OPTIMAL $\hat{\alpha}_n$

This document is supplemental to the paper *Cube–root–n and faster convergence, Laplace estimators, and uniform inference* by Jun, Pinkse, and Wan (2009). We show here that if $q = 1$ and α_n is replaced in the definition of $\hat{\boldsymbol{\theta}}$ with $\hat{\boldsymbol{\alpha}}_{\boldsymbol{n}} = \hat{\boldsymbol{c}}^{*\mathbf{2}}\sqrt[5]{n}$, then the resulting estimator $\breve{\hat{\boldsymbol{\theta}}}$ satisfies

$$n^{2/5}(\breve{\hat{\boldsymbol{\theta}}} - \theta_0) \xrightarrow{d} N\Big(\frac{b_{11}^*}{c^{*4}}, c^{*2}\mathscr{V}\Big),$$

provided that $\hat{\boldsymbol{c}}^* = c^* + o_p(1)$ and $c^* \in \operatorname{int}\mathscr{T}^*$, a compact subset of $\mathscr{R}$, entirely consisting of positive values. An implication is that the 'optimal' choice of c_α of our estimator without using a bias–correction scheme can be replaced with a consistent estimate without affecting its optimality properties.

There is nothing special about $q = 1$. The only part of the proof below that even uses q is lemma 6, whose proof is essentially already in the paper. Provided that a prior is chosen that removes the bias up to arbitrary order $\tilde{q}$, then q could be chosen equal to $\tilde{q}$, essentially without modifications other than a correction to the rate at which α_n diverges.

From hereon take $\alpha_n = \sqrt[5]{n}$. Highlighted assumptions, lemmas and equations are references to the August 2009 version of our paper.

Let

$$\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,c) = c\tilde{\boldsymbol{S}}_{\boldsymbol{n}}(t/c^2) = c\sqrt{n\alpha_n}\boldsymbol{S}_{\boldsymbol{n}}(\theta_0 + t/\alpha_n c^2) = c\sqrt{\alpha_n/n}\sum_{i=1}^{n}\tilde{\boldsymbol{g}}_{\boldsymbol{i}}(\theta_0 + t/\alpha_n c^2).$$

Under assumption G, $\breve{\boldsymbol{S}}_{\boldsymbol{n}}(\cdot,c) \xrightarrow{w} \mathbb{G}(\cdot,c) = \mathbb{G}(\cdot)$ in $\ell^\infty(\mathscr{T}_1,\mathscr{T}_2,\cdots)$ for any fixed $c > 0$. We will show that we can regard c as an extra index without losing weak convergence.

Lemma 1. *$\breve{\boldsymbol{S}}_{\boldsymbol{n}} \xrightarrow{w} \mathbb{G}$ in $\ell^\infty(\mathscr{T}_1\times\mathscr{T}^*,\mathscr{T}_2\times\mathscr{T}^*,\cdots)$.*

Proof. By theorem 1.6.1 of van der Vaart and Wellner, it suffices to show the weak convergence in $\ell^\infty(\mathscr{T}_i\times\mathscr{T}^*)$ for arbitrary i. We instead establish the equivalent result that for any i, $\breve{\breve{\boldsymbol{S}}}_{\boldsymbol{n}} \xrightarrow{w} \tilde{\mathbb{G}}$ in $\breve{\mathscr{T}}_i\times\mathscr{T}^*$, where $\breve{\breve{\boldsymbol{S}}}_{\boldsymbol{n}}(t/c^2,c) = \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,c)$, $\tilde{\mathbb{G}}(t/c^2,c) = \mathbb{G}(t,c)$, and $\breve{\mathscr{T}}_i = \{s : c^2 s\in\mathscr{T}_i,\ c\in\mathscr{T}^*\}$.

Pick some i. Note that $\breve{\breve{\boldsymbol{S}}}_{\boldsymbol{n}}(t,c) = c\tilde{\boldsymbol{S}}_{\boldsymbol{n}}(t)$ and recall from the main paper that $\tilde{\boldsymbol{S}}_{\boldsymbol{n}} \xrightarrow{w} \mathbb{G}$ in $\ell^\infty(\breve{\mathscr{T}}_i)$. Since the convergence of finite marginals is trivial, we only establish asymptotic tightness. Let T^* be an arbitrary subset of $\breve{\mathscr{T}}_i$. Then

$$\sup_{t_1,t_2\in T^*}|\breve{\breve{\boldsymbol{S}}}_{\boldsymbol{n}}(t_1,c_1) - \breve{\breve{\boldsymbol{S}}}_{\boldsymbol{n}}(t_2,c_2)| \le |c_1 - c_2|\sup_{t_1\in T^*}|\tilde{\boldsymbol{S}}_{\boldsymbol{n}}(t_1)| + |c_2|\sup_{t_1,t_2\in T^*}|\tilde{\boldsymbol{S}}_{\boldsymbol{n}}(t_1) - \tilde{\boldsymbol{S}}_{\boldsymbol{n}}(t_2)|, \qquad (1)$$

where RHS1 is $o_p(1)$ as $|c_1 - c_2| \to 0$, and RHS2 is $o_p(1)$ as the diameter of T^* tends to zero due to the tightness of $\tilde{\boldsymbol{S}}_{\boldsymbol{n}}$. Therefore, $\breve{\boldsymbol{S}}_{\boldsymbol{n}}$ is asymptotically tight in $\ell^\infty(\breve{\mathscr{T}}_i \times \mathscr{T}^*)$, and the weak convergence in $\ell^\infty(\breve{\mathscr{T}}_i \times \mathscr{T}^*)$ follows. □

Lemma 2.

$$\breve{\boldsymbol{S}}_{\boldsymbol{n}}(\cdot, \hat{\boldsymbol{c}}) - \breve{\boldsymbol{S}}_{\boldsymbol{n}}(\cdot, c^*) \xrightarrow{w} 0 \text{ in } \ell^\infty(\mathscr{T}_1, \mathscr{T}_2, \dots) \quad \text{and} \quad \sup_{t \in \mathscr{R}^d} \frac{|\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, \hat{\boldsymbol{c}}) - \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, c^*)|}{\|t\|^2 + 1} = o_p(1).$$

Proof. The proof of the second stated result is nearly identical to that of lemma A2 in view of the first stated result. We hence establish the first result (weak convergence). By theorem 1.6.1 of van der Vaart and Wellner, it suffices to establish the weak convergence in $\ell^\infty(\mathscr{T})$ for arbitrary compact set $\mathscr{T}$. Define a mapping $\mathscr{O} : \ell^\infty(\mathscr{T} \times \mathscr{T}^*) \times \mathscr{T}^* \to \ell^\infty(\mathscr{T})$ by $\mathscr{O}(\breve{g}, c) = \breve{g}(\cdot, c) - \breve{g}(\cdot, c^*)$. For all $\breve{g}, c$ for which $\sup_{t \in \mathscr{T}} |\breve{g}(t, c) - \breve{g}(t, c^*)|$ is continuous at c, $\mathscr{O}$ is continuous at $(\breve{g}, c)$ with respect to the product metric. Therefore, by the continuous mapping theorem, $\mathscr{O}(\breve{\boldsymbol{S}}_{\boldsymbol{n}}, \hat{\boldsymbol{c}}) \xrightarrow{w} \mathscr{O}(\mathbb{G}, c^*) = 0$ because $\mathbb{G}$ is a Gaussian process and hence $\sup_{t \in T} |\mathbb{G}(t, c) - \mathbb{G}(t, c^*)|$ is continuous. □

Let $\breve{Q}_n(t, c) = c^4 \alpha_n^2 Q(\theta_0 + t/\alpha_n c^2)$ and $\breve{\pi}_n(t, c) = \pi(\theta_0 + t/\alpha_n c^2)$. Let $\hat{\boldsymbol{\beta}}_{\boldsymbol{n}} = \sqrt{\hat{\boldsymbol{c}}^6 \alpha_n^3 / n}$.

Lemma 3. $\int \breve{\pi}_n(t, \hat{\boldsymbol{c}}) t \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, \hat{\boldsymbol{c}}) \exp\big(\breve{Q}_n(t, \hat{\boldsymbol{c}})\big) dt - \int \breve{\pi}_n(t, c^*) t \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, c^*) \exp\big(\breve{Q}_n(t, c^*)\big) dt = o_p(1)$.

Proof. The absolute value of the LHS of the lemma statement is for some finite order polynomial P bounded by

$$\begin{aligned} &\sup_t \frac{|\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, \hat{\boldsymbol{c}}) - \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, c^*)|}{\|t\|^2 + 1} \int \|P(t)\| |\breve{\pi}_n(t, \hat{\boldsymbol{c}})| \exp\big(\breve{Q}_n(t, \hat{\boldsymbol{c}})\big) dt \\ &\qquad + \sup_t \frac{|\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, c^*)|}{\|t\|^2 + 1} \int \|P(t)\| \big|\breve{\pi}_n(t, \hat{\boldsymbol{c}}) \exp\big(\breve{Q}_n(t, \hat{\boldsymbol{c}})\big) - \breve{\pi}_n(t, c^*) \exp\big(\breve{Q}_n(t, c^*)\big)\big| dt \\ &\qquad = o_p(1) O_p(1) + O_p(1) o_p(1), \end{aligned} \tag{2}$$

by lemma 2 and the dominated convergence theorem, noting that $\breve{\pi}_n(t, c)$ is only nonzero for values of (t, c) for which $\theta_0 + t/\alpha_n c^2 \in \Theta$, such that by lemma A5 for such values of (t, c), $\breve{Q}_n(t, c) \leq -\min(c^4 \alpha_n^2 c_q, t^\intercal V t/4)$. □

Lemma 4. *For* $j = 0, 1$, $\int \breve{\pi}_n(t, \hat{\boldsymbol{c}}) t^j \{\exp\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{n}} \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, \hat{\boldsymbol{c}})\big) - \sum_{s=0}^{j} \big(\hat{\boldsymbol{\beta}}_{\boldsymbol{n}} \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, \hat{\boldsymbol{c}})\big)^s\} \exp\big(\breve{Q}_n(t, \hat{\boldsymbol{c}})\big) dt / \hat{\boldsymbol{\beta}}_{\boldsymbol{n}}^{\boldsymbol{j+1}} = O_p(1)$.

Proof. By lemma A6, it suffices to show that

$$\int \|t\|^j |\breve{\pi}_n(t, \hat{\boldsymbol{c}})| \exp\big(|\hat{\boldsymbol{\beta}}_{\boldsymbol{n}} \breve{\boldsymbol{S}}_{\boldsymbol{n}}(t, \hat{\boldsymbol{c}})| + \breve{Q}_n(t, \hat{\boldsymbol{c}})\big) dt = O_p(1), \tag{3}$$

or indeed that

$$\sup_{c\in\mathscr{T}^*}\int \|t\|^j|\breve{\pi}_n(t,c)|\exp\big(\big|c^3\beta_n\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,c)\big|+\breve{Q}_n(t,c)\big)dt=O_p(1).$$

Let $\bar{c}=\max_{c\in\mathscr{T}^*}c$ and $\underline{c}=\min_{c\in\mathscr{T}^*}c$ and make the substitution $s=t/c^2$ to obtain

$$\sup_{c\in\mathscr{T}^*}\Big\{c^{2(d+j)}\int\|s\|^j|\pi_n(s)|\exp\big(c^4\beta_n|\tilde{\boldsymbol{S}}_{\boldsymbol{n}}(s)|+c^4Q_n(s)\big)ds\Big\}$$
$$\leq\bar{c}^{2(d+j)}\int\|s\|^j\|\pi_n(s)\|\exp\big(\bar{c}^4\beta_n|\tilde{\boldsymbol{S}}_{\boldsymbol{n}}(s)|+\underline{c}^4Q_n(s)\big)ds=O_p(1),$$

by part (ii) of lemma A9. □

Lemma 5.

$$\frac{1}{\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}}\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})t\big\{\exp\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,\hat{\boldsymbol{c}})\big)-1\big\}\exp\big(\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt\xrightarrow{d}N\big(0,C_V^2\mathscr{V}_N\big).$$

Proof. The LHS is equal to

$$\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})t\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,\hat{\boldsymbol{c}})\exp\big(\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt$$
$$+\frac{1}{\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}}\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})t\Big(\exp\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,\hat{\boldsymbol{c}})\big)-1-\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,\hat{\boldsymbol{c}})\Big)\exp\big(\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt,\quad(4)$$

where the first term converges in distribution to the stated normal distribution by lemmas 3 and lemma D3, and the second term is negligible by lemma 4. □

Lemma 6.

$$\frac{1}{\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}}\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})t\exp\big(\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt=\frac{b_{11}^*}{c^{*5}}+o_p(1).$$

Proof. Define $\breve{R}_n(t,\hat{\boldsymbol{c}})=\breve{Q}_n(t,\hat{\boldsymbol{c}})+t^{\intercal}Vt/2$ and follow the steps of the proof of theorem 5 for $q=1$. □

Lemma 7.

$$\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})\exp\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,\hat{\boldsymbol{c}})+\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt\xrightarrow{p}\pi_0C_V.$$

Proof. The LHS is equal to

$$\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})\exp\big(\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt+\int\breve{\pi}_n(t,\hat{\boldsymbol{c}})\Big(\exp\big(\hat{\boldsymbol{\beta}}_{\boldsymbol{n}}\breve{\boldsymbol{S}}_{\boldsymbol{n}}(t,\hat{\boldsymbol{c}})\big)-1\Big)\exp\big(\breve{Q}_n(t,\hat{\boldsymbol{c}})\big)dt\quad(5)$$

where the second term is negligible by lemma 4. The first term has the same limit as $\int\breve{\pi}_n(t,c^*)\exp\big(\breve{Q}_n(t,c^*)\big)dt$ by the dominated convergence theorem, which is π_0C_V by lemma D1. □

Lemma 8. *If* $q = 1$ *and scaling* $\hat{\boldsymbol{\alpha}}_{\boldsymbol{n}}$ *is used in lieu of* α_n*,*

$$n^{2/5}(\check{\hat{\boldsymbol{\theta}}} - \theta_0) \xrightarrow{d} N\Big(\frac{b_{11}^*}{c^{*4}}, c^{*2}\mathscr{V}\Big).$$

Proof. Follows from the above results. □