Semi-Nonparametric Modeling and Estimation of First-Price Auction Models with Auction-Specific Heterogeneity

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Abstract

In this paper we extend and generalize the semi-nonparametric modeling and sieve estimation approach of Bierens and Song (2011a) for independently and identically distributed first-price auctions to first-price auction models with observed auction-specific heterogeneity. The latter will be incorporated via a linear model for the log values with unknown error distribution. This distribution will be modeled semi-nonparametrically similar to the approach in Bierens (2011). It will be shown that the semi-nonparametric conditional value distribution involved can be estimated consistently by minimizing the integrated square distance between the empirical characteristic function of the actual bids and the simulated bids, together with the auction-specific covariates, via an integrated conditional moment criterion.

Keywords: Auction-specific heterogeneity, empirical characteristic function, first-price auctions, integrated conditional moment, semi-nonparametric estimation, sieve estimation, simulated method of moments.

JEL codes: C14, C21, C51, D44

1 Introduction

In many repeated auctions the objects to be auctioned off are different across auctions. Consequently, the value distributions are then different across auctions. However, if we observe the auction-specific characteristics in the form of covariates, and the value distributions conditional on these covariates have the same functional form, the conditional bid distribution given the auction-specific covariates will be the same for all auctions. The question then arises how to incorporate the observable characteristics into the auction model. Laffont et al. (1995) incorporate covariates in the value distribution by specifying a linear regression model for the log of values with zero-mean normal errors. Donald and Paarsch (1996) parameterize the upper bound of the values as a function of covariates. Li (2005) specifies the value distribution as the exponential distribution with mean a linear function of covariates. Guerre et al. (2000) propose a two-stage nonparametric kernel density estimation approach, where in the first stage the bid distribution and density conditional on the covariates are estimated nonparametrically, which then is used in inverse form to generate values given the actual bids and the covariates. The generated values are then used to estimate the conditional value distribution nonparametrically.

In this paper, we propose an alternative semi-nonparametric approach to estimate first-price auction models with observed auction-specific heterogeneity and private, symmetric and independent values conditional on a vector of auction specific covariates. This approach extends the semi-nonparametric integrated simulated moments estimation method of Bierens and Song (2011a) to the heterogenous case with observable auction-specific covariates. We consider a first-price auction model where the log value takes the form of a linear regression model conditional on covariates, with unknown error distribution. The latter distribution is modeled semi-nonparametrically similar to the approach in Bierens (2011). Given the corresponding value distribution, we generate for each auction artificial bids conditional on the auction-specific covariates. Next, we take the difference of the empirical characteristic functions of the actual bids and the simulated bids, both jointly with the covariates, as the moment conditions. Integrating the squared difference of these empirical characteristic functions yields an integrated conditional moment (ICM) objective function, similar to the ICM test statistic proposed by Bierens (1982) and Bierens and Ploberger (1997). Minimizing this ICM objective function to the regression parameters and the corresponding semi-nonparametric error distribution via a sieve estimation method then yields a consistent estimator of the conditional value distribution.

2 Model and Data-Generating Process

2.1 The equilibrium bid function and its identification

Given a vector X of auction-specific characteristics, let $\Gamma_0(v|X)$ be the conditional distribution of the private value V that each potential bidder has for the object to be auctioned off in an auction characterized by X, and let $p_0(X) > 0$ be the reservation price if it is binding, i.e., $\Gamma_0(p_0(X)|X) > 0$. If the reservation price is nonbinding then without loss of generality we may set $p_0(X) = 0$. Moreover, let $I_0(X) \geq 2$ be the number of potential bidders in this auction, which is ex-ante known to all potential bidders. The latter also applies to the reservation price.

As is well-known, the equilibrium bid function of a first-price sealed bid auctions where values are independent and private, bidders are symmetric and risk-neutral, and the reservation price is binding, takes the form

$$\beta_0(v|X) = v - \frac{1}{\Gamma_0(v|X)^{I_0(X)-1}} \int_{p_0(X)}^{v} \Gamma_0(z|X)^{I_0(X)-1} dz \qquad (1)$$

for $v > p_0(X)$. See for example Riley and Samuelson (1981) or Krishna (2002).

We will assume that all the bids in the auction characterized by X are observed by the econometrician. Then conditional on $X = x$, $x \in \mathbb{X}$, with $\mathbb{X}$ the support of X, the conditional value distribution $\Gamma_0(v|x)$ for $v \geq p_0(x)$ and the number of potential bidders $I_0(x)$ are nonparametrically identified from the conditional distribution of the bids and the actual number of bidders. This follows from Theorem 4 of Guerre et al. (2000), and under more general conditions from Bierens and Song (2011b). The identification conditions in the latter paper are:

Assumption 1.
(a) The conditional value distribution $\Gamma_0(v|x)$, *given* $X = x \in \mathbb{X} \subset \mathbb{R}^d$, *is the same in all auctions and is known to each potential bidder, and is absolutely continuous with conditional density* $\gamma_0(v|X)$ *and finite conditional expectation:* $\int_0^\infty v\gamma_0(v|X)dv < \infty$ *a.s.*
(b) Given the vector X *of auction-specific characteristics the number of potential bidders* $I_0(X)$ *and the reservation price* $p_0(X)$ *are ex-ante exactly known to all potential bidders. In other words, the pair* $(I_0(X), p_0(X))$ *is measurable with respect to the* σ*-algebra* $\mathcal{F}_X = \sigma(X)$ *generated by* X.[1]
(c) The values themselves are independent within auctions, conditional on X, *and the auction specific covariates* X *and corresponding reservation prices* $p_0(X)$ *are independent and identically distributed across auctions, with joint support* $\mathbb{X} \times \mathbb{P}$.
(d) The reservation price and all the bids in each auction are observed by the econometrician.

Note that the finite conditional expectation conditions in part (a) of Assumption 1 guarantees that the conditional bid distribution has bounded support, as follows from Guerre et al. (2000, Footnote 8) with reference to Laffont et al. (1995). The other conditions guarantee that if there exists another auction with the same auction-specific covariates X, reservation price $p_0(X)$, absolutely continuous conditional value distribution $\Gamma(v|X)$ and number of potential bidders $I(X)$ such that its bid function is a.s. the same as (1), then with probability 1, $\Gamma(v|X) = \Gamma_0(v|X)$ for $v > p_0(X)$ and $I(X) = I_0(X)$.

It has been shown in Bierens and Song (2011b) that, with $I_*(X)$ the actual number of bidders,

$$\begin{aligned} \Gamma_0(p_0(X)|X) &= \frac{E[I_*(X)^2|X]}{E[I_*(X)|X]} - E[I_*(X)|X] \\ I_0(X) &= \frac{(E[I_*(X)|X])^2}{E[I_*(X)|X] + (E[I_*(X)|X])^2 - E[I_*(X)^2|X]}, \end{aligned} \tag{2}$$

which can be estimated consistently via nonparametric kernel regression estimates of $E[I_*(X)|X]$ and $E[I_*(X)^2|X]$. Therefore, we may treat $\Gamma_0(p_0(X)|X)$ and $I_0(X)$ as known to the econometrician.

[1] See Bierens (2004, Definition 1.9, p.22), for example. Consequently, $I_0(x)$ and $p_0(x)$ are nonrandom Borel measurable functions on $\mathbb{R}^d$.

Without loss of generality we may now assume that the potential bidders with value $V \leq p_0(X)$ issue an observable zero bid, so that the bid function becomes

$$\begin{aligned} \beta_0\left(v|X\right) &= \left(v - \frac{1}{\Gamma_0(v|X)^{I_0(X)-1}} \int_{p_0(X)}^{v} \Gamma_0(z|X)^{I_0(X)-1} dz\right) \\ &\quad \times \mathbf{1}\left(v > p_0(X)\right) \end{aligned} \tag{3}$$

where $\mathbf{1}\left(.\right)$ is the indicator function. Then given X the conditional distribution function of the bids in this auction, including zero bids, is

$$\Lambda_0(b|X) = \Pr\left[\beta_0\left(V|X\right) \leq b|X\right], \tag{4}$$

where V is a random drawing from $\Gamma_0(v|X)$.

2.2 Semi-nonparametric specification of the conditional value distribution

The problem with the nonparametric two-step approach of Guerre et al. (2000) is that it is difficult to display the nonparametric estimation results for $\gamma_0(v|x)$ if the dimension of x is larger than 1. Therefore, we propose to specify the log values as a linear model in the covariates. In particular, let $V_{i,\ell}$ be the value of potential bidder i in auction $\ell \in \{1, 2, ..., L\}$ with characteristics X_ℓ and assume that

Assumption 2. *For* $i = 1, 2, ..., I_0(X_\ell)$ *and* $\ell = 1, 2, ..., L$,

$$\ln(V_{i,\ell}) = \theta_0' X_\ell + \varepsilon_{i,\ell}, \tag{5}$$

where the error terms $\varepsilon_{i,\ell}$ *are i.i.d. within and across auctions, and are independent of* X_ℓ.

Then

$$\Gamma_0(v|X) = F_0(v.\exp(-\theta_0' X)) \tag{6}$$

where $F_0(z)$ is the distribution function of $Z_{i,\ell} = \exp(\varepsilon_{i,\ell})$, and

$$\gamma_0(v|X) = \exp(-\theta_0' X) f_0(v.\exp(-\theta_0' X))$$

with $f_0(z)$ the density of $F_0(z)$.

The finite conditional expectation conditions in part (a) of Assumption 1 now reads

$$\begin{aligned}\int_0^\infty v.\gamma_0(v|X)dv &= \int_0^\infty v\exp(-\theta_0'X)f_0(v.\exp(-\theta_0'X))dv \\ &= \exp(\theta_0'X)\int_0^\infty zf_0(z)dz < \infty \text{ a.s.,}\end{aligned}$$

hence we need to require that

Assumption 3. *The error terms* $\varepsilon_{i,\ell}$ *in model* (5) *satisfy* $E[\exp(\varepsilon_{i,\ell})] < \infty$,

and that $\Pr[\exp(\theta_0'X) < \infty] = 1$. A sufficient condition for the latter is that

Assumption 4. $E[X'X] < \infty$,

because then by Chebyshev's inequality

$$\begin{aligned}&\lim\sup_{M\to\infty} \Pr[\exp(\theta_0'X) > M] = \lim\sup_{M\to\infty} \Pr[\theta_0'X > \ln(M)] \\ &\leq \lim\sup_{M\to\infty} \Pr[||\theta_0||.||X|| > \ln(M)] \leq \lim\sup_{M\to\infty} \frac{||\theta_0||^2 E[X'X]}{(\ln(M))^2} = 0.\end{aligned}$$

It remains to show that F_0 and θ_0 are identified, given $\Gamma_0(v|X)$. Suppose that there exist an absolutely continuous distribution function F with density f and a parameter vector θ such that for all $v > p_0(X)$,

$$F_0(v.\exp(-\theta_0'X)) = F(v.\exp(-\theta'X)) \text{ a.s.}$$

Then with $z = v.\exp(-\theta'X)$,

$$F_0(z.\exp((\theta-\theta_0)'X)) = F(z) \text{ a.s.}$$

for all $z > p_0(X)\exp(-\theta'X)$. Next, let $F_0^{-1}(u)$ be the inverse of F_0, i.e.,

$$F_0^{-1}(u) = \inf_{F_0(z)=u} z \text{ for } 0 < u < 1.$$

Then $\exp((\theta-\theta_0)'X) = F_0^{-1}(F(z))/z$ a.s. for all $z > p_0(X)\exp(-\theta'X)$, hence

$$\exp((\theta-\theta_0)'X) = \lim_{z\to\infty} F_0^{-1}(F(z))/z \text{ a.s.}$$

Since $c = \lim_{z\to\infty} F_0^{-1}(F(z))/z$ is nonrandom, we now have $(\theta - \theta_0)'X = \ln(c)$ a.s., hence$(\theta - \theta_0)'(X - E(X)) = 0$ a.s. and therefore

$$(\theta - \theta_0)'E\left[(X - E(X))(X - E(X))'\right](\theta - \theta_0) = 0$$

Thus, if

Assumption 5. $\det[\mathrm{Var}(X)] > 0$[2]

then $\theta = \theta_0$ and $F(z) = F_0(z)$ for $z > p_0(X)\exp(-\theta_0'X)$.

Moreover, it is easy to verify that if in addition,

Assumption 6. $\Pr[p_0(X)\exp(-\theta_0'X) \le \kappa] > 0$ *for all* $\kappa > 0$,

then $F(z) = F_0(z)$ for all $z > 0$.

Summarizing, the following identification results hold.

Lemma 1. *Consider a first price auction with auction-specific characteristics* X*, reservation price* $p_0(X)$*, number of potential bidders* $I_0(X)$ *and conditional value distribution* $\Gamma(v|X) = F(v.\exp(-\theta'X))$*, such that the conditional bid distribution involved is the same as* (4)*. Then under Assumptions 1-6,* $\theta = \theta_0$ *and* $F \equiv F_0$.

2.3 Data-generating process

Substituting (6) in (3) the equilibrium bid function now becomes

$$\begin{aligned}
&\beta_0(v|X) \\
&= \left(v - \frac{1}{F_0(v.\exp(-\theta_0'X))^{I_0(X)-1}}\int_{p_0(X)}^{v} F_0(z.\exp(-\theta_0'X))^{I_0(X)-1}dz\right) \\
&\times \mathbf{1}(v > p_0(X)) \\
&= \left(v - \frac{\exp(\theta_0'X)}{F_0(v.\exp(-\theta_0'X))^{I_0(X)-1}}\int_{p_0(X)\exp(-\theta_0'X)}^{v\exp(-\theta_0'X))} F_0(y)^{I_0(X)-1}dy\right) \\
&\times \mathbf{1}(v > p_0(X))
\end{aligned}$$

[2]Note that this condition excludes a constant in X.

and the conditional distribution of a corresponding bids B (including zero bids) given X is

$$B|X \sim \beta_0(V|X)$$

with V a random drawing from $F_0(v.\exp(-\theta_0'X))$.

As is well known, V can be generated by solving $U = F_0(V.\exp(-\theta_0'X))$, where U is a random drawing from the uniform $[0,1]$ distribution, hence

$$V = \exp(\theta_0'X)F_0^{-1}(U)$$

where $F_0^{-1}(u)$ is the inverse of F_0:

$$F_0^{-1}(u) = \inf_{F_0(z)=u} z \text{ for } 0 < u < 1.$$

Moreover, it is also well-known that $Z = F_0^{-1}(U)$ is distributed as $F_0(z)$.

Thus, with $B_{\ell,j}$ for $j = 1, ..., I_\ell$ the bids in auction ℓ with characteristics X_ℓ, and given the number of bidders (including zero bidders) $I_\ell = I_0(X_\ell)$ and reservation price $p_\ell = p_0(X_\ell)$ for $\ell = 1, ..., L$, the conditional distribution of $B_{\ell,j}$ given X_ℓ is

$$\begin{aligned} B_{\ell,j}|X_\ell &\sim \exp(\theta_0'X_\ell)\left(Z_{\ell,j} - \frac{1}{F_0(Z_{\ell,j})^{I_\ell-1}}\int_{p_\ell\exp(-\theta_0'X_\ell)}^{Z_{\ell,j}} F_0(y)^{I_\ell-1}dy\right) \\ &\quad \times \mathbf{1}\left(Z_{\ell,j} > p_\ell\exp(-\theta_0'X_\ell)\right) \\ &= \exp(\theta_0'X_\ell)\eta\left(Z_{\ell,j}|F_0, I_\ell, p_\ell\exp(-\theta_0'X_\ell)\right), \end{aligned}$$

where the $Z_{\ell,j}$'s are independent random drawings from $F_0(z)$ and

$$\begin{aligned} \eta(z|F,I,p) &= \left(z - \frac{1}{F(z)^{I-1}}\int_p^z F(y)^{I-1}dy\right)\mathbf{1}(z > p) \qquad (7) \\ &= \left(p\frac{F(p)^{I-1}}{F(z)^{I-1}} + \frac{(I-1)}{F(z)^{I-1}}\int_p^z yF(y)^{I-2}f(y)dy\right)\mathbf{1}(z > p) \end{aligned}$$

with f the density of F. The second equality follows from integration by parts. The latter implies, together with Assumption 3, that

$$\exp(\theta_0'X_\ell)\eta\left(Z_{\ell,j}|F_0, I_\ell, p_\ell\exp(-\theta_0'X_\ell)\right) \le p_\ell + (I_0(X_\ell) - 1)\int_0^\infty zf_0(z)dz$$

so that

$$\Pr\left[B_{\ell,j} \le p_\ell + (I_\ell - 1)\int_0^\infty zf_0(z)dz\right] = 1$$

2.4 Identification via characteristic functions

As is well-known, the distribution of a bounded random variable is completely determined by the shape of its characteristic function in an arbitrary neighborhood of zero. Thus, consider the same alternative first price auction as in Lemma 1, and denote a typical bid in this auction by $B(\theta, F|X)$, i.e.,

$$B(\theta, F|X) = \exp(\theta' X)\eta\left(Z(F)|F, I_0(X), p_0(X)\exp(-\theta' X)\right), \tag{8}$$

where $Z(F)$ is a random drawings from $F(z)$. Denote the conditional characteristic function of $B(\theta, F|X)$ given X by

$$\begin{aligned}
&\varphi(t|\theta, F, X) = E\left[\exp\left(\mathbf{i}.t.B(\theta, F|X)\right)|X\right] \qquad (9)\\
&= \int_0^\infty \exp\left(\mathbf{i}.t.\exp(\theta' X)\eta\left(z|F, I_0(X), p_0(X)\exp(-\theta' X)\right)\right) f(z)dz\\
&= \psi(t.\exp(\theta' X)|F, I_0(X), p_0(X)\exp(-\theta' X)),
\end{aligned}$$

where $\mathbf{i} = \sqrt{-1}$ and

$$\begin{aligned}
&\psi(\xi|F, I, p) = \int_0^\infty \exp(\mathbf{i}.\xi.\eta(z|F, I, p) f(z)dz\\
&= F(p) + \int_p^\infty \exp\left(\mathbf{i}.\xi.\left(z - \frac{1}{F(z)^{I-1}}\int_p^z F(y)^{I-1}dy\right)\right) f(z)dz \qquad (10)
\end{aligned}$$

Lemma 2. *Suppose that for an arbitrary constant $c > 0$,*

$$\sup_{|t|<c} |\varphi(t|\theta, F, X) - \varphi(t|\theta_0, F_0, X)| = 0 \ a.s. \tag{11}$$

Then under Assumptions 1-6, $\theta = \theta_0$ and $F \equiv F_0$.

Proof. Let B be a bounded random variable with characteristic function $\varphi(t) = E[\exp(\mathbf{i}.t.B)]$, and note that by the boundedness of B, $\varphi(t) = \sum_{m=0}^\infty (\mathbf{i}^m t^m/m!)E[B^m]$, hence $d^k\varphi(t)/(dt)^k\big|_{t=0} = \mathbf{i}^k E[B^k]$. Next, let B_* be a random variable with characteristic function $\psi(t) = E[\exp(\mathbf{i}.t.B_*)]$ such that for all $t \in (-c, c)$, $\varphi(t) = \psi(t)$. Then $d^k\psi(t)/(dt)^k\big|_{t=0} = \mathbf{i}^k E[B^k]$, which implies that for $k = 1, 2, 3, ...$, $E[B_*^k] = E[B^k]$. Consequently, $\psi(t) = \sum_{m=0}^\infty (\mathbf{i}^m t^m/m!)E[B^m] = \varphi(t)$ for all t, which implies that $B \sim B_*$.

This argument trivially carries over to conditional distributions and conditional characteristic functions, hence (11) implies

$$\sup_t |\varphi(t|\theta, F, X) - \varphi(t|\theta_0, F_0, X)| = 0 \text{ a.s.},$$

which in its turn implies that conditional on X, $B(\theta, F|X) \sim B(\theta_0, F_0|X)$. The latter has conditional distribution function (4), which by Lemma 1 proves Lemma 2. ■

Lemma 3. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded one-to-one mappings with Borel measurable inverse,*[3] *and denote*

$$\begin{aligned} \varphi(\xi|\theta, F) &= E\left[\varphi(\xi_1|\theta, F, X) \exp\left(\mathbf{i}.\xi_2'\Phi(X)\right)\right], \\ \xi &= (\xi_1, \xi_2')' \in \mathbb{R}^{d+1}. \end{aligned} \tag{12}$$

Let $\overline{\xi}_1, \overline{\xi}_2, ..., \overline{\xi}_{d+2}$ be arbitrary positive constants and denote

$$\Xi = \times_{m=1}^{d+1}[-\overline{\xi}_m, \overline{\xi}_m]. \tag{13}$$

If

$$\int_\Xi |\varphi(\xi|\theta, F) - \varphi(\xi|\theta_0, F_0)|^2 \, d\xi = 0 \tag{14}$$

then under Assumptions 1-6, $\theta = \theta_0$ and $F \equiv F_0$.

Proof. Note that $\varphi(\xi|\theta, F)$ is the joint characteristic function of $B(\theta, F|X)$ and $\Phi(X)$. Moreover, note that by the continuity of characteristic functions, (14) is equivalent to

$$\sup_{\xi \in \Xi} |\varphi(\xi|\theta, F) - \varphi(\xi|\theta_0, F_0)| = 0.$$

Then it follows similar to the proof Lemma 1 that

$$(B(\theta, F|X), \Phi(X)) \sim (B(\theta_0, F_0|X), \Phi(X))$$

hence, conditional on $\Phi(X)$,

$$B(\theta, F|X) \sim B(\theta_0, F_0|X). \tag{15}$$

But $\Phi(X)$ generates the same σ-algebra as X, and therefore (15) holds conditional on X as well. The lemma under review now follows from Lemma 1. ■

[3] If X is already bounded then we may choose $\Phi(x) = x$.

3 Integrated Simulated Moments Sieve Estimation

3.1 The objective function

The empirical counterparts of the characteristic functions $\varphi(\xi|\theta_0, F_0)$ and $\varphi(\xi|\theta, F)$ in (14) are

$$\widehat{\varphi}_L(\xi) = \frac{1}{L}\sum_{\ell=1}^{L}\left(\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(\mathbf{i}.\xi_1.B_{\ell,j}\right)\right)\exp\left(\mathbf{i}.\xi_2'\Phi(X_\ell)\right) \tag{16}$$

where the $B_{\ell,j}$'s are the observed bids, and

$$\widetilde{\varphi}_L(\xi|\theta, F) = \frac{1}{L}\sum_{\ell=1}^{L}\left(\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(\mathbf{i}.\xi_1.\widetilde{B}_{\ell,j}(\theta, F)\right)\right)\exp\left(\mathbf{i}.\xi_2'\Phi(X_\ell)\right) \tag{17}$$

respectively, where

$$\widetilde{B}_{\ell,j}(\theta, F) = \exp(\theta' X_\ell)\eta\left(\widetilde{Z}_{\ell,j}(F)|F, I_\ell, p_\ell\exp(-\theta' X_\ell)\right) \tag{18}$$

with the $\widetilde{Z}_{\ell,j}(F)$'s independent random drawings from $F(z)$.

It is not hard to verify that $\lim_{L\to\infty} E[|\widehat{\varphi}_L(\xi) - \varphi(\xi|\theta_0, F_0)|^2] = 0$ pointwise in ξ and similarly, $\lim_{L\to\infty} E[|\widetilde{\varphi}_L(\xi|\theta, F) - \varphi(\xi|\theta, F)|^2] = 0$ pointwise in ξ, θ and F. Therefore, it follows from Chebyshev's inequality and the dominated convergence theorem that

$$\widehat{Q}_L(\theta, F) = \int_\Xi |\widetilde{\varphi}_L(\xi|\theta, F) - \widehat{\varphi}_L(\xi)|^2 d\xi \tag{19}$$

converges in probability to

$$\overline{Q}(\theta, F) = \int_\Xi |\varphi(\xi|\theta, F) - \varphi(\xi|\theta_0, F_0)|^2\, d\xi \tag{20}$$

as $L \to \infty$, pointwise in θ and F. However, it can also be shown that

$$\widehat{Q}_L(\theta, F) \overset{\text{a.s.}}{\to} \overline{Q}(\theta, F)$$

as $L \to \infty$, uniformly in θ and F confined to compact spaces.

In view of Lemma 3 this result suggests to estimate θ_0 and F_0 by minimizing $\widehat{Q}_L(\theta, F)$ to θ and F. Of course, in practice this minimization problem is not feasible. The way to solve this problem is sieve estimation.

3.2 Sieve estimation

As usual for any nonlinear estimation method, we need to specify compact parameter spaces Θ and $\mathcal{F}$ for θ_0 and F_0, respectively. Moreover, sieve estimation requires the construction of sieve spaces of $\mathcal{F}$. Thus,

Assumption 7. $\theta_0 \in \Theta$, *where* Θ *is a compact subset of* $\mathbb{R}^d$,

and

Assumption 8.
(*a*) $F_0 \in \mathcal{F}$, *where* $\mathcal{F}$ *is a compact metric space of absolutely continuous distribution functions on* $(0, \infty)$ *endowed with the "sup" metric* $||F_1 - F_2||_{\sup} = \sup_{z>0} |F_1(z) - F_2(z)|$;
(*b*) *there exists an increasing sequence* $\{\mathcal{F}_n\}_{n=1}^{\infty}$ *of compact subspaces of* $\mathcal{F}$ *such that for each* n *the computation of the sieve estimator*

$$(\widehat{\theta}_{n,L}, \widehat{F}_{n,L}) = \arg \min_{\theta \in \Theta, F \in \mathcal{F}_n} \widehat{Q}_L(\theta, F) \tag{21}$$

is feasible, and $\{\mathcal{F}_n\}_{n=1}^{\infty}$ *is dense in* $\mathcal{F}$, *i.e.*,

$$\mathcal{F} = \overline{\cup_{n=1}^{\infty} \mathcal{F}_n}. \tag{22}$$

How to construct $\mathcal{F}$ and the sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of sieve spaces will be shown in Section 4. Note that the bar in (22) denotes the closure. Also note that the condition (22) is equivalent to the condition that for each $F \in \mathcal{F}$ there exists a sequence of distribution functions $F_n \in \mathcal{F}_n$ such that $\lim_{n\to\infty} ||F - F_n||_{\sup} = 0$.

Now suppose that

$$\sup_{\theta \in \Theta, F \in \mathcal{F}} \left| \widehat{Q}_L(\theta, F) - \overline{Q}(\theta, F) \right| \stackrel{\text{a.s.}}{\to} 0 \text{ as } L \to \infty; \tag{23}$$

$$(\theta_0, F_0) = \arg \min_{\theta \in \Theta, F \in \mathcal{F}} \overline{Q}(\theta, F) \text{ is unique}; \tag{24}$$

$$\overline{Q}(\theta, F) \text{ is continuous in } (\theta_0, F_0). \tag{25}$$

Then the following standard consistency result for sieve estimators holds.

Theorem 1. *Let n_L be an arbitrary subsequence of L such that* $\lim_{L\to\infty} n_L = \infty$. *Under Assumptions 7 and 8 and the conditions* (23), (24) *and* (25), $\widehat{\theta}_{n_L,L} \stackrel{\text{a.s.}}{\to} \theta_0$ *and* $\sup_{z>0} |\widehat{F}_{n_L,L}(z) - F_0(z)| \stackrel{\text{a.s.}}{\to} 0$ *as* $L \to \infty$.

Proof. Let $F_n \in \mathcal{F}_n$ be such that $\lim_{n\to\infty} ||F_n - F_0||_{\sup} = 0$. Then

$$\begin{aligned}
0 &\leq \overline{Q}(\widehat{\theta}_{n_L,L}, \widehat{F}_{n_L,L}) - \overline{Q}(\theta_0, F_0) \\
&= \overline{Q}(\widehat{\theta}_{n_L,L}, \widehat{F}_{n,L}) - \widehat{Q}_L(\widehat{\theta}_{n_L,L}, \widehat{F}_{n,L}) + \widehat{Q}_L(\widehat{\theta}_{n_L,L}, \widehat{F}_{n,L}) - \overline{Q}(\theta_0, F_0) \\
&\leq \sup_{\theta\in\Theta, F\in\mathcal{F}_{n_L}} \left| \widehat{Q}_L(\theta, F) - \overline{Q}(\theta, F) \right| + \widehat{Q}_L(\theta_0, F_{n_L}) - \overline{Q}(\theta_0, F_0) \\
&\leq 2 \sup_{\theta\in\Theta, F\in\mathcal{F}} \left| \widehat{Q}_L(\theta, F) - \overline{Q}(\theta, F) \right| + \overline{Q}(\theta_0, F_{n_L}) - \overline{Q}(\theta_0, F_0) \stackrel{\text{a.s.}}{\to} 0
\end{aligned}$$

where the latter follows from the conditions (23) and (25). Hence,

$$\overline{Q}(\widehat{\theta}_{n_L,L}, \widehat{F}_{n_L,L}) \stackrel{\text{a.s.}}{\to} \overline{Q}(\theta_0, F_0). \tag{26}$$

It is now a standard exercise to verify from the conditions (24) and (25) that (26) implies

$$||\widehat{\theta}_{n_L,L} - \theta_0|| + ||\widehat{F}_{n_L,L} - F_0||_{\sup} \stackrel{\text{a.s.}}{\to} 0,$$

which proves the lemma. ■

Note that condition (24) follows from Lemma 3. In the next subsection it will be shown that conditions (23) and (25) hold under Assumptions 1-7.

3.3 Continuity and uniform convergence

Let $\varphi_0(\xi) = \varphi(\xi|\theta_0, F_0)$ and observe from (16) that $\varphi_0(\xi) = E[\widehat{\varphi}_L(\xi)]$ and $\widehat{\varphi}_L(\xi) - \varphi_0(\xi) = \frac{1}{L}\sum_{\ell=1}^{L}(A_\ell(\xi) - E[A_\ell(\xi)])$, where

$$A_\ell(\xi) = \left(\frac{1}{I_\ell} \sum_{j=1}^{I_\ell} \exp\left(\mathbf{i}.\xi_1 B_{\ell,j}\right) \right) \exp\left(\mathbf{i}.\xi_2' \Phi(X_\ell)\right)$$

Since by Assumption 1(b), $I_\ell = I_0(X_\ell)$ and $p_\ell = p_0(X_\ell)$ where $I_0(x)$ and $p_0(x)$ are non-random Borel measurable functions, and the X_ℓ's are i.i.d., it follows that $\text{Re}[A_\ell(\xi)]$ and $\text{Im}[A_\ell(\xi)]$ are i.i.d. random functions, which are obviously a.s. continuous and uniformly bounded in ξ. It follows therefore from Jennrich's (1969) uniform strong law of large numbers that, with Ξ defined by (13),

$$\begin{aligned}\sup_{\xi\in\Xi}\left|\widehat{\varphi}_L(\xi)-\varphi_0(\xi)\right|^2 &\leq \sup_{\xi\in\Xi}\left(\frac{1}{L}\sum_{\ell=1}^{L}\left(\mathrm{Re}[A_\ell(\xi)]-E\left(\mathrm{Re}[A_\ell(\xi)]\right)\right)\right)^2\\ &\quad+\sup_{\xi\in\Xi}\left(\frac{1}{L}\sum_{\ell=1}^{L}\left(\mathrm{Im}[A_\ell(\xi)]-E\left(\mathrm{Im}[A_\ell(\xi)]\right)\right)\right)^2\\ &\overset{\text{a.s.}}{\to} 0\end{aligned}$$

as $L\to\infty$.

To prove that

$$\sup_{(\xi,\theta,F)\in\Xi\times\Theta\times\mathcal{F}}\left|\widetilde{\varphi}_L(\xi|\theta,F)-\varphi(\xi|\theta,F)\right|\overset{\text{a.s.}}{\to}0 \tag{27}$$

it suffices to show that

$$\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(\mathbf{i}.\xi_1.\widetilde{B}_{\ell,j}(\theta,F)\right)$$

is a.s. continuous in ξ_1, θ and F, which is true if the simulated bids $\widetilde{B}_{\ell,j}(\theta,F)$ are a.s. continuous in θ and F. Then (27) follows from Jennrich's (1969) uniform strong law of large numbers.

To show the a.s. continuity of $\widetilde{B}_{\ell,j}(\theta,F)$, recall from (7) and (18) that

$$\begin{aligned}\widetilde{B}_{\ell,j}(\theta,F) &= \exp(\theta'X_\ell)\left(Z_{\ell,j}-\frac{1}{F(Z_{\ell,j})^{I_\ell-1}}\int_{p_\ell\exp(-\theta'X_\ell)}^{Z_{\ell,j}}F(y)^{I_\ell-1}dy\right)\\ &\quad\times\mathbf{1}\left(Z_{\ell,j}>p_\ell\exp(-\theta'X_\ell)\right)\end{aligned}$$

where the $Z_{\ell,j}$'s are independent random drawings from $F(z)$, which can be generated by solving $F(Z_{\ell,j})=U_{\ell,j}$ where the $U_{\ell,j}$'s are independent random drawings from then uniform $[0,1]$ distribution. Then

$$\begin{aligned}\widetilde{B}_{\ell,j}(\theta,F) &= \exp(\theta'X_\ell)\left(F^{-1}(U_{\ell,j})-\frac{1}{U_{\ell,j}^{I_\ell-1}}\int_{p_\ell\exp(-\theta'X_\ell)}^{F^{-1}(U_{\ell,j})}F(y)^{I_\ell-1}dy\right)\\ &\quad\times\mathbf{1}\left(U_{\ell,j}>F\left(p_\ell\exp(-\theta'X_\ell)\right)\right)\end{aligned}$$

Given θ, it follows similar to Lemma 3 in Bierens and Song (2011a) that $\widetilde{B}_{\ell,j}(\theta, F)$ is a.s. continuous in F, whereas it is almost trivial that given F, $\widetilde{B}_{\ell,j}(\theta, F)$ is a.s. continuous in θ. It follows now straightforwardly that conditions (23) and (25) hold, i.e.,

Lemma 4. *Under Assumptions 1-8,* $\sup_{\theta\in\Theta,F\in\mathcal{F}}\left|\widehat{Q}_L(\theta, F) - \overline{Q}(\theta, F)\right| \overset{\text{a.s.}}{\to} 0$ *as* $L \to \infty$. *Moreover,* $\overline{Q}(\theta, F)$ *is continuous on* $\Theta \times \mathcal{F}$.

Thus, under Assumptions 1-8 the conditions of Theorem 1 hold.

4 The Compact Metric Space $\mathcal{F}$ and Its Sieve Spaces

4.1 Distribution functions on the unit interval

Any distribution function $F(z)$ on $(0, \infty)$ can be written as

$$F(z) = H(G(z))$$

where $G(z)$ is an a priori chosen absolutely continuous distribution function with support $(0, \infty)$ and

$$H(u) = F(G^{-1}(u))$$

is a distribution function on the unit interval $[0, 1]$. Moreover, if $F(z)$ is absolutely continuous with density $f(z)$ then $H(u)$ is absolutely continuous with density

$$h(u) = f(G^{-1}(u))/g(G^{-1}(u)),$$

where $g(z)$ is the density of $G(z)$.

For example, let $G(z) = 1 - \exp(-z)$. Then $G^{-1}(u) = \ln(1/(1 - u))$ and thus $H(u) = F(\ln(1/(1 - u)))$, $h(u) = f(\ln(1/(1 - u)))/(1 - u)$.

It has been shown by Bierens (2008) that any density function $h(u)$ on $[0, 1]$ can be written as

$$h(u) = \frac{(1 + \sum_{k=1}^{\infty} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{\infty} \delta_k^2} \text{ a.e. on } [0, 1] \tag{28}$$

where $\sum_{k=1}^{\infty} \delta_k^2 < \infty$ and the $\rho_k(u)$'s are orthonormal Legendre polynomials of order k. These polynomials can be constructed recursively by the three-term recursive relation

$$\rho_k(u) = \frac{\sqrt{2k-1}\sqrt{2k+1}}{n}(2u-1)\rho_{k-1}(u) - \frac{(k-1)\sqrt{2k+1}}{k\sqrt{2k-3}}\rho_{k-2}(u)$$

for $k \geq 2$, starting from $\rho_0(u) = 1$, $\rho_1(u) = \sqrt{3}(2u-1)$.

The representation (28) is based on the fact that the Legendre polynomials form a complete orthonormal sequence in the Hilbert space $L^2(0,1)$. However, the main problem with this representation is that these polynomials have to be computed recursively so that $h(u)$ has no closed form expression, and neither has the corresponding distribution function $H(u) = \int_0^u h(v)dv$. The same applies to the density and distribution function representations on the basis of Hermite polynomials advocated by Gallant and Nychka (1987).

The sequence of Legendre polynomials is not the only complete orthonormal sequence in $L^2(0,1)$. As is well-known, the Fourier series $\rho_0(u) \equiv 1$, $\rho_k(u) = \sqrt{2}\sin(2k\pi u)$ if $k \in \mathbb{N}$ is odd, $\rho_k(u) = \sqrt{2}\cos(2k\pi u)$ if $k \in \mathbb{N}$ is even, is complete in $L^2(0,1)$, and the same applies to the related cosine series $\rho_0(u) \equiv 1$, $\rho_k(u) = \sqrt{2}\cos(k\pi u)$ for $k \in \mathbb{N}$. The advantage of using the cosine series instead of the Legendre polynomials is that then the series representations of $h(u)$ and $H(u)$ have closed forms. In particular, it has been shown in Bierens (2011) that

Lemma 5. *For an arbitrary density function $h(u)$ on $[0,1]$ with corresponding distribution function $H(u)$ there exist possibly uncountable many sequences $\delta = \{\delta_m\}_{m=1}^{\infty}$ satisfying $\sum_{m=1}^{\infty} \delta_m^2 < \infty$ such that almost everywhere (a.e.) on $(0,1)$,*

$$h(u) = h(u|\delta) = \frac{\left(1 + \sum_{k=1}^{\infty} \delta_k \sqrt{2}\cos(k\pi u)\right)^2}{1 + \sum_{m=1}^{\infty} \delta_m^2}, \tag{29}$$

$$\begin{aligned} H(u) &= H(u|\delta) \\ &= u + \frac{1}{1 + \sum_{i=1}^{\infty} \delta_i^2}\left[2\sqrt{2}\sum_{k=1}^{\infty} \delta_k \frac{\sin(k\pi u)}{k\pi} + \sum_{k=1}^{\infty} \delta_k^2 \frac{\sin(2k\pi u)}{2k\pi}\right. \\ &\left. +2\sum_{k=2}^{\infty}\sum_{m=1}^{k-1} \delta_k\delta_m \frac{\sin((k+m)\pi u)}{(k+m)\pi} + 2\sum_{k=2}^{\infty}\sum_{m=1}^{k-1} \delta_k\delta_m \frac{\sin((k-m)\pi u)}{(k-m)\pi}\right]. \end{aligned} \tag{30}$$

The result for $H(u)$ follows straightforwardly from (29) and the well-known sine-cosine formulas. The reason for the non-uniqueness of the δ_k's is the square in (29), as pointed out in Bierens (2008).

Moreover, it has been shown in Bierens (2008, 2011) that the following results hold.

Lemma 6. *Given an a priori chosen positive sequence* $\{\overline{\delta}_k\}_{k=1}^{\infty}$ *satisfying* $\sum_{k=1}^{\infty} \overline{\delta}_k^2 < \infty$, *let*

$$\begin{aligned}
\Delta &= \times_{m=1}^{\infty}[-\overline{\delta}_m, \overline{\delta}_m],\ \Delta_n = \left(\times_{m=1}^{n}[-\overline{\delta}_m, \overline{\delta}_m]\right) \times \left(\times_{m=n+1}^{\infty}\{0\}\right), \\
\mathcal{D} &= \{h(u|\delta) : \delta \in \Delta\},\ \mathcal{D}_n = \{h(u|\delta) : \delta \in \Delta_n\}, \\
\mathcal{H} &= \left\{H(u) = \int_0^u h(x)dx : h \in \mathcal{D}\right\},\ \mathcal{H}_n = \left\{H(u) = \int_0^u h(x)dx : h \in \mathcal{D}_n\right\}
\end{aligned}$$

Endow the space Δ *with the metric*

$$||\delta_1 - \delta_2|| = \sqrt{\sum_{m=1}^{\infty}(\delta_{1,m} - \delta_{2,m})^2},$$

where $\delta_i = \{\delta_{i,m}\}_{m=1}^{\infty}$ *for* $i = 1, 2$, *the space* $\mathcal{D}$ *with the* L^1 *metric*

$$||h_1 - h_2||_{L^1} = \int_0^1 |h_1(u) - h_2(u)|du$$

and the space $\mathcal{H}$ *with the sup metric*

$$||H_1 - H_2||_{\sup} = \sup_{0 \le u \le 1} |H_1(u) - H_2(u)|.$$

Then Δ, $\mathcal{D}$ *and* $\mathcal{H}$ *are compact. Moreover* $\Delta = \overline{\cup_{n=1}^{\infty}\Delta_n}$, $\mathcal{D} = \overline{\cup_{n=1}^{\infty}\mathcal{D}_n}$ *and* $\mathcal{H} = \overline{\cup_{n=1}^{\infty}\mathcal{H}_n}$.

4.2 The space $\mathcal{F}$ and its sieve spaces $\mathcal{F}_n$

Consequently, if

Assumption 9. *The metric space* $\mathcal{F}$ *and its sieve spaces* $\mathcal{F}_n$ *are chosen as*

$$\mathcal{F} = \{F(.) = H(G(.)) : H \in \mathcal{H}\},\ \mathcal{F}_n = \{F(.) = H(G(.)) : H \in \mathcal{H}_n\},$$

respectively, where G is an a priori chosen absolutely continuous distribution function with support $(0, \infty)$,

then part (b) of Assumption 8 holds. Moreover, if in addition

Assumption 10. *The space* $\Delta = \times_{m=1}^{\infty}[-\overline{\delta}_m, \overline{\delta}_m]$ *is chosen such that* $H_0(u) = F_0(G^{-1}(u)) \in \mathcal{H}$,

then part (a) of Assumption 8 holds. Furthermore, recall that under Assumptions 1-8 the conditions (23), (24) and (25) in Theorem 1 hold. Thus, Theorem 1 now reads as follows.

Theorem 2. *Let n_L be an arbitrary subsequence of L such that* $\lim_{L\to\infty} n_L = \infty$. *Under Assumptions 1-7 and 9-10,* $\widehat{\theta}_{n_L,L} \stackrel{\text{a.s.}}{\to} \theta_0$ *and* $\sup_{z>0} |\widehat{F}_{n_L,L}(z) - F_0(z)| \stackrel{\text{a.s.}}{\to} 0$ *as* $L \to \infty$.

5 Concluding Remarks

So far we have only focused on the strong consistency of the sieve estimators $\widehat{\theta}_{n_L,L}$ and $\widehat{F}_{n_L,L}$ of the Euclidean parameter vector θ_0 and the distribution function F_0 in the semi-nonparametric model $\Gamma_0(v|X) = F_0(v.\exp(-\theta_0'X))$ for the conditional value distribution. Along the lines in Bierens (2011) it is possible to set forth further conditions such that $\sqrt{L}(\widehat{\theta}_{n_L,L} - \theta_0) \stackrel{\text{d}}{\to} N_d(0, \Sigma)$ as $L \to \infty$. This will be done in due course. Moreover, the conditional moment test for the validity of the first-price auction model in Bierens and Song (2011a) can be generalized to an Integrated Conditional Moment (ICM) test of the first-price auction model under review, similar to the ICM test of Bierens (1982) and Bierens and Ploberger (1997). Also this will be done in due course.

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