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Bidder collusion

Robert C. Marshall[a], Leslie M. Marx[b,*]
[a]Pennsylvania State University, USA
[b]Duke University, USA

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Abstract

We analyze bidder collusion at first-price and second-price auctions. Our focus is on less than all-inclusive cartels and collusive mechanisms that do not rely on auction outcomes. We show that cartels that cannot control the bids of their members can eliminate all ring competition at second-price auctions, but not at first-price auctions. At first-price auctions, when the cartel cannot control members' bids, cartel behavior involves multiple cartel bids. Cartels that can control bids of their members can suppress all ring competition at both second-price and first-price auctions; however, shill bidding reduces the profitability of collusion at first-price auctions.

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1. Introduction

Auctions are a prevalent mechanism of exchange.[1] It is natural for bidders to attempt to suppress rivalry and thus capture some of the rents that would be transferred to the seller if their bidding were non-cooperative. Case law is replete with examples of Section 1 violations of the Sherman Act for bid rigging—and these cases are just the bidders who were apprehended. As a casual observation, when new auction mechanisms are proposed or designed, there seems to be little attention paid to the issue of bidder collusion. Yet, bidder collusion is probably the most serious practical threat to revenue.

* Corresponding author. Fax: +1 919 660 7971.
E-mail address: marx@duke.edu (L.M. Marx).

[1] So are procurements. Our results apply to procurements, but we refer to auctions throughout the paper.

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It is commonly thought that oral ascending bid auctions, and second-price sealed-bid auctions, are more susceptible to collusion than first-price sealed-bid auctions. This contrast has lacked rigor. Within the heterogeneous independent private values model, we analyze bidder collusion at first-price and second-price auctions, allowing for within-cartel transfers. Our primary interest is in cartels, or "rings," that contain a strict subset of the bidders. We focus attention on "pre-auction mechanisms," i.e., collusive mechanisms that rely on pre-auction communication and that do not rely on information from the auction itself, such as the identity of the winner or the amount paid. [2] We consider two types of pre-auction mechanisms. In a "bid coordination mechanism," or BCM, the cartel can arrange transfers and recommend bids to the ring members, but has no power to control the bids of the ring members. In a "bid submission mechanism," or BSM, the cartel does have the power to control the bids of the ring members. One interpretation of a BSM is that the cartel actually submits a bid on behalf of each ring member, [3] and other interpretations are given in [12,18], where the center selects one ring member to attend the auction and can prevent all other ring members from bidding. We demonstrate that the auction format (first price versus second price) leads to different results in terms of the viability and profitability of collusion.

Several antitrust cases involve repeated interaction among the colluding firms, [4] but the illegal behavior described in several other cases involves only a single auction or procurement. [5] In this paper, we produce results that allow a contrast between first-price and second-price auctions in terms of their susceptibility to collusion for single auctions.

When comparing the susceptibility of first-price and second-price auctions to bidder collusion, the following intuition exists (see [13]). At a second-price auction, a ring must suppress the bids of all members except the bidder with highest value. The ring member with highest value goes to the auction and bids as he would were he acting non-cooperatively. Any ring member who thinks of breaking ranks and competing at the auction faces the highest ring bidder and the highest non-ring bidder, each submitting bids that are the same as if all were acting non-cooperatively. Thus, there is no gain to deviant behavior. The first-price auction is quite different. In order to secure a collusive gain, the ring member with the highest value must lower his bid below what he would have bid acting non-cooperatively, and other ring members must suppress their bids. But when the

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[2] For example, in the landmark antitrust case US v. Addyston Pipe and Steel Co. et al., 1897 LEXIS 2499 (E.D. Tenn. February 5, 1897) (US v. Addyston), colluding cast-iron pipe manufacturers met prior to the auction, determined which one of the colluding firms would participate in the auction, and agreed on transfer payments: "When bids are advertised for by any municipal corporation, water company, and gas company, the executive committee determines the price at which the bid is to be put in by some company in the association, and the question to which company this bid shall go is settled by the highest bonus which any one of the companies, as among themselves, will agree to pay or bid for the order. When the amount is thus settled the company to whom the right to bid upon the work is assigned sends in its estimate or bid to the city or company desiring pipe, and the amount thus bid is 'protected' by bids from such of the other members of the association as are invited to bid, and by the bidding in all instances being slightly above the one put in by the company to whom the contract is to go. ... Settlements are made at stated times of the bonus account debited against each company, where these largely offset each other, so that small sums are in fact paid by any company in balancing accounts." (p. 3) See also Addyston Pipe and Steel Company v. US, No. 51, 1899 US Supreme Court LEXIS 1559 (December 4, 1899).
Other anitrust cases involving pre-auction mechanisms include US v. Inryco, Inc., 642 F.2d 290 (1981) (US v. Inryco); US v. Raymond J. Lyons, No. 81-1287, 1982 US App. LEXIS 22194 (February 1, 1982) (US v. Lyons); US v. Metropolitan Enterprises, Inc., 728 F.2d 444 (1984) (US v. Metropolitan); US v. A-A-A Elec. Co., Inc., 788 F.2d 242 (4th Cir. 1986) (US v. A-A-A); US v. W.F. Brinkley & Son Construction Company, Inc., 783 F.2d 1157 (4th Cir. 1986) (US v. Brinkley); and Finnegan v. Campeau Corp., 722 F.Supp. 1114 (S.D.N.Y. 1989) (Finnegan v. Campeau).
See [11] for an analysis of collusion when pre-auction communication is not possible.

[3] For example, in US v. Brinkley, Brinkley turned in the bid form for at least one of his competitors.

[4] Examples include US v. Addyston, US v. Inryco, and US v. Lyons.

[5] Examples include US v. Metropolitan, US v. A-A-A, US v. Brinkley, and Finnegan v. Campeau.

highest-valuing ring member lowers his bid, the opportunity is created for a non-highest-valuing cartel member to enter a bid at the auction, either on his own or through a shill, and secure the item. This possibility jeopardizes the feasibility of a cartel at a first-price auction. In this paper, we define environments in which this intuition is borne out. In contrast to the existing literature, this paper facilitates the direct comparison of collusion at first-price versus second-price auctions.

In the literature on collusion at first-price and second-price auctions, we find only one clean comparison between the two auction formats for environments with incomplete information. [6] The results of [18] for first-price auctions and [12] for second-price auctions both cover the case of an all-inclusive cartel composed of homogeneous bidders operating a BSM satisfying ex post budget balance. [7] These papers show that, regardless of the auction format, the cartel can suppress all ring competition, sending only the highest-valuing ring member to the auction, where he wins the object for a price equal to the auctioneer's reserve. Thus, this single contrast suggests little difference between the viability or profitability of collusion at first-price and second-price auctions, despite the aforementioned intuition.

In this paper, we consider BCMs as well as BSMs, and we consider cartels that are not all-inclusive and bidders that are heterogeneous. We show that, although the profitability of collusion may depend on the auction format, the result that all ring competition can be suppressed at both first-price and second-price auctions extends to this environment when the cartel operates a BSM. [8] However, we show a distinct difference between collusion at the two types of auction when the cartel operates a BCM. Assuming ex ante budget balance, we show that a ring can suppress all ring competition using a BCM at a second-price auction, but not at a first-price auction. Thus, we provide a formalization of the intuition that collusion is more difficult at first-price than at second price-auction.

For first-price auctions, we provide a characterization for BCMs. Specifically, the mechanism must require that multiple ring members submit bids that are close together. This result has potentially important empirical implications since it provides a test for collusion. For second-price auctions, we provide results for BSMs for environments not covered by [12], such as environments in which only the highest-valuing ring member makes a payment to the ring center, environments with shills, and repeated auctions. [9]

Another contribution of this paper is to specify elements of an auction/mechanism environment that are critical to the study of collusion, but that are inconsequential for the study of non-cooperative behavior. For example, if we think of non-cooperative play within the IPV model, it

[6] In [22], the author assumes complete information among the ring members and shows that at a second-price auction all ring competition can be suppressed even in the absence of within-cartel transfer payments, but that at a first-price auction no equilibrium exists in which the highest-valuing ring member bids below the second-highest ring value.

[7] The results of [12] also apply to heterogeneous bidders in a non-all-inclusive ring. In [7], the authors also consider collusion at a second-price auction, but they assume ex ante budget balance and allow cartel transfer payments to depend on ring members' reports, the identity of the winner, and the price paid at the auction.

[8] As in [18], for first-price auctions, we are only able to establish individual rationality for particular examples because these calculations must be done numerically on a case-by-case basis. See [14] for numerical techniques.

[9] The results of [7,18] assume IPV, but a working paper by P. Lyk-Jensen (P. Lyk-Jensen, Some Suggestions on How to Cheat the Auctioneer: Collusion in Auctions When Signals Are Affiliated, University of Copenhagen, 1996) shows that an all-inclusive ring can sustain collusion using the mechanism of [7] or the mechanism of [18] in the general symmetric model with affiliated values (see [19]). In this case, efficiency can be achieved using a pre-auction mechanism that is ex ante budget balanced (see the above-mentioned working paper by Lyk-Jensen, 1996) or ex post budget balanced (see working papers P. Lyk-Jensen, Post-Auction Knock-Outs, University of Copenhagen, 1997; and P. Lyk-Jensen, Collusion at Auctions with Affiliated Signals, Université de Toulouse, 1997).

is difficult to imagine any role for a shill bidder, [10] especially if the auctioneer is non-strategic. However, in a collusive environment, ring members are asked to submit specific bids at the auction, and they may have better alternative bids. If the ring can control the bids the ring members submit, but not the bids their shills submit, then ring members may have the incentive to use a shill. When shill bidding is feasible, our results for second-price auctions are unaffected, but collusive payoffs for first-price auctions are reduced.

For repeated auctions, collusion by an all-inclusive ring can be sustained in some environments. In [6], the authors prove a folk theorem for the case in which bidders can communicate prior to each auction and can observe all ring bids but cannot make transfers. Even without communication or the ability to observe bids, working papers by Blume and Heidhues, [11] as well as [23], show that for discount factors sufficiently large, an all-inclusive ring can do better than non-cooperative play or a bid rotation scheme by using implicit transfers of equilibrium continuation payoffs. In a similar framework, but with finitely many types, a working paper by Hörner and Jamison [12] shows that an all-inclusive ring can approximate first-best profits when the discount factor is close to one using review strategies (see [21]). In [1], the author considers BCMs in a repeated auction environment with no transfers. In equilibrium the repeated play provides the opportunity for intertemporal payoff transfers through a bid rotation scheme.

The paper proceeds as follows. The model is in Section 2, and the results for second-price auctions are in Section 3, and the results for first-price auctions are in Section 4. We consider the impact of shill bidders in Section 5. We provide concluding remarks and a discussion of the collusive role of information on auction outcomes, such as the identity of the winner, in Section 6.

2. Model

We are interested in bidding rings that operate in one-shot and repeated auction environments. We focus on the heterogeneous IPV model, which is important for the study of collusion because, even if bidders are homogeneous, collusion creates heterogeneity among them. Bidding behavior and expected revenue within a heterogeneous IPV framework has been analyzed by [15]. [13] Unfortunately, in the heterogeneous IPV model, the equilibrium bid functions for a first-price auction do not have analytic representations, but are implicitly defined by a system of differential equations. Nevertheless, [3,10], and a working paper by Maskin and Riley [14] show that when each bidder's distribution has common support there exists a unique equilibrium. [15] Further, [3] implies that whatever mechanism is used by a ring at a first-price auction, if the designated ring

[10] Some literature uses "shill bidding" to mean bids submitted by the auctioneer (or seller) under the guise of being a regular bidder (see [5] and the working paper Z. Hidvégi, W. Wang, A.B. Whinston, Sequential Auctions with Shill Bidding, University of Texas at Austin, 2001). We assume a non-strategic auctioneer and use "shill bidding" to mean bids submitted by ring members under a different name that cannot be traced to them. A shill provides a way for a bidder to disguise the identity of a "second" bid.

[11] A. Blume, P. Heidhues, Modeling Tacit Collusion in Auctions, University of Pittsburgh, 2002; and A. Blume, P. Heidhues, Private Monitoring in Auctions, University of Pittsburgh, 2004.

[12] J. Hörner, J.S. Jamison, Collusion with (Almost) No Information, Northwestern University, 2004.

[13] The working paper circulated for nearly a decade prior to publication and thus influenced work published much earlier.

[14] E.S. Maskin, J.G. Riley, Uniqueness in Sealed High Bid Auctions, Harvard and UCLA, 1996.

[15] See also P. Bajari, The First Price Auction with Asymmetric Bidders: Theory and Applications, University of Minnesota Ph.D. Thesis, 1997. For results regarding non-common supports see the working paper B. Lebrun, Uniqueness of the Equilibrium in First Price Auctions, York University, 2002. The existence of equilibrium in a heterogeneous IPV setting is also demonstrated in [2,9,16].

bidders and non-ring bidders arrive at the auction with values consistent with a heterogeneous IPV model, then the equilibrium is unique.

In our model, we assume a heterogeneous IPV framework with a non-strategic seller. In the case of a tie, we assume the object is randomly allocated to one of the bidders with the high bid. There are n risk neutral bidders where bidder i independently draws a value v_i from a distribution F_i. We make the following assumption.

Assumption 1. For all i, $F_i(v_i)$ has support $[\underline{v}, \bar{v}]$, where $\underline{v} \geqslant 0$. The probability density function $f_i(v_i)$ is continuously differentiable and, for all i, $f_i(v_i)$ is bounded away from zero on $[\underline{v}, \bar{v}]$. [16]

In this environment, the literature cited above implies:

Lemma 1. *Under Assumption* 1, *an equilibrium at a first-price auction exists in pure strategies*, *the bid functions are strictly increasing and differentiable*, *and the equilibrium is unique*.

Furthermore, the unique equilibrium bid functions have the feature that a bidder with value $\underline{v}$ chooses a bid equal to $\underline{v}$, and so (regardless of the reserve price) no bidder chooses a bid less than $\underline{v}$ (see, e.g. [10]).

We begin by focusing on the case of a single-object auction in which there are $n \geqslant 3$ bidders, and k of those bidders are eligible to participate in a ring, where $2 \leqslant k \leqslant n-1$. We use indices $1, \ldots, k$ to denote ring members and $k+1, \ldots, n$ to denote outside bidders. We let $K \equiv \{1, \ldots, k\}$ denote the set of ring bidders and $\Omega \equiv \{k+1, \ldots, n\}$ denote the set of outside bidders.

The timing in the stage game is as follows: First, a ring mechanism is announced (there is commitment to the mechanism). We do not consider the equilibrium determination of the collusive mechanism, but rather ask whether mechanisms facilitating varying degrees of collusion exist. Both potential ring members and outside bidders observe the mechanism. Second, potential ring members decide whether to join. We assume they join if and only if their expected payoff from participation in the mechanism is greater than or equal to their expected payoff from non-cooperative play. All bidders observe whether all potential ring members join or not. [17] If all potential ring members join, then the ring mechanism operates and otherwise it does not, in which case all bidders participate in the auction non-cooperatively. [18] Third, bidders learn their

[16] An alternative assumption generating Lemma 1 can be found in the aforementioned working paper by Lebrun (2002). For all i, F_i has support $[\underline{v}, \bar{v}]$, where $\underline{v} \geqslant 0$ and is differentiable over $(\underline{v}, \bar{v}]$ with a derivative f_i locally bounded away from zero over this interval, and there exists $\delta > 0$ such that F_i is strictly log-concave over $(\underline{v}, \underline{v} + \delta)$. The function f_i is locally bounded away from zero if for all v in $(\underline{v}, \bar{v}]$, there exists $\varepsilon > 0$ such that $f_i(w) > 0$, for all w in $(v - \varepsilon, v + \varepsilon)$.

[17] We do not need outside bidders to observe the mechanism used by the ring, only to infer it correctly in equilibrium. However, in order to use non-cooperative bidding as the benchmark for defining our individual rationality constraint, we do require that outside bidders observe whether the ring is operating or not, i.e., whether all potential ring members chose to join.

[18] This is a common simplifying assumption in the auction literature. The assumption affects the statement of the individual rationality (IR) constraint, but is not necessary for the results of this paper. Our results for first-price auctions are not affected because the characterization results for BCMs do not rely on IR, and the results for BSMs assume IR is satisfied. The results for BSMs are accompanied by an example in which IR is satisfied, but in the example, there are only two potential ring members, so there is no ambiguity regarding the IR constraint. One might consider the alternative assumption that refusal by one potential ring member to join the ring implies that the remaining potential ring members form a ring of size $k-1$. This alternative assumption complicates the verification of individual rationality for first-price auctions with $k > 2$ because then a potential ring member may prefer to be outside a ring of $k-1$ bidders rather than inside a ring of k bidders.

values, ring members participate in the mechanism, and all bidders submit their bids. We assume that after the auction, the auctioneer announces nothing regarding the outcome of the auction. In Section 6 we discuss the possibility that ring members can observe the identity of the winner.

The ring mechanism operates as follows: Each ring member makes a report to a "center," which is a standard incentiveless mechanism agent (see [20]). Based on these reports, the center recommends a bid to be made by each ring member and requires payments from the ring members. We require that the center's budget be balanced in expectation. As discussed below, this assumption is not necessary for second-price auctions, but simplifies the analysis for first-price auctions.

We require that, given a particular collusive mechanism, the behavior of the bidders forms a Bayesian–Nash equilibrium. In particular, all bidders take the collusive mechanism as given. Conditional on all ring members choosing to join the ring, each ring member's strategy is a report to the ring (as a function of his value) and a bid (as a function of his value and the ring's recommendation and required transfer), and each outside bidder's strategy is a bid (as a function of his value). We require that each bidder's strategy be a best reply to the other bidders' strategies given their beliefs about the other bidders, and we require that bidders' beliefs be consistent with the prior that each bidder i's value is drawn from F_i and Bayesian updating. In the initial stage of the game, ring members choose whether to join or not given their prior beliefs about their own values and the other bidders' values, and given equilibrium behavior in the later stage of the game.

For both of the types of collusive mechanism we consider, BCMs and BSMs, we assume the ring can compel ring members to make their required payments, which depend on the reports made to the center. As discussed below, this assumption is not necessary for second-price auctions, but simplifies the analysis for first-price auctions. We assume these payments are made prior to the auction, so one can think of a cartel as meeting before an auction and running the mechanism, where all payments must be completed before ring members are released to attend the auction.

Thus, a collusive mechanism is $\mu = \left(\beta_1, \ldots, \beta_k, p_1, \ldots, p_k\right)$, where for all $i \in K$, $\beta_i : \mathbb{R}^k \to \mathbb{R}_+$ is ring member i's recommended bid and $p_i : \mathbb{R}^k \to \mathbb{R}$ is his required payment to the center as a function of the ring members' reports $(r_1, \ldots, r_k)$. Following the Revelation Principle, we focus on incentive compatible mechanisms. In what follows we let v^k denote the vector of values for the ring members, i.e., $v^k = (v_1, \ldots, v_k)$, v^k_{-i} denote the values of ring members other than i, and v_{-i} denote the values of all bidders other than i. Given collusive mechanism μ and $i \in K$, we let $\pi_i^{\mu,\beta_\Omega}(v, r_i, b_i)$ be the payoff to ring member i from reporting r_i and then bidding b_i when the values are v, assuming other ring members report truthfully and bid according to the recommendation of the center and outside bidders bid according to $\beta_\Omega = (\beta_j)_{j\in\Omega}$, i.e., for a first-price auction,

$$\pi_i^{\mu,\beta_\Omega}(v, r_i, b_i) \equiv (v_i - b_i)\, 1_{b_i \geqslant \max\left\{\max_{j\in K\setminus\{i\}} \beta_j(r_i, v^k_{-i}), \max_{j\in\Omega} \beta_j(v_j)\right\}} - p_i(r_i, v^k_{-i}).$$

For second-price auctions, we modify the definition of π_i^{μ,β_Ω} in the obvious ways.

Given the appropriate definition of π_i^{μ,β_Ω} for either a first-price or second-price auction, an incentive compatible BCM is defined as follows:

Definition 1. Collusive mechanism $\mu = \left(\beta_1, \ldots, \beta_k, p_1, \ldots, p_k\right)$ is an *incentive compatible BCM against outside bid functions* $\beta_\Omega = (\beta_j)_{j\in\Omega}$ if for all $i \in K$ there exists a bid $b_i(v_i, r_i, \bar{\beta}_i, \bar{p}_i)$

such that

1. (bid optimally) for all $(v_i, r_i, \bar{\beta}_i, \bar{p}_i)$ such that there exists $\hat{v}_{-i}$ satisfying $\beta_i(r_i, \hat{v}^k_{-i}) = \bar{\beta}_i$ and $p_i(r_i, \hat{v}^k_{-i}) = \bar{p}_i$,
$$b_i(v_i, r_i, \bar{\beta}_i, \bar{p}_i) \in \arg\max_{b_i} E_{\hat{v}_{-i}} \Big(\pi_i^{\mu,\beta_\Omega}(v_i, \hat{v}_{-i}, r_i, b_i) | \beta_i(r_i, \hat{v}^k_{-i}) = \bar{\beta}_i,\ p_i(r_i, \hat{v}^k_{-i}) = \bar{p}_i\Big);$$
2. (follow recommended bid) for all $(v_i, \bar{\beta}_i, \bar{p}_i)$ such that there exists $\hat{v}_{-i}$ satisfying $\beta_i(v_i, \hat{v}^k_{-i}) = \bar{\beta}_i$ and $p_i(v_i, \hat{v}^k_{-i}) = \bar{p}_i$,
$$b_i(v_i, v_i, \bar{\beta}_i, \bar{p}_i) = \bar{\beta}_i;$$
3. (report truthfully) for all v_i,
$$v_i \in \arg\max_{r_i} E_{\hat{v}_{-i}} \Big(\pi_i^{\mu,\beta_\Omega}(v_i, \hat{v}_{-i}, r_i, b_i(v_i, r_i, \beta_i(r_i, \hat{v}^k_{-i}), p_i(r_i, \hat{v}^k_{-i})))\Big).$$

The first condition in Definition 1 states that when ring member i has value v_i, reports r_i, and is told to bid $\bar{\beta}_i$ and pay $\bar{p}_i$, he finds it optimal to bid $b_i(v_i, r_i, \bar{\beta}_i, \bar{p}_i)$ at the auction, assuming all other ring members reported truthfully and that all other ring members bid according to the recommendations of the center. The second condition states that when ring member i truthfully reports his value v_i, he finds it optimal to bid according to the recommendation of the center. The third condition states that ring member i with value v_i finds it optimal to report v_i to the center.

To define an incentive compatible BSM, we drop the requirement of incentive compatibility for bidding (Condition 2 in Definition 1), and we modify the requirement of incentive compatibility for reports (Condition 3 in Definition 1) to say that for all v_i,
$$v_i \in \arg\max_{r_i} E_{\hat{v}_{-i}} \Big(\pi_i^{\mu,\beta_\Omega}(v_i, \hat{v}_{-i}, r_i, \beta_i(r_i, \hat{v}^k_{-i}))\Big).$$

We say profile (μ, β_Ω) is *strictly individually rational* if for all $i \in K$, $E_v(\pi_i^{\mu,\beta_\Omega}(v, v_i, \beta_i(v_i, v^k_{-i})))$ is greater than ring member i's ex ante expected payoff when all bidders play non-cooperatively.

As stated in the following definition, for equilibrium we require that the mechanism be individually rational, ex ante budget balanced, and incentive compatible against outside bid functions that are best replies for the outside bidders given the presence of the mechanism.

Definition 2. We say that (μ, β_Ω) is an *equilibrium BCM* (*BSM*) *profile* if it is (i) strictly individually rational, (ii) $E_{v^k}\left(\sum_{j=1}^k p_j(v^k)\right) = 0$, (iii) μ is an incentive compatible BCM (BSM) against β_Ω, and (iv) each β_j $(j \in \Omega)$ is a best response against μ and $\beta_{\Omega\setminus\{j\}}$.[19]

[19] That is, for a first-price auction we require that for all $\ell \in \Omega$,
$$\beta_\ell(v_\ell) \in \arg\max_{b_\ell} E_{v_{-\ell}} \Big((v_\ell - b_\ell) 1_{b_\ell \geqslant \max\{\max_{j\in K} \beta_j(v^k), \max_{j\in\Omega\setminus\{\ell\}} \beta_j(v_j)\}}\Big).$$

For a second-price auction, this expression is modified in the obvious way.

We say (μ, β_Ω) is *ex post efficient* if the highest-valuing bidder, whether a ring member or outside bidder, always wins the object.

3. Second-price auctions

In this section, we consider second-price auctions. We begin by considering mechanisms that result in the highest-valuing ring member bidding at the auction, but that suppress the bids of the other ring members, for example by having non-highest-valuing ring members bid $\underline{v}$ or not bid at all. We refer to mechanisms of this kind as mechanisms that *suppress all ring competition*. Note that a BCM cannot directly prevent ring members from bidding, but must provide appropriate incentives for ring members not to bid.

Proposition 1. *For a second-price auction, there exists an equilibrium BCM* (*and so also a BSM*) *profile that is ex post efficient and suppresses all ring competition.*

Proof. See the Appendix.

Proposition 1 shows that the ring can achieve its first-best outcome using a mechanism that requires no information from the auction or auctioneer. Thus, the results of [7], which involve a mechanism that relies on the identity of the winner and the amount paid at the auction, follow as a corollary to Proposition 1.

The proof of Proposition 1 is by construction, [20] and assumes that outside bidders use the weakly dominant strategy of bidding their values. [21] The collusive mechanism proposed specifies that the highest-reporting ring member pay the center an amount equal to the expected surplus that a bidder with value equal to the second-highest report would receive if he were to bid at the auction against the outside bidders. The expected value of this payment is distributed among all the ring members.

Although Proposition 1 also applies to asymmetric bidders, to gain intuition, it is useful to focus on the case of symmetric bidders. If bidders are symmetric, the expected surplus to a ring

[20] The mechanism used to prove Proposition 1 satisfies ex ante budget balance, but not ex post budget balance. As in [12, Theorem 4], for a second-price auction, one can construct a BSM (but not a BCM) that is ex post budget balanced by using the payment rule

$$p_i(r_1, \ldots, r_k) \equiv \int_{\underline{v}}^{r_i} \hat{\rho}(v)\theta_i'(v)\,dv - \frac{1}{k-1} \sum_{\ell \in K\setminus\{i\}} \int_{\underline{v}}^{r_\ell} \hat{\rho}(v)\theta_\ell'(v)\,dv,$$

where $\hat{\rho}(v_i) = E_{v_{k+1},\ldots,v_n}\left(\left(v_i - \max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j}\right)$ and $\theta_i(v_i) \equiv \Pi_{j\in K\setminus\{i\}} F_j(v_i)$. Under the mechanism of Proposition 1, truthful reporting is a weakly dominant strategy for ring members, something that is not the case in the mechanism of [12]. In addition, the mechanism of [12] may require payments from ring members other than the highest-reporting ring member, whereas the mechanism used in the proof of Proposition 1 does not.

[21] We assume non-colluding bidders follow non-weakly dominated strategies, but ring members are not so constrained. In [7,12,22], the authors also make this assumption. This is consistent with observed behavior in Finnegan v. Campeau; US v. Seville Industrial Machinery Corp., 696 F.Supp. 986 (D.N.J. 1988); US v. Ronald Pook, No. 87-274, 1988 US Dist. LEXIS 3398 (E.D. Pa. April 18, 1988); and District of Columbia, ex rel. John Payton, Corporation Counsel v. George Basiliko, et al., No. 91-2518, 1992 US Dist. LEXIS 1260 (D.C. February 10, 1992).

member with value v from being the sole ring member at the auction is

$$\tilde{p}(v) \equiv E_{v_{k+1},\ldots,v_n}\left(v - \max_{j\in\Omega} v_j \mid v \geqslant \max_{j\in\Omega} v_j\right) \Pr\left(v \geqslant \max_{j\in\Omega} v_j\right) = \int_{\underline{v}}^{v} F^{n-k}(x)\,dx.$$

So one can construct an incentive compatible mechanism by having the ring member with the highest report pay the center $\tilde{p}(r_2) - s$, where r_2 is the second-highest report and

$$s \equiv \frac{1}{k}\int_{\underline{v}}^{\bar{v}} \tilde{p}(x)k(k-1)F^{k-2}(x)(1-F(x))f(x)\,dx,$$

which is the expected value of $\tilde{p}(r_2)$. Ring members with lower reports pay $-s$, i.e., they receive a payment of s. The bid recommendations are that the bidder with the highest report bid his report at the auction and that all other ring members bid $\underline{v}$. Given this mechanism, it is an equilibrium for all ring members to report their values truthfully and follow the bid recommendations of the center. One can easily show that individual rationality is satisfied strictly. [22]

Although we assume that the mechanism can enforce transfer payments, our second-price auction mechanism can be constructed so that it is incentive compatible for the highest-valuing ring member to make its payment to the center. To do this, assume payments happen in two stages: first, the highest-valuing ring member makes his payment $\tilde{p}(r_2)$ to the center, and second, the center makes a payment of s to each ring member. If the payment $\tilde{p}(r_2)$ is not made, then no other payments are made and play reverts to non-cooperative (despite the information revealed within the ring, it remains an equilibrium for all ring members and outside bidders to bid their values at the auction). In this case, the highest-valuing ring member strictly prefers to make its transfer to the center.

Although Proposition 1 applies to a static environment, the BCM of Proposition 1 can also be used in a repeated auction environment. For example, consider the repeated auction environment in which there is an auction in each of an infinite number of periods, $t = 1, 2, \ldots$, and bidders have discount factor $\delta \in [0, 1)$. Assume the ring members remain constant in each period, but that new outside bidders arrive each period, and assume values are independently drawn in each period. As above, assume the auctioneer does not reveal information about the auction outcomes. In this environment, for all $\delta \in [0, 1)$, a ring can suppress all ring competition in every period by using the BCM of Proposition 1 prior to each auction. This result is in contrast to the results of Blume and Heidhues (2002) and [23] that the optimal collusive outcome is uniformly bounded away from efficient collusion for any discount factor. The difference in results occurs because [23] assumes no transfers and no communication, although in their models the identities of past winners are observed.

4. First-price auctions

As we now show, for a BCM there is a stark difference between profitability of collusion at a second-price versus a first-price auction. Proposition 1 shows that there is an ex post efficient mechanism that suppresses all ring competition if the auction is second price, but Proposition 2 and Corollary 1 show that all ring competition cannot be suppressed if the auction is first price.

[22] This mechanism also satisfies strict interim individual rationality, which applies if ring members make their participation decisions after learning their values.

Thus, these results formalize the intuition that the effectiveness of collusion can be reduced by using a first-price rather than a second-price auction.

Proposition 2. *For a first-price auction, there does not exist an equilibrium BCM profile that results in only one ring member submitting a bid above* $\underline{v}$.

Proof. See the Appendix.

Proposition 2 implies that there is no equilibrium in which only the highest-valuing ring member submits a bid, which gives us the following corollary.

Corollary 1. *For a first-price auction, there does not exist an equilibrium BCM profile that suppresses all ring competition.*

Because a BCM does not allow penalties for deviations from recommended bids, it cannot deter bidding by a ring member who is supposed to suppress his bid, but who can profitably deviate by competing at the auction against the highest-valuing ring member and the outside bidders. Because Proposition 2 focuses on mechanisms that suppress the bids of all but one of the ring members, the result does not rule out the possibility of profitable collusion at a first-price auction; however, it does suggest that BCMs have limited benefit because no BCM can suppress all competition among the ring members.

At a second-price auction, a ring can secure a collusive gain using a BCM that merely manipulates the second-highest ring bid; but at a first-price auction, a profitable BCM must reduce the highest ring bid *and* manipulate the second-highest ring bid. Thus, the task facing a ring is more difficult at a first-price auction than at a second-price auction.

A number of authors have considered the problem of whether a cartel can improve upon its non-cooperative payoff in a repeated first-price auction such as the one described in Section 3. With no information about auction outcomes, the game is one of private monitoring. Individual ring members can observe whether they won the object or not, but, if they did not win, they cannot perfectly disentangle the possibility that an outside bidder won the object from the possibility that another ring member deviated and won the object. In this case, even for large discount factors, a history-dependent strategy cannot deter deviations in the absence of equilibrium-path punishment phases. Obviously, if these punishment phases involve periods in which not all ring competition is suppressed, then, regardless of the discount factor $\delta \in [0, 1)$, the ring cannot suppress all ring competition in every period by using a BCM prior to each auction. [23]

This result is in contrast to the results of Blume and Heidhues (2004) for an all-inclusive cartel. Their equilibria require that a punishment phase be triggered if the cartel member designated by the equilibrium does not win the object. But if, as in our paper, the cartel is not all-inclusive, a designated cartel member who does not win cannot determine whether that is because an outside bidder won or because some other cartel member deviated, and so the equilibria of Blume and Heidhues are not equilibria in our environment.

In [23] and Blume and Heidhues (2002), the authors assume that the identities of past winners are observed and show that an all-inclusive cartel can do better than a bid rotation scheme. If,

[23] In [17], the author considers repeated games with private monitoring and two players and shows that for some environments "review strategies" may allow improvements over the repetition of the one-shot Nash equilibrium, but these review strategies involve play of the "defective review strategy" or punishment strategy on the equilibrium path.

in our model, the mechanism can condition on the identities of past winners, then the logic of Proposition 4 implies that all ring competition can be suppressed as long as individual rationality is satisfied.

4.1. First-price BCM characterization

We now characterize a profitable BCM for a first-price auction. Our characterization result has interesting empirical implications. It says that a BCM at a first-price auction must sometimes require that ring members other than the highest-valuing ring member bid at the auction. In particular, the mechanism must require that at least one other ring member submit a bid that is close to the highest ring bid.

To see the intuition for this result, note that if the center always recommends that the highest-valuing ring member bid optimally against the outside bidders and that all other ring members bid something less, then for a positive-measure set of values, some ring member has an incentive to submit a higher bid in an attempt to outbid the highest-valuing ring member. Thus, the center must sometimes recommend a bid greater than the optimal bid (against the outside bidders) for the highest-valuing ring member. But, in order for a bid that is above the highest-valuing ring member's optimal bid against the outside bidders to be incentive compatible, it must be that some other ring member also bids above the optimal bid. Loosely, to prevent deviations from non-highest-valuing ring members, the center must recommend that the highest-valuing ring member bid sufficiently high, but then to prevent deviations from that ring member, the center must recommend that some other ring member submit a bid just below his.

To formalize this result, let $\beta^*(v; \beta_\Omega)$ be the optimal bid for a ring member with value v if all other ring members submit bid $\underline{v}$ and the outside bidders bid according to bid functions β_Ω,[24] i.e.,

$$\beta^*(v; \beta_\Omega) \in \arg\max_b E_{\{v_{k+1},\ldots,v_n\}} \left((v-b)\, 1_{b \geqslant \max_{j\in\Omega} \beta_j(v_j)} \right).$$

In addition, it will be useful to define a bid function β^{in} and bid functions $\beta^{\text{out}}_{k+1}, \ldots, \beta^{\text{out}}_n$ to be the equilibrium bid functions for the case in which $n-k+1$ bidders compete against one another, where the value of the bidder using bid function β^{in} is the highest of $v_1, \ldots, v_k$ and the other bidders' values are $v_{k+1}, \ldots, v_n$. Specifically, we define these functions as follows:

$$\beta^{\text{in}}(v) \in \arg\max_b E_{v_{k+1},\ldots,v_n} \left((v-b)\, 1_{b \geqslant \max_{j\in\Omega} \beta^{\text{out}}_j(v_j)} \right)$$

and for all $i \in \Omega$,

$$\beta^{\text{out}}_i(v_i) \in \arg\max_b E_{v_{-i}} \left((v_i - b) 1_{b \geqslant \max\left\{\beta^{\text{in}}(\max_{j\in K} v_j), \max_{j\in\Omega\setminus\{i\}} \beta^{\text{out}}_j(v_j)\right\}} \right).$$

By Lemma 1, β^{in} and $\beta^{\text{out}}_{k+1}, \ldots, \beta^{\text{out}}_n$ exist and are strictly increasing and unique.

We begin with a lemma.

[24] A ring member's optimal bid against the outside bidders exists if the conjectured bid functions for the outside bidders are continuous. If the optimal bid is not unique, then Lemma 2 continues to hold if we define β^* to be the minimum of the optimal bids, and then Proposition 3 holds because there can only be multiplicity of optimal bids for a zero measure set of value realizations.

Lemma 2. *For a first-price auction, in any equilibrium BCM profile (μ, β_Ω) with β_j continuous for $j \in \Omega$, the highest ring bid is greater than or equal to $\beta^*(\max_{j\in K} v_j; \beta_\Omega)$ with probability one, and strictly greater with positive probability.*

Proof. See the Appendix.

Lemma 2 says that any BCM at a first-price auction (almost) always results in a ring bid that is at least as high as what the optimal bid would be for the highest-valuing ring member bidding against the outside bidders, and sometimes strictly greater.

Proposition 3. *For a first-price auction, in any equilibrium BCM profile (μ, β_Ω) with β_j continuous for $j \in \Omega$, at least two ring members submit bids at the auction that are greater than or equal to $\beta^*(\max_{j\in K} v_j; \beta_\Omega)$ and one bids strictly more with probability one. Furthermore, for any $\varepsilon > 0$, the highest two ring bids are within ε of each other with positive probability.*

Proof. See the Appendix.

Proposition 3 does not imply that the ring will always submit two bids where one is slightly less than the highest ring bid, only that this will happen some of the time. In Section 6 we discuss three examples from Forest Service timber sales that are consistent with Proposition 3.

To see that a mechanism satisfying the properties of Proposition 3 exists, note that the "null" collusive mechanism that recommends non-cooperative bidding and requires no transfer payments, together with non-cooperative bidding for the outside bidders, is an equilibrium BCM profile under weak individual rationality, and in equilibrium, the highest-valuing ring member bids more than β^*, i.e., more than he would bid if the bids of the other ring members were suppressed. Under this null mechanism, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the bids of the highest and second-highest ring members are within ε whenever their values are within δ.

If we assume a small, discrete bid increment, then the logic of Proposition 3 implies that for a positive-measure set of ring values, the ring center must recommend that one of the non-highest-valuing ring members bids one bid increment below the bid of the highest-valuing ring member.

To give an example of a BCM at a first-price auction that increases the ring's expected payoff relative to non-cooperative play within our environment requires complicated numerical calculations. However, we can easily construct such an example if we move to a slightly different environment: Bidders 1 and 2 are in the ring, and bidder 3 is outside. Bidder 1 has value 1 with probability 1 and bidder 2 has value .75 with probability $p \in (0, 1)$ and value .5 with probability $1 - p$. Bidder 3 has value .25 with probability 1. Let p be sufficiently close to 1 that bidder 1's non-cooperative bid is .75, with bidder 2 mixing aggressively under .75 when his value is .75. [25] Assume as in [22] that the values of ring members become common knowledge among the ring members when they join the ring. Then the ring members can achieve a collusive gain relative to non-cooperative play because, under collusion, bidder 1 bids .5 whenever bidder 2 has value .5. Note that in the collusive equilibrium it is necessary that bidder 2 submit a bid close to bidder 1's (2 mixes aggressively under .5); otherwise, bidder 1 would have an incentive to lower his bid, in which case bidder 2 would have an incentive to bid against him.

By assuming a discrete bid increment, we can give a simple example in which ring members make inferences about each other's bids based on the recommendations they receive from the ring

[25] The mixed strategy should satisfy the conditions of [8, p. 374].

center. In this example, we simplify by assuming the ring center can specify which ring member should win in the event that both ring members submit the same winning bid. Suppose there are two ring members, each with value 1 with probability one, and one outside bidder with value less than .8. Suppose the discrete bid increment is .1. Then there is one non-cooperative equilibrium in which both ring members bid 1 and there is another in which both ring members bid .9. Thus, the maximum non-cooperative surplus to the ring member is .1. Consider a BCM that makes one of six recommendations, each with equal probability. The first three possible recommendations are: (1) $\beta_1 = \beta_2 = .8$ and RM1 wins in the event both bid .8; (2) $\beta_1 = .8$, $\beta_2 = .9$, RM1 wins in the event both bid .8, and RM2 wins in the event both bid .9; (3) $\beta_1 = \beta_2 = .9$ and RM1 wins in the event both bid .9. The remaining three possible recommendations are as in 1–3, but with the roles of RM1 and RM2 reversed. Note that in all cases the ring members' bids are within one bid increment of one another.

Given these recommendations, if RM1 receives recommendation $\beta_1 = .8$, he believes that he will win with a bid of .8 with probability $\frac{1}{3}$, that he will lose with any lower bid, and that he will win with a bid of .9 with probability $\frac{2}{3}$. If RM1 receives recommendation $\beta_1 = .9$, he believes that he will win with a bid of .9 with probability $\frac{2}{3}$ and that he will lose with any lower bid. One can show that it is incentive compatible for ring members to follow the recommendation of the mechanism. So in this example, a BCM can be used to increase the payoffs of the ring members relative to non-cooperative play.

4.2. First-price BSM

In this section, we show that all ring competition can be suppressed at a first-price auction if the ring can use a BSM. We construct a mechanism in which the highest-valuing ring member bids according to the equilibrium bid function for an auction in which only the highest-valuing ring member bids against the $n-k$ outside bidders.

Let $\beta^{\text{in}}(v)$ be the equilibrium first-price bid for a ring member, whose value v is the highest in the ring, when facing only the $n-k$ outside bidders, and let $\beta_i^{\text{out}}(v)$ be the equilibrium bid for outside bidder i with value v, when facing $n-k-1$ outside bidders and only the highest valuing of k ring bidders. Equilibrium bid functions β^{in} and β_i^{out}, which are unique by Lemma 1, are defined by the conditions that for all v,

$$\beta^{\text{in}}(v) \in \arg\max_b E_{v_{k+1},\dots,v_n}\left((v-b)\, 1_{b \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)}\right)$$

and for all v and all $\ell \in \Omega$,

$$\beta_\ell^{\text{out}}(v) \in \arg\max_b E_{v_{-\ell}}\left((v-b)\, 1_{b \geqslant \max\left\{\beta^{\text{in}}(\max_{j\in K} v_j),\ \max_{j\in\Omega\setminus\{\ell\}} \beta_j^{\text{out}}(v_j)\right\}}\right).$$

Note that the equilibrium bid function for the highest-valuing ring member, β^{in}, does not depend on which ring member has the highest value.

Since β^{in} and β_ℓ^{out} for $\ell \in \Omega$ are the non-cooperative bid functions when there are $n-k+1$ bidders with one bidder drawing its values from the distribution $F_1 \dots F_k$ and $n-k$ bidders drawing their values from distributions $F_{k+1}, \dots, F_n$, it will be useful to define $\Pi_i^j(G_1, \dots, G_j)$ to be the expected non-cooperative payoff of bidder i under non-cooperative play when there are j bidders drawing their values from distributions $G_1, \dots, G_j$, respectively.

Let $\hat{p}(v)$ be the expected payoff to a ring member with value v from bidding against the outside bidders, i.e.,

$$\hat{p}(r) \equiv E_{v_{k+1},\dots,v_n}\left(\left(r - \beta^{\text{in}}(r)\right) 1_{\beta^{\text{in}}(r) \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)}\right).$$

Letting $s_1, \dots, s_k$ be such that $\sum_{i=1}^{k} s_i = E_{v^k}(\hat{p}(v)|v$ is second-highest among $v_1, \dots, v_k)$, we consider a payment rule under which each ring member i receives a payment of s_i and then the ring member with the highest report pays $\hat{p}(r)$, where r is the second-highest report. This payment rule can be implemented by distributing payments $s_1, \dots, s_k$ to the ring members and then having the ring members compete in a second-price auction for the right to be the sole ring member who bids at the main auction. The BSM enforces bids of $\underline{v}$ for ring members who do not have the highest report and a bid of $\beta^{\text{in}}(r_i)$ for the ring member i with the highest report. As shown in the proof of Proposition 4, this mechanism induces truthful revelation and suppresses all ring competition, and it is strictly individually rational for all ring members if and only if

$$\sum_{i=1}^{k} \Pi_i^n(F_1, \dots, F_n) < \Pi_1^{n-k+1}(F_1 \dots F_k, F_{k+1}, \dots, F_n). \tag{1}$$

Proposition 4. *For a first-price auction, if* (1) *holds, then there exists an equilibrium BSM profile that suppresses all ring competition.*

Proof. See the Appendix.

It remains to show that condition (1) is satisfied so that the mechanism described above is strictly individually rational. For reasons of analytic intractability, individual rationality almost always requires numerical verification given particular assumptions about parameters and distributions. To give a particular example, consider the case with values drawn from the uniform distribution on [0, 1].[26] Then the strict individual rationality condition can be written as

$$\begin{aligned} k\int_0^1\int_0^v \left(v - \beta^{\text{nc}}(v)\right)(n-1)x^{n-2}\,dx\,dv \\ < \int_0^1\int_0^{\beta^{\text{out}^{-1}}\left(\beta^{\text{in}}(v)\right)} \left(v - \beta^{\text{in}}(v)\right)(n-k)y^{n-k-1}kv^{k-1}\,dy\,dv, \end{aligned} \tag{2}$$

where β^{nc} is the non-cooperative bid function and where β^{out} and β^{in} must be calculated numerically. For the case of two bidders in the ring and one bidder outside the ring, i.e., $k = 2$ and $n = 3$, we can show that (2) is satisfied.[27]

Using Proposition 4, when the ring can directly control ring members' bids, in a repeated first-price auction for all $\delta \in [0, 1)$, a ring can suppress all ring competition in every period using a BSM prior to each auction as long as strict individual rationality is satisfied for the stage game.

[26] The conditions of Assumption 1 are not satisfied in this example since the distribution of the highest from two uniform [0, 1] random variables is $F(x) = x^2$; however, the conditions provided by Lebrun (2002) are satisfied (see Footnote 16).

[27] The difference between the right and left sides of (2) is approximately 0.034, which is easily calculated using the techniques described in the working paper W.-R. Gayle, J.-F. Richard, Numerical Solutions of Asymmetric First Price Independent Private Value Auctions, University of Pittsburgh, 2005. In addition, there are numerous other parameters for which the result holds. For example, using the numerical calculations of [14], the result holds when values are drawn from the uniform distribution on [0, 1] and $n = 5$ with either 2, 3, or 4 bidders in the ring, or $n = 101$ with either 99 or 100 bidders in the ring.

5. Shill bidders

As has been noted in the literature, the ability of ring members to use shills to place bids on their behalf can affect the profitability of a collusive mechanism. To study the impact of shill bidding on the collusive mechanisms that are the focus of this paper, we allow a ring member to submit two bids, one bid under his own name and another bid under an alias.[28] In the case of a BCM, the ring can recommend bids, but cannot control bids submitted by the ring members, regardless of whether they are submitted under the ring members' own names or under aliases. In the case of a BSM, the bid submitted under a ring member's own name is controlled by the ring, but any bid submitted under an alias is not. It remains common knowledge that there are only n bidders (k ring members and $n-k$ outside bidders) participating in the auction, but it is common knowledge that bidders have the ability to submit multiple bids.

In what follows, we refer to a collusive mechanism incorporating the feasibility of shill bidding as either a *BCM with shills* or a *BSM with shills*. In what follows we contrast results for these mechanisms with those of the BCMs and BSMs described earlier, which implicitly assume shill bidding is not possible. We refer to these as BCMs or BSMs *without shills* where the meaning is not clear. As before, we assume the ring cannot use information on the number of bids submitted or the identity of the winner; in particular, this information cannot be used to infer whether shill bidding occurred.

Formally, we define a collusive mechanism with shills to be $\mu^s = (\beta_1, \ldots, \beta_k, \beta_1^s, \ldots, \beta_k^s, p_1, \ldots, p_k)$, where for all $i \in K$, $\beta_i : \mathbb{R}^k \to \mathbb{R}_+$ is ring member i's recommended bid to be submitted under his own name, $\beta_i^s : \mathbb{R}^k \to \mathbb{R}_+$ is his recommended bid to be submitted under an alias, and $p_i : \mathbb{R}^k \to \mathbb{R}$ is his required payment to the center as a function of the ring members' reports $(r_1, \ldots, r_k)$.

We redefine π_i^{μ,β_Ω} to include a shill bid as an additional argument, i.e., given collusive mechanism with shills μ^s, outside bid functions β^Ω, and $i \in K$, we let $\hat{\pi}_i^{\mu^s,\beta_\Omega}(v, r_i, b_i, b_i^s)$ be the payoff to ring member i when the values are v from reporting r_i, bidding b_i under his own name, and bidding b_i^s under an alias, assuming other ring members report truthfully and bid according to the recommendation of the center and outside bidders bid according to $\beta_\Omega = (\beta_j)_{j\in\Omega}$. For a first-price auction, we have

$$\begin{aligned}
&\hat{\pi}_i^{\mu^s,\beta_\Omega}(v, r_i, b_i, b_i^s) \\
&\quad \equiv \left(v_i - \max\left\{b_i, b_i^s\right\}\right) \\
&\qquad \times 1_{\max\{b_i,b_i^s\} \geqslant \max\left\{\max_{j\in K\setminus\{i\}} \beta_j(r_i,v_{-i}^k), \max_{j\in K\setminus\{i\}} \beta_j^s(r_i,v_{-i}^k), \max_{j\in\Omega} \beta_j(v_j)\right\}} \\
&\qquad - p_i(r_i, v_{-i}^k).
\end{aligned}$$

Replacing π_i^{μ,β_Ω} with $\hat{\pi}_i^{\mu^s,\beta_\Omega}$, the definition of an incentive compatible BCM for an environment in which shill bidding is feasible is the similar to Definition 1. For clarity, we state the new definition below.

Definition 3. Collusive mechanism with shills $\mu^s = \left(\beta_1, \ldots, \beta_k, \beta_1^s, \ldots, \beta_k^s, p_1, \ldots, p_k\right)$ is an *incentive compatible BCM with shills against outside bid functions* $\beta_\Omega = (\beta_j)_{j\in\Omega}$ if for all $i \in K$

[28] Allowing multiple shill bids does not affect our results, and allowing outside bidders to submit shill bids does not affect our results.

there exist bids $b_i(v_i, r_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i)$ and $b_i^s(v_i, r_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i)$ such that

1. (bid optimally) for all $(v_i, r_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i)$ such that there exists $\hat{v}_{-i}$ satisfying $\beta_i(r_i, \hat{v}_{-i}^k) = \bar{\beta}_i$, $\beta_i^s(r_i, \hat{v}_{-i}^k) = \bar{\beta}_i^s$, and $p_i(r_i, \hat{v}_{-i}^k) = \bar{p}_i$,
$$\begin{aligned}&\left(b_i(v_i, r_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i), b_i^s(v_i, r_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i)\right)\\ &\quad \in \arg\max_{(b_i, b_i^s)} E_{\hat{v}_{-i}}(\hat{\pi}_i^{\mu^s, \beta_\Omega}(v_i, \hat{v}_{-i}, r_i, b_i, b_i^s) | \beta_i(r_i, \hat{v}_{-i}^k) = \bar{\beta}_i,\ \beta_i^s(r_i, \hat{v}_{-i}^k)\\ &\qquad = \bar{\beta}_i^s,\ p_i(r_i, \hat{v}_{-i}^k) = \bar{p}_i);\end{aligned}$$
2. (follow recommended bid) for all $(v_i, \bar{\beta}_i, \bar{p}_i)$ such that there exists $\hat{v}_{-i}$ satisfying $\beta_i(v_i, \hat{v}_{-i}^k) = \bar{\beta}_i$, $\beta_i^s(r_i, \hat{v}_{-i}^k) = \bar{\beta}_i^s$, and $p_i(v_i, \hat{v}_{-i}^k) = \bar{p}_i$,
$$b_i(v_i, v_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i) = \bar{\beta}_i \quad \text{and} \quad b_i^s(v_i, v_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i) = \bar{\beta}_i^s;$$
3. (report truthfully) for all v_i,
$$v_i \in \arg\max_{r_i} E_{\hat{v}_{-i}} \left(\hat{\pi}_i^{\mu^s, \beta_\Omega} \left(\begin{array}{l} v_i, \hat{v}_{-i}, r_i, b_i(v_i, r_i, \beta_i(r_i, \hat{v}_{-i}^k), \beta_i^s(r_i, \hat{v}_{-i}^k), \\ p_i(r_i, \hat{v}_{-i}^k)), \\ b_i^s(v_i, r_i, \beta_i(r_i, \hat{v}_{-i}^k), \beta_i^s(r_i, \hat{v}_{-i}^k), p_i(r_i, \hat{v}_{-i}^k)) \end{array} \right) \right).$$

To define an incentive compatible BSM with shills, we drop the first part of Condition 2 since the ring can control bids submitted by ring members under their own names and require only that $b_i^s(v_i, v_i, \bar{\beta}_i, \bar{\beta}_i^s, \bar{p}_i) = \bar{\beta}_i^s$. In addition, we modify Condition 3 to say that for all v_i,
$$v_i \in \arg\max_{r_i} E_{\hat{v}_{-i}} \left(\hat{\pi}_i^{\mu^s, \beta_\Omega} \left(\begin{array}{c} v_i, \hat{v}_{-i}, r_i, \beta_i(r_i, \hat{v}_{-i}^k), \\ b_i^s(v_i, r_i, \beta_i(r_i, \hat{v}_{-i}^k), \beta_i^s(r_i, \hat{v}_{-i}^k), p_i(r_i, \hat{v}_{-i}^k)) \end{array} \right) \right).$$

In a BCM without shills, the ring has no direct control over the bids of the ring members. A ring member's payoff from submitting bid b_i under his own name and bid b_i^s under an alias is the same as if the ring member submitted bid $\max\{b_i, b_i^s\}$ under his own name. Thus, ring members have no incentive to submit an additional bid under an alias. This implies any outcome that can be achieved with a BCM without shills can also be achieved with a BCM with shills. For example, as in Proposition 1, at a second-price auction, a BCM with shills can suppress all ring competition.

Corollary 2. *For a second-price auction, there exists an equilibrium BCM with shills* (*and so also a BSM with shills*) *profile that is ex post efficient and suppresses all ring competition.*

In contrast, shill bidding does have an effect on the payoffs from BSMs at first-price auctions. In a BSM without shills, the ring can control the bids submitted by ring members, but in a BSM with shills, the ring can only control bids submitted under ring members' own names. The ring cannot control bids submitted under an alias. Thus, payoffs that can be a with a BSM without shills cannot necessarily be achieved with a BSM with shills. Although Section 4.2 shows that for a first-price auction there exists an equilibrium BSM without shills profile that suppresses all ring competition, the following proposition shows that this outcome cannot be achieved with a BSM with shills.

Proposition 5. *For a first-price auction, there does not exist an equilibrium BSM with shills profile that suppresses all ring competition.*

Proof. See the Appendix.

In addition, if we allow the possibility that the center can submit a bid, possibly through a shill,[29] then a BSM with shills performs no better than a BCM (with or without shills).

Proposition 6. *For a first-price auction, if the ring center can submit a bid, for any equilibrium BSM with shills profile, there exists an equilibrium BCM (with or without shills) profile that results in at least as high an expected payoff to the ring.*

Proof. See the Appendix.

Proposition 6 implies that at a first-price auction, the ability of ring members to use shills negates any advantage of a BSM.

6. Discussion

Our focus is on mechanisms that do not rely on information from the auctioneer, such as the identity of the winner or the amount paid. We refer to these mechanisms as pre-auction mechanisms. This type of collusive mechanism is necessary when information on the auction outcome is not available to losing bidders. For example, when using competitive procurements to acquire factor inputs, many firms do not reveal the losing bids or the price paid by the winning bidder.

Within the class of pre-auction mechanisms, we identify two types. The first type of mechanism, a BCM, gathers information from the ring members regarding their values for the object, specifies transfers among ring members, and makes recommendations on how they should bid. The second type of mechanism, a BSM, gathers the same information from the bidders and arranges for transfers, but instead of merely recommending bids to the ring members, the ring center controls the bids submitted by the ring members. We consider the effectiveness of these two types of pre-auction mechanisms in facilitating collusion at first- and second-price auctions.

We show that at a second-price auction, a BCM allows the ring to suppress all competition among ring members. In contrast, at a first-price auction this is not the case—a BCM cannot suppress all competition among ring members. Bidders at a first-price auction are limited in their ability to profitably collude by the facts that (i) a ring may not be able to suppress competition among its members, and (ii) individual rationality may be impossible to satisfy. Despite these limitations, collusion is possible at a first-price auction in some cases. We provide a characterization of BCMs at a first-price auction, and show the existence of BSMs for first-price auctions that suppress all ring competition.

Our characterization result for BCMs for a first-price auction provides us with an important empirical implication. In particular, Proposition 3 tells us that a ring using a BCM must sometimes

[29] We assume that the center has zero value for the object and that in the off-equilibrium case in which the center wins the object, it disposes of it in a way that provides no value to any of the bidders. By submitting a bid just below the recommended bid of the high-valuing ring member, the center can prevent downward deviations from recommended bids at a BCM with shills.

Table 1
Bids from three first-price forest service timber sales

Auction	Reserve	High bid	2nd bid	3rd bid	4th bid	5th bid	6th bid
1	$21,272	**$62,270**	**$62,072**	$51,770	$42,080	$38,504	$28,310
2	$37,523	$85,365	**$77,213**	**$76,860**	$65,847	**$57,960**	**$57,683**
3	$27,208	$129,240	$70,000	**$46,800**	**$46,480**	–	–

require multiple bids from the ring members—a high bid and another that is just below it. One other auction environment in which one sees clustered bids is a first-price auction with *complete information* where bidders act non-cooperatively. In this case, equilibrium behavior involves the highest-valuing bidder submitting a bid equal to the second-highest value, while the second-highest-valuing bidder aggressively mixes under his value (see [8, p. 374] for the conditions that must be satisfied by the mixing distribution). This implies that the two high bids will be very close to one another. But a key feature differentiates the "close" collusive bids of Proposition 3 from the "close" non-cooperative bids of a complete information environment. The close collusive bids are ring bids and so may or may not be the highest two bids submitted at the auction, depending on the bids of the outside bidders. But the close non-cooperative bids of a complete information environment are always the highest and second-highest bids. In other words, a prediction of Proposition 3 that is unique to collusion is that we will observe pairs of non-winning bids that are close to one another. This provides a way to detect collusion that requires little information about the bidders or the items being sold. [30]

As an illustration, in Table 1 we provide the bids from three first-price sealed-bid auctions conducted by the Forest Service in the late 1980s in Region 1, which consists primarily of Montana, North Dakota, and part of Idaho. [31] Although empirical analysis conducted as part of future research may reveal that the bids are consistent with non-collusive behavior, the bids are suggestive that Proposition 3 is worthy of further empirical exploration. In the first auction, the highest and second-highest bids are close (less than 1% difference). In the second auction, the second- and third-highest bids are close, while in the third auction, the third- and fourth-highest bids are close. In fact, for the second auction the fifth- and sixth-highest bids are also close, suggesting that two separate coalitions may have been functioning. Note that for each auction, the bids are well above the reserve, thus the closeness of the bid pairs is not driven by the reserve. Also, note that for any given auction, there is substantial dispersion in the bids, excluding the pairs mentioned above. This suggests that some information common to all bidders is not the cause of the compression in the spacings for the identified pairs.

When we consider all of the sealed-bid auctions held in Region 1 in years 1983–1992, a total of 434 auctions, we find that in 43 of these auctions, there are two bids that are within 1% of one another and are at least 20% greater than the reserve price, and we find that there are 54 bidding pairs in which the bids are within 1% of one another and both bids are at least 20% greater than the reserve price. We leave a more complete analysis of this data for future research.

[30] One might expect to see close bids under non-cooperative bidding when values are close, but then with a discrete bid increment, one would also expect to see ties occasionally, something that would not be expected under efficient collusion.

[31] US Senate hearings on *Timber Sales Bidding Procedures*: *Hearings before the Senate Subcommittee on Public Lands and Resources* (95th Cong., 1st sess., 1977) identified collusion as a potential problem at Forest Service sales. Also, [4] finds evidence of collusion at Forest Service timber sales conducted via English auction.

Our results show that a bidding ring can suppress all ring competition if (i) the auction is second price and the ring is weak in the sense of not being able to control ring members' bids or (ii) the auction is first price and the ring is strong in the sense of being able to control ring members' bids. However, one can show that a bidding ring can suppress all ring competition at a first-price auction even if it cannot control ring members' bids if the auctioneer announces the identity of the winner, or if that information becomes known to the ring in a way that allows the ring to condition transfer payments on it. If the ring can condition on the identity of the winner, additional announcements by the auctioneer, such as the amount paid or vector of bids submitted, play no additional role in facilitating collusion. In a repeated first-price auction environment, an announcement of the identity of the winner can facilitate collusion even if transfer payments within the ring cannot be conditioned on this announcement because the identity of the winner can still affect the ring members' future participation decisions. For example, bidding might revert to non-cooperative bidding if it is ever observed that a ring member other than the highest-reporting ring member won the auction. Then for δ sufficiently large, the ring can suppress all ring competition using a BCM prior to each auction.

A policy implication of our results seems clear—if collusion is a major concern for auction designers, then a first-price auction should be used rather than a second-price auction, and, if possible, information on auction outcomes should not be made public. Auctioneers (or procurement agents) should maintain a record of all bids, not just those of winners.

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Appendix A. Proofs

Proof of Proposition 1. Consider the following bidding rule: if bidder i's report is not highest, the center recommends a bid of $\underline{v}$, and if bidder i's report is highest, the center recommends a bid equal to his report. Note that if the bidders report truthfully, there is no incentive for any bidder to deviate from the center's recommendation. Consider the following payment rule: if bidder i's report is not highest, bidder i pays the center $-s$, but if bidder i's report is highest and r_2 is the second-highest report, then bidder i pays the center $\hat{p}(r_2) - s$, where $\hat{p}(r_2) \equiv E_{v_{k+1},\ldots,v_n}\left(\left(r_2 - \max_{j\in\Omega} v_j\right) 1_{r_2 \geqslant \max_{j\in\Omega} v_j}\right)$ and $s \equiv \frac{1}{k} E_{v^k}\left(\hat{p}(v) | v \text{ is second highest in } \{v_1, \ldots, v_k\}\right)$. Note that under this payment rule, the center has zero expected revenue.

Suppose the other $k-1$ ring members report truthfully. Then ring member i with value v_i who reports r_i and then bids optimally in the continuation game has expected interim payoff of

$E_{v^k_{-i}}\left(\hat{\pi}_i(v_i, v^k_{-i}, r_i)\right)$, where

$$\hat{\pi}_i(v_i, v^k_{-i}, r_i) \equiv \begin{cases} E_{v_{k+1},\ldots,v_n}\left(\left(v_i - \max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j}\right) - \hat{p}(\max_{j\in K\setminus\{i\}} v_j) + s \\ \quad \text{if } r_i \geqslant \max_{j\in K\setminus\{i\}} v_j, \\ E_{v_{k+1},\ldots,v_n}\left(\left(v_i - \max_{j\in(K\setminus\{i\})\cup\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in(K\setminus\{i\})\cup\Omega} v_j}\right) + s \\ \quad \text{if } r_i < \max_{j\in K\setminus\{i\}} v_j. \end{cases}$$

Letting $f_i(v_i, v^k_{-i}) \equiv E_{v_{k+1},\ldots,v_n}\left(\left(v_i - \max_{j\in(K\setminus\{i\})\cup\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in(K\setminus\{i\})\cup\Omega} v_j}\right)$ and using the definition of $\hat{p}$, we can rewrite $\hat{\pi}_i(v_i, v^k_{-i}, r_i)$ as

$$\hat{\pi}_i(v_i, v^k_{-i}, r_i) = \begin{cases} \hat{p}(v_i) - \hat{p}(\max_{j\in K\setminus\{i\}} v_j) + s & \text{if } r_i \geqslant \max_{j\in K\setminus\{i\}} v_j, \\ f_i(v_i, v^k_{-i}) + s & \text{if } r_i < \max_{j\in K\setminus\{i\}} v_j. \end{cases}$$

We now show that for all r_i, $E_{v^k_{-i}}\left(\hat{\pi}_i(v_i, v^k_{-i}, v_i)\right) \geqslant E_{v^k_{-i}}\left(\hat{\pi}_i(v_i, v^k_{-i}, r_i)\right)$ by considering the six cases corresponding to the possible orderings of v_i, r_i, and $\max_{j\in K\setminus\{i\}} v_j$.

Cases (1) $X =$"$\max_{j\in K\setminus\{i\}} v_j \leqslant v_i < r_i$" and (2) $X =$"$\max_{j\in K\setminus\{i\}} v_j \leqslant r_i < v_i$": Ring member i has the highest report and the highest value, so $E_{v^k_{-i}}(\hat{\pi}_i(v_i, v^k_{-i}, r_i)|X) = E_{v^k_{-i}}(\hat{\pi}_i(v_i, v^k_{-i}, v_i)|X) = \hat{p}(v_i) - E_{v^k_{-i}}\left(\hat{p}(\max_{j\in K\setminus\{i\}} v_j)|X\right) + s$.

Cases (3) $X =$"$v_i < r_i \leqslant \max_{j\in K\setminus\{i\}} v_j$" and (4) $X =$"$r_i < v_i \leqslant \max_{j\in K\setminus\{i\}} v_j$": Ring member i has neither the highest report nor the highest value, so $E_{v^k_{-i}}(\hat{\pi}_i(v_i, v^k_{-i}, r_i)|X) = E_{v^k_{-i}}(\hat{\pi}_i(v_i, v^k_{-i}, v_i)|X) = s$.

Case (5) $X =$"$v_i < \max_{j\in K\setminus\{i\}} v_j < r_i$": Ring member i has the highest report, but not the highest value, so since $\hat{p}$ is increasing,

$$\begin{aligned} E_{v^k_{-i}}\left(\hat{\pi}_i(v_i, v^k_{-i}, r_i)|X\right) &= \hat{p}(v_i) - E_{v^k_{-i}}\left(\hat{p}\left(\max_{j\in K\setminus\{i\}} v_j\right)\middle| X\right) + s < s \\ &= E_{v^k_{-i}}(\hat{\pi}_i(v_i, v^k_{-i}, v_i)|X). \end{aligned}$$

Case (6) $X =$"$r_i < \max_{j\in K\setminus\{i\}} v_j < v_i$": Ring member i has the highest value, but not the highest report. In this case, one can show that $E_{v^k_{-i}}(f_i(v_i, v_{-i})|X) = \hat{p}(v_i) - E_{v^k_{-i}}\left(\hat{p}\left(\max_{j\in K\setminus\{i\}} v_j\right)|X\right)$, so

$$\begin{aligned} E_{v^k_{-i}}\left(\hat{\pi}_i(v_i, v^k_{-i}, r_i)|X\right) &= E_{v^k_{-i}}(f_i(v_i, v_{-i})|X) + s \\ &= \hat{p}(v_i) - E_{v^k_{-i}}\left(\hat{p}\left(\max_{j\in K\setminus\{i\}} v_j\right)\middle| X\right) + s \\ &= E_{v^k_{-i}}(\hat{\pi}_i(v_i, v^k_{-i}, v_i)|X). \end{aligned}$$

To see that $E_{v^k_{-i}}(f_i(v_i, v_{-i})|X) = \hat{p}(v_i) - E_{v^k_{-i}}\left(\hat{p}\left(\max_{j\in K\setminus\{i\}} v_j\right)|X\right)$, first note that

$$\begin{aligned} & E_{v^k_{-i}}\left(E_{v_{k+1},\ldots,v_n}\left(\left(\max_{j\in(K\setminus\{i\})\cup\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in(K\setminus\{i\})\cup\Omega} v_j}\right)|X\right) \\ &= E_{v^k_{-i}}\left(E_{v_{k+1},\ldots,v_n}\left(\begin{array}{c} \left(\max_{j\in K\setminus\{i\}} v_j\right) 1_{v_i \geqslant \max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j} \\ + \left(\max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j \geqslant \max_{j\in K\setminus\{i\}} v_j} \end{array}\right)\middle| X\right) \end{aligned}$$

$$= E_{v_{-i}^k}\left(E_{v_{k+1},\ldots,v_n}\left(\begin{array}{l}\left(\max_{j\in K\setminus\{i\}} v_j\right) 1_{v_i \geqslant \max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j} \\ \quad + \left(\max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j} \\ \quad - \left(\max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j}\end{array}\right)\middle| X\right)$$

$$= E_{v_{-i}^k}\left(E_{v_{k+1},\ldots,v_n}\left(\begin{array}{l}\left(\max_{j\in K\setminus\{i\}} v_j\right) 1_{\max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j} \\ \quad + \left(\max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j} \\ \quad - \left(\max_{j\in\Omega} v_j\right) 1_{\max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j}\end{array}\right)\middle| X\right), \tag{A.1}$$

where the first equality breaks $\max_{j\in(K\setminus\{i\})\cup\Omega} v_j$ into two components involving $\max_{j\in K\setminus\{i\}} v_j$ and $\max_{j\in\Omega} v_j$, the second equality breaks $\max_{j\in\Omega} v_j$ into two components based on whether $\max_{j\in\Omega} v_j$ is less than $\max_{j\in K\setminus\{i\}} v_j$ or not, and the third equality notes that, given our conditioning event X, there is probability one that $v_i \geqslant \max_{j\in K\setminus\{i\}} v_j$. Using (A.1), we can show that

$$\begin{aligned}
&E_{v_{-i}^k}\left(f_i(v_i, v_{-i})|X\right)\\
&= E_{v_{-i}^k}\left(E_{v_{k+1},\ldots,v_n}\left(\left(v_i - \max_{j\in(K\setminus\{i\})\cup\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in(K\setminus\{i\})\cup\Omega} v_j}\right)\middle| X\right)\\
&= E_{v_{-i}^k}\left(E_{v_{k+1},\ldots,v_n}\left((v_i)\, 1_{v_i \geqslant \max_{j\in\Omega} v_j} - \left(\max_{j\in(K\setminus\{i\})\cup\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in(K\setminus\{i\})\cup\Omega} v_j}\right)\middle| X\right)\\
&= E_{v_{-i}^k}\left(E_{v_{k+1},\ldots,v_n}\left(\begin{array}{l}(v_i)\, 1_{v_i \geqslant \max_{j\in\Omega} v_j} - \left(\max_{j\in K\setminus\{i\}} v_j\right)\\ \quad \times 1_{\max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j}\\ \quad - \max_{j\in\Omega} v_j 1_{v_i \geqslant \max_{j\in\Omega} v_j} + \left(\max_{j\in\Omega} v_j\right)\\ \quad \times 1_{\max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j}\end{array}\right)\middle| X\right)\\
&= E_{v_{-i}^k}\\
&\quad \times \left(E_{v_{k+1},\ldots,v_n}\left(\begin{array}{l}\left(v_i - \max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j}\\ \quad - \left(\max_{j\in K\setminus\{i\}} v_j - \max_{j\in\Omega} v_j\right) 1_{\max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j}\end{array}\right)\middle| X\right)\\
&= E_{v_{k+1},\ldots,v_n}\left(\left(v_i - \max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j}\right)\\
&\quad - E_{v_{-i}^k}\left(E_{v_{k+1},\ldots,v_n}\left(\left(\max_{j\in K\setminus\{i\}} v_j - \max_{j\in\Omega} v_j\right) 1_{\max_{j\in K\setminus\{i\}} v_j \geqslant \max_{j\in\Omega} v_j}\right)\middle| X\right)\\
&= \hat{p}(v_i) - E_{v_{-i}^k}\left(\hat{p}\left(\max_{j\in K\setminus\{i\}} v_j\right)\middle| X\right),
\end{aligned}$$

where the first equality uses the definition of f_i, the second equality separates two terms and uses the fact that, given the conditioning event X, the probability that $v_i \geqslant \max_{j\in K\setminus\{i\}} v_j$ is one, the third equality uses (A.1), the fourth equality collects terms, the fifth equality uses the fact that the term $\left(v_i - \max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\in\Omega} v_j}$ is not affected by the conditioning event X and does not involve variables v_{-i}^k, and the final equality uses the definition of $\hat{p}$.

This completes the proof that bidders report truthfully. It remains to show that individual rationality is satisfied. If bidder i does not join the ring, play is non-cooperative, and bidder i has expected payoff $E_v\left(\left(v_i - \max_{j\neq i} v_j\right) 1_{v_i \geqslant \max_{j\neq i} v_j}\right)$. If bidder $i \in K$ joins the ring, he expects payoff

$$E_v\left(\left(v_i - \max_{j\in\Omega} v_j\right) 1_{v_i \geqslant \max_{j\neq i} v_j}\right) - E_v\left(\hat{p}\left(\max_{j\in K\setminus\{i\}} v_j\right) 1_{v_i \geqslant \max_{j\in K\setminus\{i\}} v_j}\right) + s. \tag{A.2}$$

Substituting the definition of $\hat{p}$ and rearranging, (A.2) is equal to

$$
\begin{aligned}
&E_v\begin{pmatrix} \left(v_i-\max_{j\in\Omega} v_j\right)1_{v_i\geqslant \max_{j\neq i} v_j} \\ -\left(\max_{j\in K\setminus\{i\}} v_j-\max_{j\in\Omega} v_j\right)1_{v_i\geqslant \max_{j\in K\setminus\{i\}} v_j\geqslant \max_{j\in\Omega} v_j}\end{pmatrix}+s \\
&=E_v\begin{pmatrix} \left(v_i-\max_{j\in\Omega} v_j\right)1_{v_i\geqslant \max_{j\in\Omega} v_j\geqslant \max_{j\in K\setminus\{i\}} v_j} \\ +\left(v_i-\max_{j\in K\setminus\{i\}} v_j\right)1_{v_i\geqslant \max_{j\in K\setminus\{i\}} v_j\geqslant \max_{j\in\Omega} v_j}\end{pmatrix}+s \\
&=E_v\left(\left(v_i-\max_{j\neq i} v_j\right)1_{v_i\geqslant \max_{j\neq i} v_j}\right)+s,
\end{aligned}
$$

which is bidder i's expected payoff if he does not join the ring plus s. Because $s>0$, individual rationality is satisfied strictly. □

Proof of Proposition 2. Suppose there is a BCM in which non-highest-valuing ring members bid $\underline{v}$. Then ring members truthfully report their values and bid according to the recommendations of the center. Let $\imath$ be the index of the ring member (randomly selected in the case of a tie) with the highest report. Because ring member $\imath$'s payment to the ring does not depend upon his bid at the auction, his recommended bid must be optimal in the auction subgame. In particular, it must be that the center's recommended bid to bidder $\imath$, $\beta_\imath(v_1,\ldots,v_k)$, and the bids of the outside bidders, $\beta_i(v_i)$ for $i\in\Omega$, satisfy

$$
\begin{aligned}
\beta_\imath(v^k)&\in\arg\max_b E_{v_{-\imath}}\left((v_\imath-b)1_{b\geqslant \max_{j\in\Omega}\beta_j(v_j)}\,\middle|\, v_\imath=\max_{j\in K} v_j\right)\\
&=\arg\max_b E_{v_{-\imath}}\left((v_\imath-b)1_{b\geqslant \max_{j\in\Omega}\beta_j(v_j)}\right)
\end{aligned}
$$

and for $i\in\Omega$,

$$
\beta_i(v_i)\in\arg\max_b E_{v_{-i}}\left((v_i-b)1_{b\geqslant \max\left\{\beta_\imath(v^k),\max_{j\in\Omega\setminus\{i\}}\beta_j(v_j)\right\}}\,\middle|\, v_\imath=\max_{j\in K} v_j\right).
$$

Notice that the recommended bid to bidder $\imath$ depends only on bidder $\imath$'s value and not on the other ring members' values or the identity of the highest-valuing ring member, so we can define function $\hat{\beta}$ such that $\beta_\imath(v^k)=\hat{\beta}(v_\imath)$. Note also that for all $v>\underline{v}$, $\hat{\beta}(v)<v$.

Suppose ring member i has value $v_i\in(\hat{\beta}(v_\imath),v_\imath)$, which is a positive probability event. Let I_i be bidder i's information, if any, about the values of the other ring members as a result of learning his required payment to the center and his recommended bid. Given this information, bidder i forms beliefs (correct in equilibrium) about the values of the other ring members. Because the center recommends that ring member i bid $\underline{v}$, ring member i's belief must be that the probability that his value is highest is zero. Using the monotonicity of $\hat{\beta}$, this implies that i believes there is zero probability that a bid of $\hat{\beta}(v_i)$ would win the auction. Furthermore, since $v_i\in(\hat{\beta}(v_\imath),v_\imath)$ and beliefs are correct in equilibrium, ring member i must believe that there is positive probability that the center's highest recommended bid is less than i's value, i.e., $E_{v_{-i}}\left(1_{\hat{\beta}(v_\imath)<v_i}\,\middle|\, I_i\right)>0$. Thus, ring member i believes he has zero probability of winning the auction with a bid of $\hat{\beta}(v_i)$, but believes there exists some bid less than his value, but above $\hat{\beta}(v_i)$, with positive probability of winning. Thus, ring member i can profitably deviate from his recommended bid, a contradiction. □

Proof of Lemma 2. Let $(\mu,\hat{\beta}_\Omega)$ be an equilibrium BCM profile, where $\mu=(\beta_1,\ldots,\beta_k,p_1,\ldots,p_k)$.

Case 1. To show that with probability one, the highest ring bid is greater than or equal to $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$, we begin by supposing that, to the contrary, there exists a positive-measure set of ring members' values such that the highest ring bid is less than $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$. Given ring members' values in this set, after the ring announcements, but prior to bidding, there exists a ring member i whose information is such that he believes that there is positive probability that his value is highest, in which case all other ring members have recommended bids less than $\beta^*(v_i;\hat{\beta}_\Omega)$. Such a ring member can profitably deviate by bidding $\beta^*(v_i;\hat{\beta}_\Omega)$. To see this, suppose that ring member i's recommended bid is highest in the ring. Then by increasing his bid to be an optimal bid against the outside bidders, the ring member increases his expected payoff. If ring member i's recommended bid is not highest in the ring, then by increasing his bid, he at least weakly increases his expected payoff. This provides a contradiction.

Case 2. We know from Case 1 that the highest ring bid is less than $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$ only for a zero-measure set of ring members' values. Thus, the highest ring bid must either be equal to $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$ for a positive-measure set of ring members' values, or be greater than $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$ for a positive-measure set of ring members' values. To show that for the highest bid from the ring is greater than $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$ for a positive-measure set of ring members' values, we begin by supposing that, to the contrary, for all but a zero-measure set of values for ring members, the highest bid from the ring is equal to $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$.

Because the proof is long, we try to indicate some of the steps:

Step 1. Show the high ring bid is submitted by the highest-valuing ring member. As a first step in this case, we show that under our supposition, for all but a zero-measure set of values for ring members, the highest ring bid is submitted by the highest-valuing ring member. To prove this, suppose to the contrary that the highest ring bid is submitted by a ring member whose value is not highest in the ring. Then the highest-valuing ring member receives a recommended bid that is less than $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$. In this case, the highest-valuing ring member's information is such that he believes he is the highest-valuing ring member with positive probability, he believes he will win with his recommended bid with probability zero, and he believes that he will win with a bid greater than $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$ but less than $\max_{j\in K}\ v_j$ (his value) with positive probability, implying that the highest-valuing ring member has a profitable deviation, a contradiction.

Step 2. Write the expression for the optimal bid for a ring member. Since the highest ring bid of $\beta^*(\max_{j\in K}\ v_j;\hat{\beta}_\Omega)$ is submitted by the highest-valuing ring member, the definitions of β^{in} and $\beta^{\text{out}}_{k+1},\ldots,\beta^{\text{out}}_n$ given in the text imply that $\beta^*=\beta^{\text{in}}$ and for all $i\in\Omega$, $\hat{\beta}_i=\beta^{\text{out}}_i$.

Because $\beta^{\text{in}}(v_i)$ is an optimal bid for ring member i when he has no competition from other ring members, ring member i, given his information, never strictly prefers to bid less than $\beta^{\text{in}}(v_i)$ when there is competition from other ring members, holding fixed the bids of the outside bidders. In addition, ring member i never strictly prefers to bid more than v_i. This allows us to restrict attention to bids in the interval $\left[\beta^{\text{in}}(v_i),v_i\right]$. Thus, ring member i chooses his bid to solve:

$$\max_{b\in\left[\beta^{\text{in}}(v_i),v_i\right]} E_{v_{-i}}\left((v_i-b)\,1_{b\geqslant\max\left\{\max_{j\in K\setminus\{i\}}\ \beta_j(v^k),\max_{j\in\Omega}\ \beta^{\text{out}}_j(v_j)\right\}}\,\middle|\, I_i\right), \tag{A.3}$$

where $I_i=\left(v_i,\beta_i(v^k),p_i(v^k)\right)$ and for $j\in K$, $\beta_j(v^k)$ is the ring's recommended bid for ring member j.

Comment on remainder of proof. The remainder of the proof shows that for a positive-measure set of ring members' values, there exists a ring member i whose value is not highest in the ring, and whose recommended bid is $\beta^{\text{in}}(v_i)$, but for whom the maximand in (A.3) is increasing in b at $b = \beta^{\text{in}}(v_i)$, implying that this ring member can profitably deviate from the ring's recommendation, which contradicts $(\mu, \hat{\beta}_\Omega)$ being an equilibrium BCM profile and completes our proof.

Step 3. Rewrite the expression for the optimal bid for a ring member. Note that for $b \geqslant \beta^{\text{in}}(v_i)$,

$$\begin{aligned} &E_{v_{-i}}\left((v_i - b)\, 1_{b \geqslant \max\left\{\max_{j\in K\setminus\{i\}} \beta_j(v^k), \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)\right\}} \middle| I_i, \beta^{\text{in}}(v_i) > \max_{j\in K\setminus\{i\}} \beta_j(v^k)\right) \\ &\quad = E_{v_{-i}}\left((v_i - b)\, 1_{b \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)}\right), \end{aligned}$$

where the equality holds because, conditional on $\beta^{\text{in}}(v_i) > \max_{j\in K\setminus\{i\}} \beta_j(v^k)$, ring member i wins the auction with a bid $b \geqslant \beta^{\text{in}}(v_i)$ if and only if his bid b is greater than the bids of the outside bidders. Using this equality and letting

$$A_i(b) \equiv E_{v_{-i}}\left((v_i - b)\, 1_{b \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)}\right)$$

and

$$B_i(b; I_i) \equiv E_{v_{-i}}\left(\begin{array}{c} (v_i - b)\, 1_{b \geqslant \max\left\{\max_{j\in K\setminus\{i\}} \beta_j(v^k), \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)\right\}} \\ |I_i, \beta^{\text{in}}(v_i) \leqslant \max_{j\in K\setminus\{i\}} \beta_j(v^k) \end{array}\right),$$

the rules for conditional probabilities allow us to rewrite ring member i's problem in (A.3) as:

$$\begin{aligned} &\max_{b\in\left[\beta^{\text{in}}(v_i), v_i\right]} A_i(b) \Pr\left(\beta^{\text{in}}(v_i) > \max_{j\in K\setminus\{i\}} \beta_j(v^k) \middle| I_i\right) \\ &\quad + B_i(b; I_i) \Pr\left(\beta^{\text{in}}(v_i) \leqslant \max_{j\in K\setminus\{i\}} \beta_j(v^k) \middle| I_i\right), \end{aligned} \tag{A.4}$$

where the probabilities are with respect to v_{-i}. Note that by the definition of β^{in}, $\left.\frac{\partial A_i(b)}{\partial b}\right|_{b=\beta^{\text{in}}(v_i)} = 0$ (Assumption 1 implies differentiability). Note that $B_i(\beta^{\text{in}}(v_i); I_i) = 0$ and for all $b \in [\beta^{\text{in}}(v_i), v_i]$, $B_i(b; I_i) \geqslant 0$.

Comment on remainder of proof. To complete the proof, we need to show that the maximand in (A.3) is increasing in b at $b = \beta^{\text{in}}(v_i)$. Since $\left.\frac{\partial A_i(b)}{\partial b}\right|_{b=\beta^{\text{in}}(v_i)} = 0$, it is sufficient to show that for a positive-measure set of values, $\Pr\left(\beta^{\text{in}}(v_i) \leqslant \max_{j\in K\setminus\{i\}} \beta_j(v^k) \middle| I_i\right) > 0$ and $\left.\frac{\partial B_i(b; I_i)}{\partial b}\right|_{b=\beta^{\text{in}}(v_i)} > 0$. So we start by defining a set of values such that the first inequality holds. This inequality is satisfied, for example, if ring member i's value is not the highest in the ring. Then we rewrite the expression for B_i and differentiate.

Step 4. Set up an environment in which ring member i does not have the highest value in the ring. The next step in the proof is to consider a ring member i with value v_i and the set of other ring members' values such that the maximum value for the other ring members is greater than or equal to v_i, but the maximum bid for the other ring members using bid function β^{in} is less than v_i. We can easily show that ring member i's prior distribution must give positive weight to this possibility. And then we can show that there must be a positive-measure set of values for ring members other than i such than ring member i's information results in a posterior that gives positive weight to this possibility.

Let $G(\cdot)$ be the joint distribution of the values of the ring members other than i, and let $\bar{G}(\cdot)$ be the distribution of the highest from the values of the ring members other than i. Let $g(\cdot)$ and $\bar{g}(\cdot)$ be the corresponding densities. To conserve on notation, let $X_i(v_i)$ be the set of values for ring members other than i such that the maximum value for the other ring members is greater than or equal to v_i, but the maximum bid for the other ring members using bid function β^{in} is less than v_i, i.e.,

$$X_i(v_i) \equiv \left\{ v_{-i}^k \,\middle|\, \beta^{\text{in}}\left(\max_{j\in K\setminus\{i\}} v_j\right) \leqslant v_i \leqslant \max_{j\in K\setminus\{i\}} v_j \right\}.$$

Step 5. Define notation needed to describe ring member i's inference problem. Let $V(v_i, x, y)$ be the set of values for ring members other than i that are consistent with ring member i's receiving a recommended bid of x and required payment of y, i.e.,

$$V(v_i, x, y) \equiv \left\{ v_{-i}^k | (v_i, v_{-i}^k) \in \beta_i^{-1}(x) \cap p_i^{-1}(y) \right\}.$$

Then the probability that a ring member with value v_i has information $I_i = (v_i, x, y)$, conditional on $v_{-i}^k \in X_i(v_i)$, is

$$P(x, y) \equiv \int_{V(v_i,x,y)} g(\tilde{v}_{-i}^k | v_{-i}^k \in X_i(v_i))\, d\tilde{v}_{-i}^k,$$

which is well defined because we assume β_i and p_i are measurable functions. Note that because beliefs are correct in equilibrium, the unconditional beliefs must be equal to the integral of the conditional beliefs over all possible information.

Step 6. Rewrite B_i and differentiate with respect to b. Using the assumption of independent values, for $b \in \left[\beta^{\text{in}}(v_i), v_i\right]$, we can rewrite B_i as

$$\begin{aligned} B_i(b; I_i) &= (v_i - b)\,\bar{G}\left(\beta^{\text{in}^{-1}}(b) \,\middle|\, I_i, \beta^{\text{in}}(v_i) \leqslant \max_{j\in K\setminus\{i\}} \beta_j(v^k)\right) \Pr\left(b \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)\right) \\ &= (v_i - b)\,\bar{G}\left(\beta^{\text{in}^{-1}}(b) \,\middle|\, I_i, v_i \leqslant \max_{j\in K\setminus\{i\}} v_j^k\right) \Pr\left(b \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)\right) \\ &= (v_i - b)\,\bar{G}\left(\beta^{\text{in}^{-1}}(b) \,\middle|\, I_i, v_{-i}^k \in X_i(v_i)\right) \Pr\left(\beta^{\text{in}}\left(\max_{j\in K\setminus\{i\}} v_j\right) \leqslant v_i \,\middle|\, I_i\right) \\ &\quad \times \Pr\left(b \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)\right), \end{aligned}$$

where the second equality holds because the highest ring bid is $\beta^{\text{in}}(\max_{j\in K} v_j)$ and is submitted by the highest-valuing ring member, and the third equality holds because the expected payoff from a bid $b \in \left[\beta^{\text{in}}(v_i), v_i\right]$ is zero conditional on $v_i < \beta^{\text{in}}(\max_{j\in K\setminus\{i\}} v_j)$. Note that $\bar{G}\left(\beta^{\text{in}^{-1}}(b)\middle| I_i, v_{-i}^k \in X_i(v_i)\right)$ is equal to zero when evaluated at $b = \beta^{\text{in}}(v_i)$. Thus,

$$\begin{aligned} \frac{\partial B_i(\beta^{\text{in}}(v_i); I_i)}{\partial b} &= \left(v_i - \beta^{\text{in}}(v_i)\right) \bar{g}\left(v_i \,\middle|\, I_i, v_{-i}^k \in X_i(v_i)\right) \left.\frac{\partial \beta^{\text{in}^{-1}}(b)}{\partial b}\right|_{b=\beta^{\text{in}}(v_i)} \\ &\quad \times \Pr\left(\beta^{\text{in}}(\max_{j\in K\setminus\{i\}} v_j) \leqslant v_i | I_i\right) \Pr\left(\beta^{\text{in}}(v_i) \geqslant \max_{j\in\Omega} \beta_j^{\text{out}}(v_j)\right). \end{aligned}$$

Step 7. Derive sufficient conditions for $\frac{\partial B_i(\beta^{\text{in}}(v_i); I_i)}{\partial b} > 0$. Using the expression above for $\frac{\partial B_i(\beta^{\text{in}}(v_i); I_i)}{\partial b}$, we see that it is positive for $v_i > \underline{v}$ if and only if

$$\bar{g}\left(v_i \,\middle|\, I_i, v_{-i}^k \in X_i(v_i)\right) > 0 \tag{A.5}$$

and

$$\Pr\left(\beta^{\text{in}}\left(\max_{j\in K\setminus\{i\}} v_j\right) \leqslant v_i \middle| I_i\right) > 0. \tag{A.6}$$

Step 8. Show that the sufficient condition (A.5) *holds.* Note that

$$\bar{G}\left(\beta^{\text{in}^{-1}}(b)\,\middle|\, v_{-i}^k \in X_i(v_i)\right) = \frac{\int_{v_i}^{\beta^{\text{in}^{-1}}(b)} \bar{g}(v)\,dv}{\bar{G}(\beta^{\text{in}^{-1}}(v_i)) - \bar{G}(v_i)},$$

which is increasing in b for $b \geqslant \beta^{\text{in}}(v_i)$, i.e.,

$$\bar{g}\left(\beta^{\text{in}^{-1}}(b)\,\middle|\, v_{-i}^k \in X_i(v_i)\right) > 0. \tag{A.7}$$

Using (A.7) (evaluated at $b = \beta^{\text{in}}(v_i)$),

$$\begin{aligned}0 &< \bar{g}\left(v_i \,\middle|\, v_{-i}^k \in X_i(v_i)\right)\\ &= \int_{X_i(v_i)} \bar{g}\left(v_i \,|\, I_i = \left(\beta_i(v_i, \tilde{v}_{-i}^k), p_i(v_i, \tilde{v}_{-i}^k)\right), v_{-i}^k \in X_i(v_i)\right)\\ &\quad \times P(\beta_i(v_i, \tilde{v}_{-i}^k), p_i(v_i, \tilde{v}_{-i}^k)) g\left(\tilde{v}_{-i}^k \,\middle|\, v_{-i}^k \in X_i(v_i)\right) d\tilde{v}_{-i}^k,\end{aligned}$$

which implies that for a positive measure subset of $X_i(v_i)$, call it $\tilde{X}_i(v_i)$, ring member i's information is such that

$$\bar{g}(v_i | I_i, v_{-i}^k \in X_i(v_i)) > 0. \tag{A.8}$$

Thus, sufficient condition (A.5) holds on this set.

Step 9. Show that ring member i's recommended bid is $\beta^{\text{in}}(v_i)$ and that sufficient condition (A.6) *holds.* Let v^k be such that $v_i > \underline{v}$ and $v_{-i}^k \in \tilde{X}_i(v_i)$ (a positive probability event). In this case, ring member i, given his information, must place positive probability weight on the event that a bid of less than his value wins the auction. If ring member i's recommended bid is not equal to $\beta^{\text{in}}(v_i)$, then ring member i knows with probability one that his recommendation is not highest and that a bid equal to his recommended bid has zero probability of winning the auction, implying that the ring member can profitably deviate by bidding some amount greater than his recommended bid, but less than his value, a contradiction. Thus, it must be that ring member i's recommended bid is equal to $\beta^{\text{in}}(v_i)$. In addition, ring member i's information must be such that (A.6) holds.

Step 10. Conclude by showing ring member i has an incentive to deviate. Since we have shown there exists a positive-measure set of values such that $\Pr\left(\beta^{\text{in}}(v_i) \leqslant \max_{j\in K\setminus\{i\}} \beta_j(v^k) \middle| I_i\right) > 0$ and such that (A.5) and (A.6) hold, which implies that $\frac{\partial B_i(\beta^{\text{in}}(v_i); I_i)}{\partial b} > 0$. Thus, referring to (A.4), ring member i can profitably deviate by bidding some amount greater than $\beta^{\text{in}}(v_i)$. Since ring member i's recommended bid is $\beta^{\text{in}}(v_i)$, this is a contradiction. □

Proof of Proposition 3. Let $\hat{\beta}_\Omega$ be the equilibrium bid functions for the outside bidders. Let $B(v^k)$ be the equilibrium highest ring bid as a function of $v_1, \ldots, v_k$. By Lemma 2, for all but a zero-measure set of ring members' values, $B(v^k) \geqslant \beta^*(\max_{j\in K} v_j; \hat{\beta}_\Omega)$, with a strict inequality for a positive-measure set of values.

To improve readability, we use RM as an abbreviation for ring member in the remainder of the proof.

Step 1. Suppose that for all but a zero-measure set of value realizations, one RM bids $B(v^k)$ and all other RMs bid some amount less than $\beta^*(\max_{j\in K} v_j; \hat{\beta}_\Omega)$. In this case, we can show two things: (1) If RM i's recommended bid is $B(v^k)$ then $B(v^k) \leqslant v_i$. (2) It must be the RM with the highest value whose recommended bid is $B(v^k)$.

First, suppose that for a given vector of RMs' values, RM i is the RM whose recommended bid is $B(v^k)$, and suppose that $v_i < B(v^k)$. Then RM i's beliefs place positive probability on his winning the auction with a bid equal to $B(v^k)$, and so RM i can profitably deviate by bidding v_i, a contradiction. Thus, for all RMs' values, if RM i's recommended bid is $B(v^k)$ then $B(v^k) \leqslant v_i$.

Second, suppose that the RM whose recommended bid is $B(v^k)$ is not the RM with the highest value. If RM i is the RM with the highest value, then RM i's recommended bid is less than $\beta^*(v_i; \hat{\beta}_\Omega)$, and so RM i's beliefs place probability one on his recommended bid not being the highest among the RMs, and RM i's beliefs place positive probability on the highest recommended bid from among the RMs being less than his value. Thus, RM i can profitably deviate by submitting a bid greater than $\beta^*(v_i; \hat{\beta}_\Omega)$ but less than v_i, a contradiction. Thus, for all RMs' values, it must be the RM with the highest value whose recommended bid is $B(v^k)$.

Step 2. Continuing with the supposition that for all but a zero-measure set of value realizations, one RM bids $B(v^k)$ and all other RMs bid some amount less than $\beta^*(\max_{j\in K} v_j; \hat{\beta}_\Omega)$, we can now show that at least one RM must have a profitable deviation, which is the desired contradiction. Let RM i be a RM such that with positive probability RM i's value is highest in the ring and his recommended bid is strictly greater than $\beta^*(\max_{j\in K} v_j; \hat{\beta}_\Omega)$ (at least one such RM must exist). In such cases, RM i has a recommended bid greater than $\beta^*(v_i; \hat{\beta}_\Omega)$ and believes that (i) if his value is highest, then all other RMs' recommended bids are less than $\beta^*(v_i; \hat{\beta}_\Omega)$, in which case, by the definition of β^*, RM i strictly prefers to deviate from his recommended bid by bidding $\beta^*(v_i; \hat{\beta}_\Omega)$, and (ii) if his value is not highest, then his recommended bid has probability zero of winning, so he is no worse off by bidding $\beta^*(v_i; \hat{\beta}_\Omega)$. In such cases, RM i's information must be such that he believes his value is highest with positive probability, so the deviation to $\beta^*(v_i; \hat{\beta}_\Omega)$ is profitable, a contradiction. Thus, at least one RM in addition to the highest-valuing RM must bid greater than or equal to $\beta^*(\max_{j\in K} v_j; \hat{\beta}_\Omega)$.

Step 3. To complete the proof, suppose that there exists $\varepsilon > 0$ such that there is zero probability that the two highest ring bids are within ε of each other. Suppose the highest recommended bid to any RM is to RM i, and suppose the recommendation, β_i, satisfies $\beta_i > \beta^*(v_i; \hat{\beta}_\Omega)$, which is a positive probability event. Then RM i can profitably deviate by bidding $b \in (\max\{\beta^*(v_i; \hat{\beta}_\Omega), \beta_i - \varepsilon\}, \beta_i)$, a contradiction. □

Proof of Proposition 4. Let r^1 be the highest report and r^2 the second-highest report in the ring. Let v^1 be the highest value and v^2 the second-highest value in the ring. Consider the following bidding rule: the center recommends that the ring member with the highest report bid $\beta^{\text{in}}(r^1)$ and that all others bid $\underline{v}$. Consider the following payment rule: if ring member i has the highest report, he pays the center $\hat{p}(r^2) - s_i$, and all other ring members $j \neq i$ pay $-s_j$. Consider whether ring members report truthfully. If all other ring members report truthfully and a ring member with value $\hat{v}$, where $\hat{v} < v^1$, reports $\hat{r} > \hat{v}$, causing him to have the highest report, i.e., $\hat{v} < v^1 < \hat{r}$, then his expected payoff from participating in the auction is $\hat{p}(\hat{v})$, but his payment to the center increases by $\hat{p}(v^1) > \hat{p}(\hat{v})$, so the deviation is not profitable. If a ring member with value v^1

reports $\hat{r} < v^1$, causing him not to have the highest report, i.e., $\hat{r} < v^2 < v^1$, then his expected payoff from the auction is zero, but if he reports truthfully, his expected payoff from the auction minus the payment to the center of $\hat{p}(v^2)$ is positive, so the deviation is not profitable. Because all other deviations have zero expected payoff, this establishes that no ring member has an incentive to misrepresent his value to the center.

We now show that strict individual rationality holds. Let G be the distribution of the highest valuation from the k ring members, and let G_{-i} be the distribution of the highest valuation from the $k-1$ ring members other than ring member i. Note that ring member i's expected payoff from participation in the mechanism can be written as $s_i + \int_0^{\bar{v}} \int_0^{v_i} \left(\hat{p}(v_i) - \hat{p}(x)\right) dG_{-i}(x)\, dF_i(v_i)$ since ring member i receives payment s_i and then, if ring member i's value is highest in the ring, he has expected payoff $\hat{p}(v_i)$ from participation in the auction, but must make payment $\hat{p}(x)$ to the ring center, where x is the highest value from among the other $k-1$ ring members. Ex post budget balance requires that $\sum_{i=1}^k s_i = \sum_{i=1}^k \int_0^{\bar{v}} \int_0^{v_i} \hat{p}(x)\, dG_{-i}(x)\, dF_i(v_i)$. Thus, strict individual rationality can be satisfied for all ring members if and only if

$$
\begin{aligned}
\sum_{i=1}^{k} \Pi_i^n(F_1, \ldots, F_n) &< \sum_{i=1}^{k} \left(s_i + \int_0^{\bar{v}} \int_0^{v_i} \left(\hat{p}(v_i) - \hat{p}(x)\right) dG_{-i}(x)\, dF_i(v_i) \right) \\
&= \sum_{i=1}^{k} \int_0^{\bar{v}} \int_0^{v_i} \hat{p}(v_i)\, dG_{-i}(x)\, dF_i(v_i) \\
&= \int_0^{\bar{v}} \hat{p}(y)\, dG(y) \\
&= \Pi_1^{n-k+1}(F_1 \ldots F_k, F_{k+1}, \ldots, F_n). \qquad \square
\end{aligned}
$$

Proof of Proposition 5. Because the use of a shill does not affect a ring member's payment to the center in a BSM with shills, the proof of Proposition 2 implies that for any collusive mechanism for a first-price auction in which non-highest-valuing ring members do not submit bids above $\underline{v}$ (either under their own names or under aliases), there is positive probability that a non-highest-valuing ring member can increase his expected payoff by using an alias to submit a bid greater than $\underline{v}$. In particular, a ring member i with value $v_i \in \left(\underline{v}, \max_{j \in K \setminus \{i\}} v_j\right)$ whose recommended bids are $\underline{v}$ can profitably deviate by using a shill to bid $v_i - \varepsilon$ for some $\varepsilon \in \left(0, v_i - \underline{v}\right)$. $\square$

Proof of Proposition 6. We show that the payoff from an equilibrium BSM with shills profile can be replicated with an equilibrium BCM with shills profile if the center can submit a bid. Let (μ^s, β_Ω) be an equilibrium BSM with shills profile, where $\mu^s = (\beta_1, \ldots, \beta_k, \beta_1^s, \ldots, \beta_k^s, p_1, \ldots, p_k)$. Define $\tilde{\mu}^s = (\tilde{\beta}_1, \ldots, \tilde{\beta}_k, \tilde{\beta}_1^s, \ldots, \tilde{\beta}_k^s, \tilde{p}_1, \ldots, \tilde{p}_k)$ as follows: for all v and all $i \in K$, let $\tilde{\beta}_i(v) \equiv \max\left\{\beta_i(v), \beta_i^s(v)\right\}$, $\tilde{\beta}_i^s(v) \equiv \underline{v}$, and $\tilde{p}_i \equiv p_i$. Note that $(\tilde{\mu}^s, \beta_\Omega)$ is also an equilibrium BSM with shills profile, and the ring payoff is the same as in (μ^s, β_Ω). We now show that $(\tilde{\mu}^s, \beta_\Omega)$ is also an equilibrium *BCM* with shills profile if we allow the ring center to submit a bid using a mixed strategy that mixes aggressively just below the highest recommended bid for the ring members $\max_{i \in K} \tilde{\beta}_i(v)$ (for example, have the center mix according to a distribution that satisfies the conditions of [8, p. 374]). Given this behavior by the center, under BCM with shills $(\tilde{\mu}^s, \beta_\Omega)$, no ring member can profitably deviate by submitting a bid less than his recommended bid, either under his own name or under an alias. In addition, no ring member can profitably deviate by submitting a bid greater than his recommended bid—if he could, then that deviation would also have been profitable (and feasible if made under an alias) under a BSM with shills, which contradicts $(\tilde{\mu}^s, \beta_\Omega)$ being an equilibrium BSM with shills profile. Thus, $(\tilde{\mu}^s, \beta_\Omega)$ is also an equilibrium BCM with shills profile and payoffs are the same with equilibrium BSM with shills (μ^s, β_Ω). Finally, since any outcome that can be achieved with a BCM with shills can also be achieved with a BCM without shills, the result holds for BCMs without shills as well. □

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