \lambda^{(r_2+1)}(\nu t x)\, d\nu \right),
\end{aligned}$$

where $0 \leq r_1 + r_2 \leq R$. Using the Lebesgue Dominated Convergence Theorem and the $R+1$ continuous differentiability of $\lambda(\cdot)$ on $I\!R_+$, it can be checked that the above two functions are continuous on $I\!R_+ \times [0,1]$, thereby establishing part (i). Part (ii) follows from $V'(\alpha;t) = v'(\alpha t)$ for $(\alpha,t) \in [0,1]^2$, where $v'(\cdot)$ is R continuously differentiable on $[0,1]$ because $F(\cdot|\cdot) \in \mathcal{F}_R$. □

We now introduce some functional operators associated with the differential equation (B.6). Let $\mathbf{C}_0$ be the set of functions $\pi(\cdot)$ from $[0,1]$ to $I\!R$ that are continuous on $[0,1]$. As is well-known, $\mathbf{C}_0$ is a Banach space equipped with the norm $\|\pi\|_0 = \sup_{\alpha\in[0,1]} |\pi(\alpha)|$. Similarly, $\mathbf{C}_1^0$ as defined earlier is a Banach space equipped with the norm $\|\pi\|_1 = \max_{r=0,1} \sup_{\alpha\in[0,1]} |\pi^{(r)}(\alpha)|$ $= \sup_{\alpha\in[0,1]} |\pi^{(1)}(\alpha)|$.[26] In particular, $\mathbf{\Sigma}$ is an open subset of $\mathbf{C}_1^0$ since the open ball $\mathcal{V}(\pi;\epsilon) = \{\zeta \in \mathbf{C}_1^0; \|\zeta - \pi\|_1 < \epsilon\} \subset \mathbf{\Sigma}$ for any $\pi \in \mathbf{\Sigma}$ and $\epsilon = \epsilon_\pi$ small enough. Moreover, for every $t \in [0,1]$, it can be checked that $\Lambda(\pi(\alpha);t)/\alpha$ and $V'(\alpha;t)$ are continuous in $\alpha \in [0,1]$ whenever $\pi(\cdot) \in \mathbf{C}_1^0$. Thus, for every $t \in [0,1]$, we can view the solutions of the differential equation (B.6) as the zeros of the functional operator $\mathbf{E}(\cdot;t)$ from $\mathbf{C}_1^0$ to $\mathbf{C}_0$ (see Lemma B3-(i) below), where

$$\mathbf{E}(\cdot;t) : \pi(\cdot) \to \mathbf{E}(\pi;t)(\alpha) = \pi^{(1)}(\alpha) + \frac{I-1}{\alpha}\Lambda(\pi(\alpha);t) - V'(\alpha;t)\ ,\ \alpha \in [0,1]\ .$$

[26] To see that $\|\pi\|_1 = \|\pi^{(1)}\|_0$, note that $|\pi(\alpha)| = \left|\int_0^\alpha \pi^{(1)}(u)du\right| \leq \sup_{u\in[0,1]} |\pi^{(1)}(u)|$ for all $\alpha \in [0,1]$.

In what follows $\mathbf{E}^{r_1 r_2}(\pi;t) = \partial^{r_1+r_2}\mathbf{E}(\pi;t)/\partial\pi^{r_1}\partial t^{r_2}$ denotes the Fréchet partial derivatives of $\mathbf{E}(\pi;\alpha)$ (see e.g. Zeidler, 1985), which are linear operators from $(\mathbf{C}_1^0)^{r_1} \times I\!R^{r_2}$. For a linear operator $L : \mathcal{C}_1 \mapsto \mathcal{C}_0$ with Banach spaces $\mathcal{C}_i$ equipped with norms N_i, $\rho(L) = \sup_{x\in\mathcal{C}_1, N_1(x)=1} N_0(L(x))$ is the operator norm of L.

Lemma B3: *If $[U,F] \in \mathcal{U}_R \times \mathcal{F}_R$, then for every $I \in \mathcal{I}$*
(i) $\mathbf{E}(\pi;t) \in \mathbf{C}_0$ for all $(\pi,t) \in \mathbf{C}_1^0 \times [0,1]$,
(ii) $\mathbf{E}(\pi;t)$ is R Fréchet differentiable in $(\pi,t) \in \mathbf{\Sigma} \times [0,1]$ with Fréchet partial derivatives $\mathbf{E}^{r_1 r_2}(\pi;t)$, $0 \le r_1 + r_2 \le R$ that are uniformly continuous over $\mathbf{\Sigma} \times [0,1]$,
(iii) The Fréchet partial derivative $\mathbf{E}^{10}(\pi;t)$ at $(\pi,t) \in \mathbf{\Sigma} \times [0,1]$, maps $\eta \in \mathbf{C}_1^0$ to $\mathbf{E}^{10}(\pi;t)(\eta) \in \mathbf{C}_0$ defined as $\mathbf{E}^{10}(\pi;t)(\eta)(\alpha) = \eta^{(1)}(\alpha) + \frac{I-1}{\alpha}\Lambda^{(1)}(\pi(\alpha);t)\eta(\alpha)$, for $\alpha \in [0,1]$. Moreover, $\mathbf{E}^{10}(\pi;t)$ is one-to-one (bijective) from $\mathbf{C}_1^0$ to $\mathbf{C}_0$ with an inverse of bounded operator norm uniformly in $(\pi,t) \in \mathbf{\Sigma} \times [0,1]$.

Proof of Lemma B3: Throughout, fix $I \in \mathcal{I}$.

(i) Fix $(\pi,t) \in \mathbf{C}_1^0 \times [0,1]$. It is sufficient to study $\Lambda(\pi(\cdot);t)$, which is clealry continuous on $(0,1]$. As $\alpha \downarrow 0$, $\pi(\alpha) = \pi'(0)\alpha + o(\alpha)$ since $\pi(0) = 0$ by definition of $\mathbf{C}_1^0$. For $t > 0$, it follows that $\Lambda(\pi(\alpha);t)/\alpha = \lambda(t\pi(\alpha))/(\alpha t) = \lambda'(0)\pi'(0) + o(1)$, the last expansion being also true for $t = 0$. Thus, $\mathbf{E}(\pi;t) \in \mathbf{C}_0$.

(ii) We first consider the Gâteaux derivatives of $\mathbf{E}(\pi;t)$. From e.g. Zeilder (1985) these are obtained in two steps: In a first step $\partial^{r_1+r_2}\mathbf{E}(\pi + u\eta;t)/\partial u^{r_1}\partial t^{r_2}$ is computed, where $\eta \in \mathbf{C}_1^0$, and in a second step the term η^{r_1} arising in this expression is changed into $\eta_1 \times \cdots \times \eta_{r_1}$, where the η_r are in $\mathbf{C}_1^0$. For $1 \le r_1 + r_2 \le R$ and $\eta_1,\ldots,\eta_R$ in $\mathbf{C}_1^0$, the Gâteaux derivatives are

$$\begin{aligned}
\mathbf{E}^{10}(\pi;t)(\eta_1)(\alpha) &= \eta_1^{(1)}(\alpha) + (I-1)\Lambda^{(1)}(\pi(\alpha);t)\frac{\eta_1(\alpha)}{\alpha}, \qquad \text{(B.7)}\\
\mathbf{E}^{0r_2}(\pi;t)(\alpha) &= \frac{I-1}{\alpha}\frac{\partial^{r_2}\Lambda(\pi(\alpha);t)}{\partial t^{r_2}} - \frac{\partial^{r_2}V'(\alpha;t)}{\partial t^{r_2}}, \text{ for } r_2 \ge 1,\\
\mathbf{E}^{r_1 r_2}(\pi;t)(\eta_1,\ldots,\eta_{r_1})(\alpha) &= (I-1)\frac{\partial^{r_1+r_2}\Lambda(\pi(\alpha);t)}{\partial x^{r_1}\partial t^{r_2}}\frac{\eta_1(\alpha)}{\alpha}\eta_2(\alpha) \times \cdots \times \eta_{r_1}(\alpha), \text{ otherwise.}
\end{aligned}$$

Note that $\eta_1(\alpha)/\alpha$ belongs to $\mathbf{C}_0$ since $\eta_1 \in \mathbf{C}_1^0$. It follows that $\mathbf{E}^{r_1 r_2}(\pi;t)(\eta_1,\ldots,\eta_{r_1}) \in \mathbf{C}_0$ for $1 \le r_1 + r_2 \le R$ by Lemma B2.

We now show that $\mathbf{E}(\pi;t)$ is R Fréchet continuously differentiable over $\mathbf{\Sigma} \times [0,1]$, with Fréchet partial derivatives $\mathbf{E}^{r_1 r_2}(\pi;t)$. Note that $\mathbf{E}^{0r_2}(\pi;t)$ is uniformly continuous over $\mathbf{\Sigma} \times [0,1]$ by Lemma B2. Thus, from Proposition 4.8 in Zeilder (1985), Part (ii) is proven if for $r_1 \ge 1$

(ii.a) The map $(\eta_1,\ldots,\eta_{r_1}) \to \mathbf{E}^{r_1 r_2}(\pi;t)(\eta_1,\ldots,\eta_{r_1})$ is a continuous multilinear operator, (ii.b) The map $(\pi, t) \to \mathbf{E}^{r_1 r_2}(\pi; t)$ is uniformly continuous over $\mathbf{\Sigma} \times [0,1]$.

We show these for $(r_1, r_2) = (1, 0)$ only, the other cases being similar. Recall that if $\pi \in \mathbf{\Sigma}$, then $\pi(\cdot)$ takes its values in $[0, \bar{v}']$, which is compact. For (ii.a), we have to show that the operator norm $\rho_1(\mathbf{E}^{10}(\pi; t)) \equiv \sup_{\eta_1 \in \mathbf{C}_1^0, \|\eta_1\|_1 = 1} \|\mathbf{E}^{10}(\pi; t)(\eta_1)\|_0 < \infty$ for all $(\pi, t) \in \mathbf{\Sigma} \times [0,1]$. Since $\|\eta_1(\alpha)/\alpha\|_0 \leq \|\eta_1\|_1$, it follows from (B.7) and Taylor inequality that

$$\begin{aligned}
\left\|\mathbf{E}^{10}(\pi; t)(\eta_1)\right\|_0 &= \left\|\eta_1^{(1)}(\alpha) + (I-1)\Lambda^{(1)}(\pi(\alpha); t)\frac{\eta_1(\alpha)}{\alpha}\right\|_0 \\
&\leq \|\eta_1\|_1\left(1 + (I-1)\sup_{(x,t)\in[0,\bar{v}']\times[0,1]}\left|\Lambda^{(1)}(x; t)\right|\right),
\end{aligned}$$

so that $\rho_1(\mathbf{E}^{10}(\pi; t)) < \infty$ by Lemma B2. For (ii.b), we have for any (π_0, t_0) and (π, t) in $\mathbf{\Sigma} \times [0,1]$

$$\begin{aligned}
\left\|\mathbf{E}^{10}(\pi_0; t_0)(\eta_1) - \mathbf{E}^{10}(\pi; t)(\eta_1)\right\|_0 &= (I-1)\left\|\left(\Lambda^{(1)}(\pi_0(\alpha); t_0) - \Lambda^{(1)}(\pi(\alpha); t)\right)\frac{\eta_1(\alpha)}{\alpha}\right\|_0 \\
&\leq (I-1)\|\eta_1\|_1\left\|\Lambda^{(1)}(\pi_0(\alpha); t_0) - \Lambda^{(1)}(\pi(\alpha); t)\right\|_0.
\end{aligned}$$

It follows from Lemma B2 that $\rho_1(\mathbf{E}^{10}(\pi_0; t_0) - \mathbf{E}^{10}(\pi; t))$ can be made arbitrarily small by choosing $\|\pi_0 - \pi\|_1$ and $|t_0 - t|$ small enough.

(iii) Fix (π, t) in $\mathbf{\Sigma} \times [0,1]$ and abbreviate $\mathbf{E}^{10}(\pi; t)$ into $\mathbf{E}^1$. The first part of (iii) has been established in (B.7). To show that this operator is one-to-one from $\mathbf{C}_1^0$ to $\mathbf{C}_0$, consider ζ in $\mathbf{C}_0$. Finding an $\eta \in \mathbf{C}_1^0$ with $\mathbf{E}^1(\eta) = \zeta$ amounts to solving the linear differential equation

$$E_\zeta^1 \;:\; \eta^{(1)}(\alpha) + (I-1)\Lambda^{(1)}(\pi(\alpha); t)\frac{\eta(\alpha)}{\alpha} = \zeta(\alpha) \text{ with } \eta(0) = 0. \tag{B.8}$$

Proceeding as in Step 1 of the proof of Lemma B1 yields that the unique candidate solution is

$$\eta_\zeta(\alpha) = \int_0^\alpha \zeta(u)\frac{R(u)}{R(\alpha)}du \text{ where } R(\alpha) = \exp\left(-(I-1)\int_\alpha^1 \frac{\Lambda^{(1)}(\pi(u); t)}{u}du\right).$$

Note that $\Lambda^{(1)}(x; t) = \lambda'(tx) \geq 1$ and $\Lambda^{(1)}(\pi(u); t)/u = \lambda'(0)/u + O(1)$ when $u \downarrow 0$. Thus,

$$0 \leq \frac{R(u)}{R(\alpha)} = \exp\left(-(I-1)\int_u^\alpha \frac{\Lambda^{(1)}(\pi(\tau); t)}{\tau}d\tau\right) \leq 1 \text{ for } 0 \leq u \leq \alpha \text{ and } \lim_{\alpha\to 0} R(\alpha) = 0.$$

It follows that η_ζ is defined and continuously differentiable over $(0,1]$. Observe now that $|\eta_\zeta(\alpha)| \leq \alpha\|\zeta\|_0$ so that setting $\eta_\zeta(0) = 0$ gives a continuous function over $[0,1]$. For differentiability at 0, note that for $0 \leq u \leq \alpha$ we have $(\ln\alpha - \ln u)/(\alpha - u) \to +\infty$ as $\alpha \downarrow 0$. Thus, as $\alpha \downarrow 0$ we have

$$\frac{R(u)}{R(\alpha)} = \exp\left[-(I-1)\int_u^\alpha\left(\frac{\lambda'(0)}{\tau} + O(1)\right)d\tau\right] = \exp\left(-(I-1)\lambda'(0)\ln\frac{\alpha}{u} + O(\alpha - u)\right)$$

$$
\begin{aligned}
&= \exp\left(-(I-1)\lambda'(0)\ln\frac{\alpha}{u} + \frac{\alpha-u}{\ln\alpha - \ln u}O(1)\right) = \left(\frac{u}{\alpha}\right)^{(I-1)\lambda'(0)}(1+o(1)), \\
\eta_\zeta(\alpha) &= \int_0^\alpha (\zeta(0)+o(1))\left(\frac{u}{\alpha}\right)^{(I-1)\lambda'(0)}(1+o(1))\,du = \frac{\zeta(0)}{(I-1)\lambda'(0)+1}\alpha + o(\alpha).
\end{aligned}
$$

Hence, η_ζ is differentiable at 0 with $\eta_\zeta^{(1)}(0) = \zeta(0)/[(I-1)\lambda'(0)+1]$. To check that $\eta_\zeta^{(1)}$ is continuous at 0, observe that (B.8) gives for $\alpha \downarrow 0$

$$
\eta_\zeta^{(1)}(\alpha) = \zeta(0) - (I-1)\left(\lambda'(0)+o(1)\right)\left(\frac{\zeta(0)}{(I-1)\lambda'(0)+1} + o(1)\right) = \eta_\zeta^{(1)}(0) + o(1).
$$

Hence, $\eta_\zeta \in C_1^0$. Thus, $\mathbf{E}^1 : \mathbf{C}_1^0 \mapsto \mathbf{C}_0$ is one-to-one with $\left[\mathbf{E}^1\right]^{-1}(\zeta) = \eta_\zeta$ for any $\zeta \in \mathbf{C}_0$.

Lastly, recall that $|\eta_\zeta(\alpha)| \le \alpha\|\zeta\|_0$ and $0 \le \pi(\alpha) \le \bar{v}'$ for any $\pi \in \mathbf{\Sigma}$. This gives

$$
\begin{aligned}
\left\|\left[\mathbf{E}^1\right]^{-1}(\zeta)\right\|_1 &= \left\|\eta_\zeta^{(1)}\right\|_0 = \left\|\zeta(\alpha) - (I-1)\Lambda^{(1)}(\pi(\alpha);t)\frac{\eta(\alpha)}{\alpha}\right\|_0 \\
&\le \|\zeta\|_0 + (I-1)\sup_{(x,t)\in[0,\bar{v}']\times[0,1]} \lambda'(tx)\|\zeta\|_0 = \left(1 + (I-1)\sup_{x\in[0,\bar{v}']}\lambda'(x)\right)\|\zeta\|_0.
\end{aligned}
$$

Hence the operator norm $\rho\left(\left[\mathbf{E}^1(\pi;t)\right]^{-1}\right)$ is bounded uniformly in $(\pi,t) \in \mathbf{\Sigma}\times[0,1]$. □

We now prove Theorems B1 and B2. To prove Theorem B1, we use the following continuation argument in Zeidler (1985, Proposition 6.10).

Theorem Z1 (Continuation argument): *Let $\mathcal{C}_1$ and $\mathcal{C}_0$ be some Banach spaces. For $\pi \in \mathcal{C}_1$, let $\mathcal{V}(\pi;\epsilon)$ denote the ϵ-neighborhood of π in $\mathcal{C}_1$. Suppose that*
(i) The map $(\pi,t) \in \mathcal{C}_1 \times [0,1] \mapsto \mathbf{E}(\pi;t) \in \mathcal{C}_0$ is continuous,
(ii) (A priori condition) There exists an open subset $\mathcal{S}$ of $\mathcal{C}_1$ and a number $\epsilon > 0$ such that, if (π_t,t) verifies $\mathbf{E}(\pi_t;t) = 0$, then $\mathcal{V}(\pi_t;\epsilon) \subset \mathcal{S}$ for all $t \in [0,1]$,
(iii) For any $t \in [0,1]$, the operator $\mathbf{E}$ has a Fréchet derivative $\mathbf{E}_\pi$ with respect to $\pi \in \mathcal{S}$. The operators $(\pi,t) \mapsto \mathbf{E}(\pi;t)$ and $(\pi,t) \mapsto \mathbf{E}_\pi(\pi;t)$ are uniformly continuous over $\mathcal{S}\times[0,1]$,
(iv) The linear operator $\eta \in \mathcal{C}_1 \mapsto \mathbf{E}_\pi(\pi;t)(\eta) \in \mathcal{C}_0$ is one-to-one, and for some constant C, $\rho(\mathbf{E}_\pi(\pi;t)^{-1}) \le C$ for all $(\pi,t) \in \mathcal{S}\times[0,1]$.
If $\mathbf{E}(\pi;0) = 0$ has a unique solution π_0, then $\mathbf{E}(\pi;1) = 0$ has a unique solution π_1.

Proof of Theorem B1: Fix $I \in \mathcal{I}$. Part (ii) follows from Lemma B1, Corollary B1 and part (i). Thus, it suffices to show the latter. In view of Corollary B1-(ii), it remains to show the existence and uniqueness of the solution of (B.1), i.e $E(B;1) = 0$ or equivalently $\tilde{E}(\pi;1) = 0$. We apply Theorem Z1, where $\mathcal{C}_1 = \mathbf{C}_1^0$, $\mathcal{C}_0 = \mathbf{C}_0$, and $\mathcal{S} = \mathbf{\Sigma}$. Lemma B3 shows that conditions (i), (iii) and (iv) of Theorem Z1 hold. Hence, it remains to check condition (ii).

We begin with some inequalities. Let $\bar{v}'$ be as in Corollary B2 and define $\underline{v}' = \inf_{\alpha\in[0,1]} v'(\alpha)$, $\underline{\lambda}' = \inf_{x\in[0,\bar{v}']} \frac{\lambda(x)}{x}$, $\bar{\lambda}' = \sup_{x\in[0,\bar{v}']} \frac{\lambda(x)}{x}$, where $0 < \underline{v}' \leq \bar{v}' < \infty$ and $1 \leq \underline{\lambda}' \leq \bar{\lambda}' < \infty$ because $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. Recall that $V'(\alpha; t) = v'(\alpha t)$ for $(\alpha, t) \in [0, 1]^2$, while $\pi(\cdot) \in \mathbf{\Sigma}$ takes its values in $[0, \bar{v}')$. Thus, for any $(\alpha, t) \in [0, 1]^2$ and $\pi(\cdot) \in \mathbf{\Sigma}$ we have $\underline{v}' \leq V'(\alpha; t) \leq \bar{v}'$ and $\underline{\lambda}'\pi(\alpha) \leq \Lambda(\pi(\alpha); t) \leq \bar{\lambda}'\pi(\alpha)$ since $\Lambda(x; t) = \lambda(xt)/t$ for $(x, t) \in I\!R_+ \times (0, 1]$, $\Lambda(x; 0) = \lambda'(0)x$ for $x \in I\!R_+$ and $\underline{\lambda}' \leq \lambda'(0) \leq \bar{\lambda}'$. For $t \in [0, 1]$, let $\pi(\cdot; t)$ be a solution of $\tilde{E}(\pi; t) = 0$ so that $\pi(\cdot) \in \mathbf{\Sigma}$ by Corollary B2. Since $\tilde{E}(\pi; t) = 0$ writes $\pi'(\alpha) + (I - 1)\Lambda(\pi(\alpha); t)/\alpha = V'(\alpha; t)$, the above inequalities yields, for all $(\alpha, t) \in [0, 1]^2$,

$$\underline{v}' \leq \pi'(\alpha; t) + (I-1)\bar{\lambda}'\frac{\pi(\alpha; t)}{\alpha} \quad \text{and} \quad \pi'(\alpha; t) + (I-1)\underline{\lambda}'\frac{\pi(\alpha; t)}{\alpha} \leq \bar{v}' \quad \text{with} \quad \pi(0; t) = 0.$$

Setting $\bar{C}(\alpha; t) = \pi(\alpha; t)\alpha^{(I-1)\bar{\lambda}'}$ so that $\bar{C}'(\alpha; t) = \alpha^{(I-1)\bar{\lambda}'}[\pi'(\alpha; t) + (I-1)\bar{\lambda}'\pi(\alpha; t)/\alpha]$ yields $\underline{v}'\alpha^{(I-1)\bar{\lambda}'} \leq \bar{C}'(\alpha; t)$ from the first differential inequality. Thus, integrating and using $\bar{C}(0; t) = 0$ gives $\underline{v}'\alpha/[(I-1)\underline{\lambda}'+1] \leq \pi(\alpha)$ for all $(\alpha, t) \in [0, 1]^2$. Setting $\underline{C}(\alpha; t) = \pi(\alpha; t)\alpha^{(I-1)\underline{\lambda}'}$, proceeding similarly with the second differential inequality, and combining yield

$$\frac{\underline{v}'\alpha}{(I-1)\bar{\lambda}'+1} \leq \pi(\alpha; t) \leq \frac{\bar{v}'\alpha}{(I-1)\underline{\lambda}'+1} < \bar{v}' \text{ for all } (\alpha, t) \in [0, 1]^2. \tag{B.9}$$

We now check condition (ii) of Theorem Z1. We have to show that $\mathcal{V}(\pi(\cdot; t); \epsilon) \subset \mathbf{\Sigma}$ for $\epsilon > 0$ small enough and all $t \in [0, 1]$. Recall that the neighborhood $\mathcal{V}(\pi(\cdot; t); \epsilon)$ of $\pi(\cdot; t)$ in $\mathbf{C}_1^0$ consists of functions $\zeta(\cdot) \in \mathbf{C}_1^0$ with $\sup_{\alpha\in[0,1]} |\zeta'(\alpha) - \pi'(\alpha; t)| < \epsilon$. In particular, $\zeta'(0) > \pi'(0; t) - \epsilon$, where $\pi'(0; t) = V'(0; t) - B'(0; t) = v'(0)[1 - s'(v(0); t)] = v'(0)/[(I-1)\lambda'(0)+1] > 0$ by Lemma B1. Moreover, integrating and using $\pi(0; t) = \zeta(0) = 0$ give $\pi(\alpha; t) - \epsilon\alpha < \zeta(\alpha) < \pi(\alpha; t) + \epsilon\alpha$ for all $\alpha \in [0, 1]$. Hence, for $\epsilon > 0$ small enough, $\zeta'(0) > \pi'(0; t) - \epsilon > 0$, while (B.9) yields

$$0 < \left(\frac{\underline{v}'}{(I-1)\bar{\lambda}'+1} - \epsilon\right)\alpha < \zeta(\alpha) < \left(\frac{\bar{v}'}{(I-1)\underline{\lambda}'+1} + \epsilon\right)\alpha < \bar{v}' \text{ for all } \alpha \in (0, 1],$$

for all $t \in [0, 1]$. That is, there exists $\epsilon > 0$ such that $\mathcal{V}(\pi(\cdot; t); \epsilon)$ is a subset of $\mathbf{\Sigma}$ for all $t \in [0, 1]$ and the a priori condition (ii) of Theorem Z1 is proven. □

To prove Theorem B1, we use the following Implicit Functional Theorem in Zeidler (1985, Theorem 4.B).

Theorem Z2 (Implicit Functionial Theorem): *Let* (π_0, t_0) *be in* $\mathcal{C}_1 \times [0, 1]$*, where* $\mathcal{C}_1$ *is a Banach space, and consider an* R *continuously Fréchet differentiable operator* $\mathbf{E}(\cdot, \cdot)$ *defined on a neighborhood of* (π_0, t_0) *with values in a Banach space* $\mathcal{C}_0$ *such that* $\mathbf{E}(\pi_0, t_0) = 0$*. If the Fréchet derivative* $\mathbf{E}_\pi(\pi, t)$ *of* $\mathbf{E}(\pi, t)$ *with respect to* π *is such that* $\mathbf{E}_\pi(\pi_0, t_0)$ *is one-to-one, then* *there exists a neighborhood $\mathcal{O}(t_0)$ of t_0 such that, for $t \in \mathcal{O}(t_0)$, the equation $\mathbf{E}(\pi, t) = 0$ has a unique solution $\pi(t)$, which is R continuously differentiable on $\mathcal{O}(t_0)$.*

Proof of Theorem B2: Fix $I \in \mathcal{I}$. Part (ii) follows from part (i) since $s(v) = b(F(v))$ by Corollary B1-(ii) and $F \in \mathcal{F}_R$. Thus, it suffices to show part (i). Let $\mathcal{C}_1 = \mathbf{C}_1^0$ and $\mathcal{C}_0 = \mathbf{C}_0$. For any $t_0 \in [0,1]$, note that $\tilde{E}(\pi; t_0) = 0$ has a unique solution $\pi_0(\cdot) = \pi(\cdot; t_0)$ as it suffices to consider the flow of differential equations $\{\tilde{E}_0(\pi; u) = 0; u \in [0,1]\}$, where $\tilde{E}_0(\pi; u) \equiv \tilde{E}(\pi; ut_0)$ and to follow the proof of Theorem B1-(i) with $\mathcal{S} = \mathbf{\Sigma}$. As $\pi_0 \in \mathbf{\Sigma}$ by Corollary B2, while $\mathbf{\Sigma} \times [0,1]$ is a neighborhood of (π_0, t_0), Lemma B3 and Theorem Z2 yield that $\pi(t) = \pi(\cdot; t)$ is R continuously differentiable with respect to t in a neighborhood $\mathcal{O}(t_0)$ of t_0, and hence at t_0. As t_0 is arbitrary in $[0,1]$, then $\pi(\cdot; t)$ is R continuously differentiable in $t \in [0,1]$.

For $t \in (0,1]$, note that $\pi(\alpha; t) = [v(\alpha t) - b(\alpha t)]/t$ for $\alpha \in [0,1]$, where $b(\cdot)$ is the solution of (B.1). To see this, it suffices to verify that such a $\pi(\cdot; t)$ verifies (B.6) using $\pi'(\alpha; t) = v'(\alpha t) - b'(\alpha t)$, $V'(\alpha; t) = v'(\alpha t)$, $\Lambda(x; t) = \lambda(tx)/t$ for $x \geq 0$, and (B.1). Similarly, for $t = 0$, let $\pi(\alpha; 0) = [v'(0) - b'(0)]\alpha$ for $\alpha \in [0,1]$, which can be seen to verify (B.6). In particular, $\pi'(1; t) = v'(t) - b'(t)$ for $t \in [0,1]$. Thus, using $v'(t) = V'(1; t)$ and (B.6) at $\alpha = 1$ give $b'(t) = (I-1)\Lambda(\pi(1; t); t)$ for $t \in [0,1]$, where $\Lambda(\cdot; \cdot)$ is R continuously differentiable on $I\!R_+ \times [0,1]$ by Lemma B2-(i) and $\pi(1; \cdot)$ is R continuously differentiable on $[0,1]$. Hence, $b'(\cdot)$ is R continuously differentiable on $[0,1]$ implying that $b(\cdot)$ is $R+1$ continuously differentiable on $[0,1]$ as desired. Lastly, using (B.1) shows that $b'(\cdot)$ is $R+1$ continuously differentiable on $(0,1]$. □

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Figure 1: Identification with $s(\cdot)$ increasing in competition

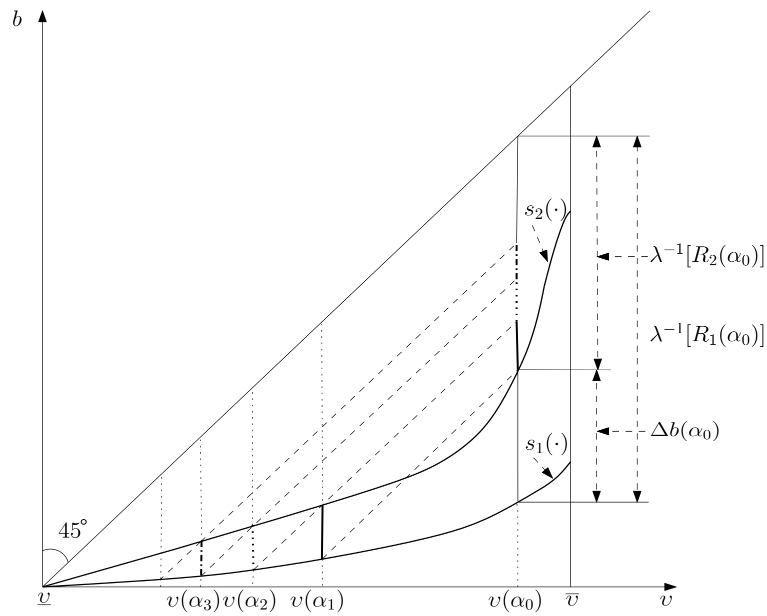

Figure 2: Identification with $s(\cdot)$ nonincreasing in competition, case (i)

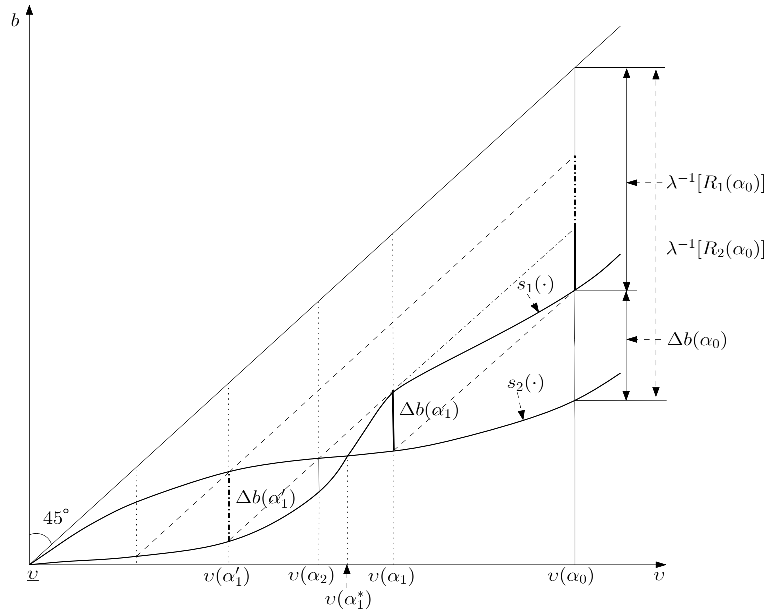

Figure 3: Identification with $s(\cdot)$ nonincreasing in competition, case (ii)