

EMBEDDING AN ANALYTIC EQUIVALENCE RELATION IN THE TRANSITIVE CLOSURE OF A BOREL RELATION

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ABSTRACT. The transitive closure of a reflexive, symmetric, analytic relation is an analytic equivalence relation. Does some smaller class contain the transitive closure of every reflexive, symmetric, closed relation? An essentially negative answer is provided here. Every analytic equivalence relation on an arbitrary Polish space is Borel embeddable in the transitive closure of the union of two smooth Borel equivalence relations on that space. In the case of the Baire space, the two smooth relations are closed and the embedding is homeomorphic.

1. INTRODUCTION

This note answers a question in descriptive set theory that arises in the context of the Bayesian theory of decisions and games. It concerns the notion of common knowledge, formalized by Robert Aumann [1976]. For an event A that is represented as a subset of a measurable space Ω , Aumann defines the event that an agent *knows* A to be the event $A \setminus [\Omega \setminus A]_{\mathcal{P}}$, where \mathcal{P} is the agent's *information partition* of Ω .¹ If \mathcal{P} is the meet of individual agents' information partitions (in the lattice of partitions where $\mathcal{P}' \leq \mathcal{P}'' \iff \mathcal{P}''$ refines \mathcal{P}'), then Aumann defines

$$(1.1) \quad A \setminus [\Omega \setminus A]_{\mathcal{P}}$$

to be the event that A is common knowledge among the agents.²

Aumann restricts attention to the case that Ω is countable (or that the Borel σ -algebra on Ω is generated by the elements of a countable partition), so that measurability issues do not arise. But, otherwise, the passage from information partitions to a common-knowledge partition is very badly behaved, as is the passage from an information partition \mathcal{P} and an event A to the related event that A is known according to \mathcal{P} . For example, let X be an arbitrary subset of $(0, 1)$, and let $\Omega = [0, 2]$. Consider two agents, whose information partitions are $\mathcal{P}_1 = \{\{\omega, \omega + 1\} \mid$

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¹ $[A]_{\mathcal{P}}$ denotes $\bigcup\{\pi \mid \pi \in \mathcal{P} \text{ and } \pi \cap A \neq \emptyset\}$, the saturation of A with respect to \mathcal{P} . If E is an equivalence relation, then $[A]_E$ denotes the saturation of A with respect to the partition induced by E . Aumann's definition corresponds to the truth condition for $\Box A$ in Kripke [1959].

²Aumann sketches an argument—reminiscent of a general principle in proof theory (cf. Pohlers [2009, Lemma 6.4.8, p. 89])—that this definition is equivalent to the intuitive, recursive definition of common knowledge: that A has occurred and that, for all $n \in \mathbb{N}$, both agents know... that both agents know (n times) that A has occurred.

$\omega \in X\} \cup \{\{\omega\} \mid \omega \notin X\}$ and $\mathcal{P}_2 = \{\{\omega\} \mid \omega < 1\} \cup \{[1, 2]\}$. Then $X \cup [1, 2]$ is the block of the common-knowledge partition that includes the block $[1, 2]$ in \mathcal{P}_2 . From information partitions composed of the simplest events—singletons, pairs, and a closed interval—we have passed to a common-knowledge partition with a block that, depending on what is X , might even be outside the projective hierarchy. Knowledge of an event by individual agent is likewise problematic. In the present example, the event that agent 1 knows $(0, 1)$ is $(0, 1) \setminus X$.

These measurability problems dictate that information partitions should be represented as equivalence relations. If E_1 and E_2 are Σ_1^1 (that is, analytic) equivalence relations, then the meet of the partitions that they induce is induced by the transitive closure of their union. This transitive closure is also a Σ_1^1 equivalence relation.³ Moreover, if an information partition is represented by a Σ_1^1 relation and an event is Π_1^1 (that is, co-analytic), then knowledge of the event according to the information partition is also a Π_1^1 event.⁴

This observation implies that knowledge of a Borel event is a universally measurable event—surely a threshold condition for incorporating the analysis of knowledge into Bayesian theory. However, it is easy to envision a noncooperative Bayesian game situation in which a player must condition on the event that some other event is common knowledge.⁵ Then the game theorist needs to model the event that both players' receipt of their respective signals is common knowledge as being a Δ_1^1 (that is, Borel) event, not just a Π_1^1 event. It might be thought possible to avoid this difficulty by making tighter modeling assumptions regarding both the agents' information structures and also the event to which the common-knowledge operator is to be applied.

In particular, in most applications to Bayesian decision theory and game theory, it is reasonable to specify each agent's information as a Δ_1^1 equivalence relation, or even as a smooth or closed Borel relation rather than as an arbitrary Σ_1^1 equivalence

³Composition is defined with a single existential quantifier, and thus takes a pair of Σ_1^1 relations to a Σ_1^1 relation. The countable union of Σ_1^1 relations is Σ_1^1 . Cf. Moschovakis [2009, Theorem 2B.2, p. 54].

⁴This is equivalent, by (1.1), to the fact that the saturation of a Σ_1^1 set with respect to a Σ_1^1 equivalence relation is Σ_1^1 . This latter fact is true because the saturation is defined with a single existential quantifier.

⁵In a game-theoretic analysis of the “coordinated-attack problem” (cf. Rubinstein [1989]), there is an equilibrium in which two players will take complementary actions iff it is common knowledge that each has received a signal. The two signals are privately observed by the respective players and are highly, but imperfectly, correlated. Thus the players must exchange infinitely many messages to one another (that is, ‘I have received the signal’, ‘I have received your confirmation of receipt of the signal’, ‘I have received your confirmation of receipt of my confirmation of receipt of the signal’, . . .) in order to attain common knowledge. If the communications channel closes at any stage, then common knowledge is not reached, and the action will not be taken. Closure of the channel is irreversible. The game can be elaborated by supposing that the stochastic distribution of payoffs from coordination is dependent on whether common knowledge has been achieved. (For example, the communication channel may have to be used, and therefore must not be broken, in order to accomplish the task. The probability that the channel being open at that time is positive if common knowledge has been achieved, but is zero otherwise.) Thus, in order for the game theorist to prove that coordination conditional on attaining common knowledge is an equilibrium, the expected payoff from coordination conditional on common knowledge being attained must be well defined.

relation.⁶ Thus it may be asked: if the graphs of E_1 and E_2 are in Δ_1^1 or in some smaller class, then how is the graph of the transitive closure of $E_1 \cup E_2$ restricted?

It will be shown here that no significant restriction of the common-knowledge partition is implied by such restriction of agents' information partitions. This finding is not surprising, since restricting the complexity of individuals' equivalence relations does not obviate the use of an existential quantifier to define the transitive closure of a relation. Nevertheless, the syntactic form of a specific description of a set does not determine the intrinsic complexity of the set, so it needs to be shown that common-knowledge equivalence relations derived from Borel equivalence relations are not lower in the projective hierarchy, as a class, than their definition would suggest. Moreover, proposition 2.1 will show that being the union of finitely many (in fact, of fewer than 2^{\aleph_0}) Borel equivalence relations—that is, representability in the form, of which the transitive closure is an equivalence relation specifying common-knowledge—is a stronger property than representation as the transitive closure of an arbitrary Borel, reflexive, symmetric relation.

To define the transitive closure of $R \subseteq \Omega \times \Omega$, let $R^{(1)} = R$ and $R^{(n+1)} = RR^{(n)}$ (that is, the composition of relations R and $R^{(n)}$). Denote the transitive closure of R by $R^+ = \bigcup_{n \in \mathbb{N}_+} R^{(n)}$. It will be proved here that, if Ω is a Polish space and $E_0 \subseteq \Omega \times \Omega$ is a Σ_1^1 equivalence relation, then there are smooth Δ_1^1 equivalence relations E_1 and E_2 and a Δ_1^1 subset Z of Ω , such that $(E_1 \cup E_2)^+ \upharpoonright Z$ is Borel equivalent to E_0 .⁷ If Ω is the Baire space, then E_1 and E_2 can be taken to be closed, Z can be taken to be open, and the Borel equivalence can be taken to be a homeomorphic equivalence.

2. THE CASE OF THE BAIRE SPACE

First take Ω to be the Baire space, $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$.⁸ Define subsets X and Y of \mathcal{N} by $X = \{\alpha \mid \alpha_0 > 0\}$ and $Y = \{\alpha \mid \alpha_0 = 0\}$. X and Y are both homeomorphic to \mathcal{N} , and homeomorphisms $f: X \rightarrow Y$ and $g: Y \times Y \times Y \rightarrow Y$ are routine to construct.⁹ Each of X and Y is both open and closed in \mathcal{N} . It follows that, if Z is either X or Y , then $A \subseteq Z$ is open (resp. closed, Borel, Σ_1^1) as a subset of A iff it is open (resp. closed, Borel, Σ_1^1) as a subset of Z . This invariance to the ambient space extends to product spaces. (For example a subset of $X \times Y$ is closed in $X \times Y$ iff it is closed in $\mathcal{N} \times \mathcal{N}$.) In subsequent discussions, subsets of these subspaces will be characterized (for example, as being closed) without mentioning the subspace.

Theorem 2.1. *If $E \subseteq X \times X$ is a Σ_1^1 equivalence relation, then there are equivalence relations I and J on $\mathcal{N} \times \mathcal{N}$, each of which has a closed graph, such that $E = (I \cup J)^+ \upharpoonright X$.*

Before proceeding to the proof of this theorem, note that $I \cup J$ is a closed, reflexive, symmetric relation. Thus, theorem 2.1 has the following corollary.

⁶Smoothness (also called tameness) and closedness are co-extensive for equivalence relations on standard Borel spaces. Cf. Harrington et al. [1990, proof of Theorem 1.1, p. 920]. Standard Borel spaces are defined below, in section 4.

⁷ $R \upharpoonright Z = R \cap (Z \times Z)$. Let restriction take precedence over Boolean operations. For example, $X \cup R \upharpoonright Z \cap Y$ means $X \cup (R \upharpoonright Z) \cap X$.

⁸ $\mathbb{N} = \{0, 1, \dots\}$. \mathcal{N} is topologized as the product of discrete spaces.

⁹Since Y is homeomorphic with \mathcal{N} , g can be constructed from the function described by Moschovakis [2009, p. 31].

Corollary 2.2. *If $E \subseteq X \times X$ is a Σ_1^1 equivalence relation, then there is a closed, reflexive, symmetric relation R on $\mathcal{N} \times \mathcal{N}$, such that $E = R^+ \upharpoonright X$.*

Theorem 2.1 would follow from corollary 2.2 if every closed, reflexive, symmetric relation were the union of two closed equivalence relations, but that is not the case.

Proposition 2.1. *Let $\alpha \in \mathcal{N}$. Define $R = D \cup (\{\alpha\} \times \mathcal{N}) \cup (\mathcal{N} \times \{\alpha\})$, and define*

$$\mathcal{E} = \bigcup \{(D \cup \{(\alpha, \beta), (\beta, \alpha)\}) \mid \beta \in \mathcal{N} \setminus \{\alpha\}\}.$$

$R = \bigcup \mathcal{E}$; every $E \in \mathcal{E}$ is an equivalence relation; R is closed, reflexive, and symmetric; and 2^{\aleph_0} is the cardinality of \mathcal{E} . There is no other set \mathcal{F} of equivalence relations such that $R = \bigcup \mathcal{F}$. Thus, R is not a union of fewer than 2^{\aleph_0} equivalence relations.

Proof. The assertions regarding \mathcal{E} are obvious from its construction. To obtain a contradiction from supposing that \mathcal{E} were not unique, suppose that R were also the union of a set $\mathcal{F} \neq \mathcal{E}$ of equivalence relations. Not $\mathcal{F} \subseteq \mathcal{E}$. So, there must be some $E \in \mathcal{F} \setminus \mathcal{E}$. By symmetry, there must be three distinct points, α, β, γ such that $\{(\beta, \alpha), (\alpha, \gamma)\} \subseteq E$. Since E is transitive, $(\beta, \gamma) \in E \setminus R$, contrary to $R = \bigcup \mathcal{F}$. \square

3. PROOF OF THE THEOREM

Denote the diagonal (that is, identity) relation in $\mathcal{N} \times \mathcal{N}$ by $D = \{(\alpha, \alpha) \mid \alpha \in \mathcal{N}\}$. D is closed.

If $1 \leq i < j \leq k$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathcal{N}^k$, then a transposition mapping is defined by $t_{ij}(\vec{\alpha}) = (\alpha_1, \dots, \alpha_{i-1}, \alpha_j, \alpha_{i+1}, \dots, \alpha_{j-1}, \alpha_i, \alpha_{j+1}, \dots, \alpha_k)$.¹⁰ The abbreviation $\widetilde{A} = t_{12}(A) = \{t_{12}(\alpha) \mid \alpha \in A\}$ will sometimes be used. Each t_{ij} is a homeomorphism of \mathcal{N}^k with itself. Note that $t_{ij} \upharpoonright X$ and $t_{ij} \upharpoonright Y$ map X^k and Y^k homeomorphically onto themselves.

Recall that a relation $E \subseteq X \times X$ is Σ_1^1 iff there is a set F such that

$$(3.1) \quad F \subseteq X \times X \times \mathcal{N} \text{ is closed, and } (\alpha, \beta) \in E \iff \exists \gamma (\alpha, \beta, \gamma) \in F.$$

Lemma 3.1. *If $E \subseteq X \times X$ is symmetric, then E is Σ_1^1 iff there is a closed, t_{12} -invariant set $F \subset X \times X \times X$ that satisfies (3.1).*

Proof. Let F_0 satisfy (3.1). Let h be a homeomorphism from \mathcal{N} to X , and define $F_1 \subseteq X \times X \times X$ by $(\alpha, \beta, \gamma) \in F_0 \iff (\alpha, \beta, h(\gamma)) \in F_1$. F_1 also satisfies (3.1), then, and it is closed. By symmetry of E , \widetilde{F}_1 is another closed set that satisfies (3.1). Consequently, $F = F_1 \cup \widetilde{F}_1$ is a t_{12} -invariant closed set that satisfies (3.1). \square

Let i denote the identity function on \mathcal{N} . If K, L, M, N are any sets, and $p: K \rightarrow L$ and $q: M \rightarrow N$, then denote the product mapping by $p \times q: K \times M \rightarrow L \times N$.

¹⁰A sub-sequence of subscripted alphas distinct from α_i and α_j having subscripts that are not increasing, which occurs if $i = 1$ or $j = i + 1$ or $j = k$, denotes the empty sequence.

The two closed equivalence relations that theorem 2.1 asserts to exist are defined from the homeomorphisms f and g introduced in section 2, and the closed, t_{12} -invariant set F guaranteed to exist by lemma 3.1, as follows.

$$\begin{aligned}
 (3.2) \quad & j(\alpha, \beta, \gamma) = g(f(\alpha), f(\beta), f(\gamma)). \\
 & G = \{(\alpha, j(\alpha, \beta, \gamma)) \mid (\alpha, \beta, \gamma) \in F\} \subseteq X \times Y; \\
 & H = \{(j(\alpha, \beta, \gamma), j(\beta, \alpha, \gamma)) \mid (\alpha, \beta, \gamma) \in X \times X \times X\}; \\
 & I = D \cup G \cup \tilde{G} \cup \tilde{G}G; \\
 & J = D \cup H.
 \end{aligned}$$

Lemma 3.2. $D, G, \tilde{G}, H,$ and J are closed.

Proof. D is closed because \mathcal{N} is a metric space.

The function $i \times j$ is a homeomorphism from $X \times X \times X \times X$ to $X \times Y$. Being a homeomorphism, it is an open mapping (which takes closed sets to closed sets). $G = [i \times j](((D \upharpoonright X \times X \times X) \cap (X \times F)))$. $D \upharpoonright X \times X \times X$ and $X \times F$ are both closed subsets of $X \times X \times X \times X$, so G is closed. \tilde{G} is closed, as the image of G under t_{12} , a self-homeomorphism of $\mathcal{N} \times \mathcal{N}$.

$j \times j$ is a homeomorphism from $(X \times X \times X) \times (X \times X \times X)$ to $Y \times Y$. The image under $j \times j$ of a closed subset of its domain is therefore closed in its range. $\{((\alpha, \beta, \gamma), (\beta, \alpha, \gamma)) \mid (\alpha, \beta, \gamma) \in X \times X \times X\}$ is $t_{23} \circ t_{25}(D \upharpoonright X \times D \upharpoonright X \times D \upharpoonright X)$, which is closed. H , the image of this set under $j \times j$, is therefore closed.

J , the union of two closed sets, is closed. \square

Lemma 3.3. $G\tilde{G} = D \upharpoonright X$. $\tilde{G}G = \{(j(\alpha, \beta, \gamma), j(\alpha, \delta, \epsilon)) \mid (\alpha, \beta, \gamma) \in F \text{ and } (\alpha, \delta, \epsilon) \in F\}$. $H = \tilde{H}$. $H^{(2)} = D \upharpoonright Y$. $GH = \{(\alpha, j(\beta, \alpha, \gamma)) \mid (\alpha, \beta, \gamma) \in F\}$. $GH\tilde{G} = E$. $\tilde{G}G$ and I are closed.

Proof. All assertions except the one regarding closedness of $\tilde{G}G$ and I are verified by straightforward calculations. That F is invariant under t_{12} is used to show that $H = \tilde{H}$ and that $E \subseteq GH\tilde{G}$.

The proof that $\tilde{G}G$ is closed is parallel to the proof that H is closed. According to the first part of this lemma, $\tilde{G}G = [j \times j](t_{24}(D \upharpoonright X \times X \times X \times X) \cap (F \times F))$.

I , the union of four closed sets, is closed. \square

Lemma 3.4. I and J are equivalence relations.

Proof. These relations are reflexive and symmetric, so their transitive closures are equivalence relations. Thus, the lemma is equivalent to the assertion that $I = I^+$ and $J = J^+$. For any relation K , $K^{(2)} = K$ is sufficient for $K = K^+$. In the following calculations of $I^{(2)}$ and $J^{(2)}$, composition of relations is distributed over unions. Terms that evaluate by identities that were calculated in lemma 3.3 to a previous term or its sub-relation, are omitted from the expansion by terms in the

pentultimate step of each calculation.

$$\begin{aligned}
I^{(2)} &= (D \cup G \cup \tilde{G} \cup \tilde{G}G)(D \cup G \cup \tilde{G} \cup \tilde{G}G) \\
&= (D \cup G \cup \tilde{G} \cup \tilde{G}G) \cup (G \cup G\tilde{G} \cup G\tilde{G}G) \cup (\tilde{G} \cup \tilde{G}G \cup \tilde{G}\tilde{G} \cup \tilde{G}\tilde{G}G) \\
&\quad \cup (\tilde{G}G \cup \tilde{G}GG \cup \tilde{G}G\tilde{G} \cup \tilde{G}G\tilde{G}G) \\
&= D \cup G \cup \tilde{G} \cup \tilde{G}G \\
(3.3) \quad &= I.
\end{aligned}$$

$$\begin{aligned}
J^{(2)} &= (D \cup H)(D \cup H) \\
&= (D \cup H) \cup (H \cup H^{(2)}) \\
&= D \cup H \\
&= J.
\end{aligned}$$

□

Proof of theorem 2.1. Lemmas 3.2-3.4 show that each of the relations I and J on $\mathcal{N} \times \mathcal{N}$, is an equivalence relation that has a closed graph. It remains to be shown that that $E = (I \cup J)^+ \cap (X \times X)$. Note that, since $D \subseteq I \cup J$, $I \cup J \subseteq (I \cup J)^{(2)} \subseteq (I \cup J)^{(3)} \subseteq \dots$. Hence, if $(I \cup J)^{(n)} = (I \cup J)^{(n+1)}$, then $(I \cup J)^{(n)} = (I \cup J)^+$.

The following calculation shows that $(I \cup J)^{(5)} = (I \cup J)^{(6)}$. The calculation is done recursively, according to the following recipe at each stage $n > 1$:

- (1) Begin with the equation $(I \cup J)^{(n+1)} = (I \cup J)(I \cup J)^{(n)}$.
- (2) Rewrite $(I \cup J)$ as $D \cup G \cup \tilde{G} \cup \tilde{G}G \cup H$ according to (3.2), rewrite $(I \cup J)^{(n)}$ according to the result of the previous step, and then distribute composition of relations over union in the resulting equation.
- (3) For each identity stated in lemma 3.3, and for each identity that, for some $K \in \{G, \tilde{G}, H\}$, equates a composition KD or DK of K and D (or a restriction of D to a product set of which K is a subset) to K , do as follows: Going from left to right, apply the identity wherever possible.¹¹ Repeat this entire step (consisting of one pass per identity) until no further simplifications are possible.
- (4) Delete compositions of relations that include terms KL such that the range of K and the domain of L (viewed as correspondences) are disjoint, in which case the term denotes the empty relation. Delete $D \upharpoonright X$ (occurring as a term by itself), of which D is a superset.
- (5) Delete each term of form $[K]\tilde{G}[L]$ (resp. $[K]G[L]$) from a union in which the corresponding term for its superset, $[K]\tilde{G}E[L]$ (resp. $[K]EG[L]$) also appears. (One or both of the bracketed sub-terms may be absent from both terms in the pair.) Delete D (occurring as a term by itself) from every union that contains both $D \upharpoonright Y$ and E , since $D \subseteq D \upharpoonright Y \cup E$.

¹¹Let $P = D \upharpoonright X$ and $Q = D \upharpoonright Y$. Identities are applied in the following order at each stage of the recursion: $DD = D$, $DE = E$, $DG = G$, $D\tilde{G} = \tilde{G}$, $DH = H$, $DP = P$, $DQ = Q$, $ED = E$, $EE = E$, $EP = E$, $GD = G$, $G\tilde{G} = P$, $GH\tilde{G} = E$, $GQ = G$, $\tilde{G}D = \tilde{G}$, $\tilde{G}P = \tilde{G}$, $HD = H$, $HH = Q$, $HQ = H$, $PD = P$, $PE = E$, $PG = G$, $PP = P$, $QD = Q$, $Q\tilde{G} = \tilde{G}$, $QH = H$, $QQ = Q$.

- (6) Reorder terms lexicographically, in the order $D < D \upharpoonright Y < E < G < \tilde{G} < H$. Delete repeated terms.

$$(I \cup J) = D \cup G \cup \tilde{G} \cup \tilde{G}G \cup H$$

$$(I \cup J)^{(2)} = D \cup D \upharpoonright Y \cup G \cup GH \cup \tilde{G} \cup \tilde{G}G \cup \tilde{G}GH \\ \cup H \cup H\tilde{G} \cup H\tilde{G}G$$

$$(I \cup J)^{(3)} = D \upharpoonright Y \cup E \cup EG \cup GH \cup \tilde{G}E \cup \tilde{G}EG \cup \tilde{G}GH \\ \cup H \cup H\tilde{G} \cup H\tilde{G}G \cup H\tilde{G}GH$$

(3.4)

$$(I \cup J)^{(4)} = D \upharpoonright Y \cup E \cup EG \cup EGH \cup \tilde{G}E \cup \tilde{G}EG \cup \tilde{G}EGH \\ \cup H \cup H\tilde{G}E \cup H\tilde{G}EG \cup H\tilde{G}GH$$

$$(I \cup J)^{(5)} = D \upharpoonright Y \cup E \cup EG \cup EGH \cup \tilde{G}E \cup \tilde{G}EG \cup \tilde{G}EGH \\ \cup H \cup H\tilde{G}E \cup H\tilde{G}EG \cup H\tilde{G}EGH \\ = (I \cup J)^{(6)}$$

Thus $(I \cup J)^+ = (I \cup J)^{(5)}$. Note that $D \upharpoonright Y$, G , \tilde{G} , H and all relations of form or $\tilde{G}Q$ or HQ or QG or QH (where variable Q ranges over compositions of G , \tilde{G} , H , and E), are disjoint from $X \times X$. Therefore, from the calculation in (3.4) of $(I \cup J)^{(5)}$ as a union of E with such relations, it follows that $(I \cup J)^+ \cap (X \times X) = E$. \square

4. THE GENERAL CASE OF A STANDARD BOREL SPACE

In this concluding section, theorem 2.1 is generalized in two ways to an arbitrary standard Borel space. A *standard Borel space* is a pair $\Omega_0 = (\Omega_0, \mathcal{B}_0)$ such that, for some pair $\Omega = (\Omega, \mathcal{B})$, \mathcal{B} is the σ -algebra of Borel subsets of the set Ω under some Polish topology, $\Omega_0 \in \mathcal{B}$, and $\mathcal{B}_0 = \{B_0 \mid \exists B [B \in \mathcal{B} \text{ and } B_0 = B \cap \Omega_0]\}$. A *Borel isomorphism* of standard Borel spaces Ω_0 and Ω is a Δ_1^1 function $k: \Omega_0 \rightarrow \Omega$ such that $k^{-1}: \Omega \rightarrow \Omega_0$ exists and is also Δ_1^1 . A Δ_1^1 subset of a standard Borel space is also a standard Borel space, and every two uncountable standard Borel spaces are isomorphic.¹²

In both generalizations, the concept of smoothness of a Borel equivalence relation substitutes for the concept of closedness that appears in theorem 2.1. If $E \subseteq \Omega \times \Omega$ is a Δ_1^1 equivalence relation, and if there is a set $\{Y_n\}_{n \in \mathbb{N}}$ of Δ_1^1 sets such that $(\omega, \omega') \in E \iff \forall n [\omega \in Y_n \iff \omega' \in Y_n]$, then E is a *smooth* equivalence relation. By Harrington et al. [1990, proof of Theorem 1.1, p. 920], every equivalence relation with closed graph is smooth. If $k: \Omega_0 \rightarrow \Omega$ is Δ_1^1 and $E \subseteq \Omega \times \Omega$ is a smooth Δ_1^1 equivalence relation, then $E_0 \subseteq \Omega_0 \times \Omega_0$ defined by $(\psi, \omega) \in E_0 \iff (k(\psi), k(\omega)) \in E$ is also smooth, with E_0 -equivalence determined by $\{k^{-1}(Y_n)\}_{n \in \mathbb{N}}$.

¹²Mackey [1957, pp. 338–9]. Henceforth, \mathcal{B} will be implicit and the structure Ω will be identified with the set Ω on which it is defined.

The first generalization of theorem 2.1 asserts Borel embeddability of an arbitrary Σ_1^1 equivalence relation. If Ω_0 and Ω are standard Borel spaces, and $E_0 \subseteq \Omega_0 \times \Omega_0$ and $E \subseteq \Omega \times \Omega$ are Σ_1^1 equivalence relations, then a *Borel embedding* of E_0 into E is a Borel isomorphism $e: \Omega_0 \rightarrow Z \subseteq \Omega$ that extends naturally to a Borel isomorphism from E_0 to $E \upharpoonright Z$. That is, $(\psi, \omega) \in E_0 \iff (e(\psi), e(\omega)) \in E$.

Corollary 4.1. *Let Ω_0 and Ω be standard Borel spaces, and let $E_0 \subseteq \Omega_0 \times \Omega_0$ be a Σ_1^1 equivalence relation. There are smooth Δ_1^1 equivalence relations $E_1 \subseteq \Omega \times \Omega$ and $E_2 \subseteq \Omega \times \Omega$ such that E_0 is Borel embeddable in $(E_1 \cup E_2)^+$.*

Proof. If Ω_0 is countable, then E_1 and E_2 can both be taken to be the image of E_0 under an arbitrary injection of Ω_0 into Ω . Otherwise, there is a Borel isomorphism $k_0: \Omega_0 \rightarrow X$ (where X is as in theorem 2.1), and there is a Borel isomorphism $k: \Omega \rightarrow X$. Define $e = k^{-1} \circ k_0$ and define $Z \subseteq \Omega$ by $Z = e(\Omega_0)$. If $E \subset X \times X$ is defined by $(\alpha, \beta) \in E \iff (k_0^{-1}(\alpha), k_0^{-1}(\beta)) \in E_0$, then E is a Σ_1^1 equivalence relation.¹³ Let I and J be the closed equivalence relations defined in (3.2), and define $(\psi, \omega) \in E_1 \iff (k(\psi), k(\omega)) \in I$ and $(\psi, \omega) \in E_2 \iff (k(\psi), k(\omega)) \in J$. E_1 and E_2 are smooth. Now the corollary follows immediately from theorem 2.1. \square

The second generalization of theorem 2.1 applies to a Σ_1^1 equivalence relation that, in a sense, does not occupy the entire product space $\Omega \times \Omega$. Specifically, the set of points, the singletons of which are blocks of the partition induced by the relation, must have an uncountable Δ_1^1 subset.¹⁴

Corollary 4.2. *Suppose Ω is a standard Borel space and that $E \subseteq \Omega \times \Omega$ is a Σ_1^1 equivalence relation such that, for some uncountable Δ_1^1 set $B \subseteq \Omega$, $E \upharpoonright B = D \upharpoonright B$. Define $\Omega_0 = \Omega \setminus B$. Then there are smooth Δ_1^1 relations E_1 and E_2 , such that $E \setminus D \subseteq E \upharpoonright \Omega_0 \cup D \upharpoonright B$*

Finally, corollary 4.2 provides a negative answer to the question, raised in the introduction, of whether the saturations of Borel sets (or even of singletons) with respect to the transitive closures of unions of smooth Borel equivalence relations lie within any significantly restricted sub-class of Σ_1^1 .

Corollary 4.3. *Suppose Ω is a standard Borel space and that $S \subseteq \Omega$ is a Σ_1^1 set such that, for some Δ_1^1 set Ω_0 , $S \subseteq \Omega_0$ and $\Omega \setminus \Omega_0$ is uncountable. Then there are smooth Δ_1^1 relations E_1 and E_2 , such that for every non-empty $A \subseteq S$, $[A]_{(E_1 \cup E_2)^+} \cap \Omega_0 = S$.*

Proof. Define $(\psi, \omega) \in E \iff [\{\psi, \omega\} \subseteq S \text{ or } \psi = \omega]$, specify $B = \Omega \setminus \Omega_0$, and apply corollary 4.2. For some block, π , of the partition induced by $(E_1 \cup E_2)^+$, $\pi \cap \Omega_0 = S$. Therefore, if $\emptyset \neq A \subseteq S$, then $[A]_{(E_1 \cup E_2)^+} \cap \Omega_0 = S$. \square

¹³Moschovakis [2009, Theorem 2B.2, p. 54].

¹⁴If E is a Σ_1^1 equivalence relation, then the set of all such points is a Π_1^1 subset of Ω . One sufficient condition for an uncountable Π_1^1 set, W , to have an uncountable Δ_1^1 subset is that there should be a nonatomic measure, μ , on Ω such that $\mu^*(\Omega \setminus W) < \mu(\Omega)$ (where μ^* is outer measure). Another sufficient condition is that W should have a perfect (hence both uncountable and Δ_1^1) subset. Two sufficient conditions for every uncountable Π_1^1 set to have a non-empty perfect subset—albeit conditions that are independent of ZFC set theory (if ZFC is consistent)—are provided by Moschovakis [2009, Exercise 6G.10, p. 288; and Corollary 8G.4, p. 419]. (Jech [2002, Theorem 25.38, p. 499] states the “boldface” implication of Corollary 8G.4.) It is provable in ZFL that there is an uncountable Π_1^1 set (in fact, a Π_1^1 set) without a non-empty perfect subset. Moschovakis [2009, Exercise 5A.8, p. 212].

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