On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games*

Philip J. Reny
Department of Economics
University of Chicago

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Abstract

We extend and strengthen both Athey's (2001) and McAdams' (2003) results on the existence of monotone pure strategy equilibria in Bayesian games. We allow action spaces to be compact locally-complete metrizable semilattices, type spaces to be partially ordered complete separable metric spaces, and can handle both a weaker form of quasisupermodularity than is employed by McAdams and a weaker single-crossing property than is required by both Athey and McAdams. Our proof is based upon contractibility rather than convexity of best reply sets. Finally, we do not require the Milgrom-Weber (1985) absolute continuity condition on the joint distribution of types. Several examples illustrate the scope of the result, including new applications to multiunit auctions with risk-averse bidders.

1. Introduction

In an important paper, Athey (2001) demonstrates that a monotone pure strategy equilibrium exists whenever a Bayesian game satisfies a Spence-Mirlees single-crossing property. Athey's result is now a central tool for establishing the existence of monotone pure strategy equilibria in auction theory (see e.g., Athey (2001), Reny and Zamir (2004)). Recently, McAdams (2003) has shown that Athey's results, which exploit the assumed total ordering of the players' one-dimensional type and action spaces, can be extended to settings in which type and action spaces are multi-dimensional and only partially ordered. This permits new existence results in auctions with multi-dimensional types and multi-unit demands (see McAdams (2004)). The techniques employed by Athey and McAdams, while ingenious, have their limitations and do not appear to easily extend beyond the environments they consider. We therefore introduce a new approach.

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The approach taken here exploits an important unrecognized property of a large class of Bayesian games. In these games, the players pure-strategy best-reply sets, while possibly nonconvex, are always contractible.[1] This observation permits us to generalize the results of Athey and McAdams in several directions. First, we permit infinite-dimensional type spaces and infinite-dimensional action spaces. Both can occur, for example, in share-auctions where a bidder's type is a function expressing his marginal valuation at any quantity of the good, and where a bidder's action is a downward-sloping demand schedule. Second, even when type and action spaces are subsets of Euclidean space, we permit more general joint distributions over types, allowing moving supports,[2] as well as permitting positive probability on lower dimensional subsets, which can be useful when modeling random demand in auctions. Third, our approach allows general partial orders on both type spaces and action spaces. This can be rather helpful in establishing existence because, while single-crossing may fail for one partial order, it might nonetheless hold for another, in which case our existence result can still be applied. Finally, we also weaken the single-crossing assumption, thereby further expanding the class of games covered by our analysis.

The key to our approach is to employ a more powerful fixed point theorem than those employed in Athey (2001) and McAdams (2003). Both Athey and McAdams apply a fixed-point theorem to the product of the players' best-reply correspondences — Athey applies Kakutani's theorem, McAdams applies Glicksberg's theorem. In both cases, essentially all of the effort is geared toward proving that sets of monotone pure-strategy best replies are convex. Our central observation is that this impressive effort is unnecessary and, more importantly, that the additional structure imposed to achieve the desired convexity (i.e., *Euclidean* type spaces with the *coordinatewise* partial order, *Euclidean sublattice* action spaces, *absolutely continuous* type distributions), is unnecessary as well.

The fixed point theorem upon which our approach is based is due to Eilenberg and Montgomery (1946) and does not require the correspondence in question to be convex-valued. Rather, the correspondence need only be contractible-valued. Consequently, we need only demonstrate that monotone pure-strategy best-reply sets are contractible. While this task need not be straightforward in general, it turns out to be essentially trivial in the class of Bayesian games of interest here. To gain a sense of this, note first that a pure strategy — a function from types to actions — is a best reply for a player if and only if it is a pointwise best reply for almost every type of that player. Consequently, any piecewise combination of two best replies — i.e., a strategy equal to one of the best replies on some subset of types and equal to the other best reply on the remainder of types — is also a best reply. Thus, by reducing the set of types on which the first best reply is employed and increasing the set of

[1] A set is contractible if it can be continuously deformed, within itself, to a single point. Convex sets are contractible, but contractible sets need not be convex (e.g., the symbol "+" viewed as a subset of $\mathbb{R}^2$).

[2] That is, when some player has private information about the support of another player's private information.

types on which the second is employed, it is possible to move from the first best reply to the second, all the while remaining within the set of best replies. With this simple observation, the set of best replies can be shown to be contractible.[3]

Because contractibility of best-reply sets follows almost immediately from the pointwise almost everywhere optimality of best replies, we are able to expand the domain of analysis well beyond Euclidean type and action spaces, and most of our additional effort is directed here. In particular, we require and prove two new results about the space of monotone functions from partially ordered complete separable metric spaces endowed with an appropriate probability measure into compact metric semilattices. The first of these results (Lemma A.10) is a generalization of Helley's selection theorem, stating that any sequence of monotone functions possess a pointwise almost everywhere convergent subsequence. The second result (Lemma A.15) states that the space of monotone functions is an absolute retract, a property that, like convexity, renders a space amenable to fixed point analysis. In contrast, both of these results would be straightforward to establish with the additional structure imposed by Athey and McAdams.

Our main result, Theorem 4.1, is as follows. Suppose that action spaces are compact locally-complete metric semilattices, that type spaces are partially ordered complete separable metric spaces, that payoffs are continuous in actions for each type vector, and that the joint distribution over types induces marginals for each player assigning probability zero to any set with no strictly ordered points.[4] If, whenever the others employ monotone pure strategies, each player's set of monotone pure-strategy best replies is nonempty and join-closed,[5] then a monotone pure strategy equilibrium exists.

We provide several applications yielding new existence results. First, we consider both uniform-price and discriminatory multi-unit auctions with independent private values. We depart from standard assumptions by permitting bidders to be risk averse. Under risk aversion, monotonicity of best replies is known to fail under the standard coordinatewise partial order over types. Nevertheless, by employing an alternative, yet natural, partial order over types, we are able to demonstrate the existence of a monotone pure strategy equilibrium with respect to this partial order. In the uniform-price auction, no additional assumptions are required, while in the discriminatory auction we require each bidder to have CARA preferences. Our second application considers a price-competition game between firms selling differentiated products. Firms have private information about their constant marginal cost as well as private information about market demand. While it is natural to assume that costs may be affiliated, in the context we consider it is less natural to assume that

[3] Because we are concerned with *monotone* pure strategy best replies, some care must be taken to ensure that one maintains monotonicity throughout the contraction. Further, continuity of the contraction requires appropriate assumptions on the distribution over players' types. In particular there can be no atoms.

[4] Two points are strictly ordered if every point in some neighborhood of one is greater than every point in some neighborhood of the other.

[5] That is, the pointwise supremum of any pair of best replies is also a best reply.

information about market demand is affiliated. Nonetheless, and again through a judicious choice of a partial order over types, we are able to establish the existence of a pure strategy equilibrium that is monotone in players' costs, but not necessarily monotone in their private information about demand. Our final application establishes the existence of monotone *mixed* strategy equilibria when type spaces have atoms.[6]

If in addition to our assumptions on payoffs, the actions of distinct players are strategic complements, Van Zandt and Vives (2006) have shown that even stronger results can be obtained. They prove that monotone pure strategy equilibria exist under somewhat more general distributional, type-space and action-space assumptions than we impose here, and demonstrate that such an equilibrium can be obtained through iterative application of the best reply map.[7] In our view, Van Zandt and Vives (2006) obtain perhaps the strongest possible results for the existence of monotone pure strategy equilibria in Bayesian games when strategic complementarities are present. Of course, while many interesting economic games exhibit strategic complements, many do not. Indeed, many auction games satisfy the hypotheses required to apply our result here, but fail to satisfy the strategic complements condition.[8] The two approaches are therefore complementary.

The remainder of the paper is organized as follows. Section 2 presents the essential ideas as well as the corollary of Eilenberg and Montgomery's (1946) fixed point theorem that is central to our approach. Section 3 describes the formal environment, including semilattices and related issues. Section 4 contains our main result and a corollary which itself strictly generalizes the results of both Athey and McAdams. Section 5 provides several illustrative examples, and Section 6 contains the proof of our main result.

2. The Main Idea[9]

As mentioned in the introduction, the proof of our main result is based upon a fixed point theorem that permits the correspondence for which a fixed point is sought — in our case, the product of the players' monotone pure best reply correspondences — to have contractible rather than convex values.

In this section, we introduce this fixed point theorem and also illustrate the ease with which the contractibility of sets of monotone pure strategy best replies can be established, focussing on the most basic case in which type spaces are $[0, 1]$, action spaces are subsets of

[6] A player's mixed strategy is monotone if all actions in the support of one of his types are weakly greater than all actions in the support of any lower type.

[7] Related results can be found in Milgrom and Roberts (1990) and Vives (1990).

[8] In a first-price IPV auction, for example, a bidder might increase his bid if his opponent increases her bid slightly when her private value is high. However, for sufficiently high increases in her bid at high private values, the bidder might be better off reducing his bid (and chance of winning) to obtain a higher surplus when he does win. Such nonmonotonic responses to changes in the opponent's strategy are not possible under strategic complements.

[9] Readers more interested in applying our main result than in its proof can skip the present section.

$[0, 1]$, and the marginal distribution over each player's type space is atomless.

A subset X of a metric space is *contractible* if for some $x_0 \in X$ there is a continuous function $h : [0, 1] \times X \to X$ such that for all $x \in X$, $h(0, x) = x$ and $h(1, x) = x_0$. We then say that h is a *contraction* for X.

Note that every convex set is contractible since, choosing any point x_0 in the set, the function $h(\tau, x) = (1 - \tau)x + \tau x_0$ is a contraction. On the other hand, there are contractible sets that are not convex (e.g., the symbol "+"). Hence, contractibility is a strictly more permissive condition than convexity.

A subset X of a metric space Y is said to be a *retract* of Y if there is a continuous function mapping Y onto X leaving every point of X fixed. A metric space (X, d) is an *absolute retract* if for every metric space (Y, δ) containing X as a closed subset and preserving its topology, X is a retract of Y. Examples of absolute retracts include closed convex subsets of Euclidean space or of any metric space, and many nonconvex sets as well (e.g., any contractible polyhedron).[10] The fixed point theorem we make use of is the following corollary of an even more general result due to Eilenberg and Montgomery (1946).[11]

Theorem 2.1. *Suppose that a compact metric space (X, d) is an absolute retract and that $F : X \twoheadrightarrow X$ is an upper hemicontinuous, nonempty-valued, contractible-valued correspondence. Then F has a fixed point.*

For our purposes, the correspondence F is the product of the players' monotone pure strategy best reply correspondences and X is the product of their sets of monotone pure strategies. While we must eventually establish all of the properties necessary to apply Theorem 2.1, our modest objective for the remainder of this section is to show, with remarkably little effort, that in the simple $[0, 1]$ type and action space environment considered here, F is contractible-valued, i.e., that monotone pure best reply sets are contractible.

Fix monotone pure strategies for other players, and suppose that $s^* : [0, 1] \to A$ is a monotone best reply for player 1, where $A \subseteq [0, 1]$ is player 1's action set.[12] We shall provide a contraction that shrinks player 1's entire set of monotone best replies, within itself, to the function s^*. The simple, but key, observation is that a pure strategy is a best reply for player 1 if and only if it is a pointwise best reply for almost every type $t \in [0, 1]$ of player 1.

Consider the following candidate contraction map. For $\tau \in [0, 1]$ and any monotone best

[10] Indeed, a compact subset, X, of Euclidean space is an absolute retract if and only if it is contractible and locally contractible. The latter means that for every $x_0 \in X$ and every neighborhood U of x_0, there is a neighborhood V of x_0 and a continuous $h : [0, 1] \times V \to U$ such that $h(0, x) = x$ and $h(1, x) = x_0$ for all $x \in V$.

[11] Theorem 2.1 follows directly from Eilenberg and Montgomery (1946) Theorem 1, because every absolute retract is a contractible absolute neighborhood retract (Borsuk (1966), V (2.3)) and every nonempty contractible set is acyclic (Borsuk (1966), II (4.11)).

[12] We consider the empty set contractible by convention. Hence, to establish contractibility, it is without loss to suppose that the set of monotone best replies is nonempty.

reply, s, for player 1, define $h(\tau, s) : [0,1] \to A$ as follows:

$$h(\tau, s)(t) = \begin{cases} s(t), & \text{if } t \le |1-2\tau| \text{ and } \tau < 1/2, \\ s^*(t), & \text{if } t \le |1-2\tau| \text{ and } \tau \ge 1/2, \\ \max(s^*(t), s(t)), & \text{if } t > |1-2\tau|. \end{cases}$$

Note that $h(0,s) = s$, that $h(1,s) = s^*$, and that $h(\tau, s)(t)$ is always either $s^*(t)$ or $s(t)$, and so is a best reply for almost every t. Hence, by the key observation in the previous paragraph, $h(\tau, s)(\cdot)$ is a best reply. The pure strategy $h(\tau, s)(\cdot)$ is also clearly monotone. It can also be shown that, so long as the marginal distribution over player 1's type is atomless, the monotone pure strategy $h(\tau, s)(\cdot)$ varies continuously in the arguments τ and s, when the distance between two strategies of player 1 is defined to be the integral with respect to his type distribution of their absolute pointwise difference (see Section 6). Consequently, h is a contraction, and so player 1's set of monotone best replies is contractible. It's that simple.

Figure 2.1 shows how the contraction works when player 1's set of actions A happens to be finite, so that his set of monotone best replies cannot be convex in the usual sense unless it is a singleton. Three monotone functions are shown in each panel, where 1's actions are on the vertical axis and 1's types are on the horizontal axis. The thin dashed line step function (black) is s, the thick solid line step function (green) is s^*, and the very thick solid line step function (red) is the step function determined by the contraction h.

In panel (a), $\tau = 0$ and so the very thick (red) step function coincides with s. The position of the vertical line (blue) appearing in each panel represents the value of τ. When $\tau = 0$ the vertical line is at the far right-hand side, as shown in panel (a). As indicated by the arrow, the vertical line moves continuously toward the origin as τ moves from 0 to 1/2. The very thick (red) step function determined by the contraction h is $s(t)$ for values of t to the left of the vertical line and is $\max(s^*(t), s(t))$ for values of t to the right; see panels (a)-(c). Note that this step function therefore changes continuously with τ, in a pointwise sense, and that when $\tau = 1/2$ this function is $\max(s^*(\cdot), s(\cdot))$.

In panels (d)-(f), τ increases from 1/2 to 1 and the vertical line moves from the origin continuously to the right. For these values of τ, the very thick (red) step function determined by the contraction h is now $s^*(t)$ for values of t to the left of the vertical line and is $\max(s^*(t), s(t))$ for values of t to the right. Hence, when $\tau = 1$, the contraction yields $s^*(\cdot)$; see panel (f). So altogether, as τ moves continuously from 0 to 1, the image of the contraction moves continuously from s to s^*.

Two points are worth mentioning before moving on. First, as just illustrated, single-crossing plays no role in establishing the contractibility of sets of monotone best replies. As we shall see, single-crossing is needed only to help ensure the existence of monotone pure strategy best replies. Thus, the present approach clarifies the role of single-crossing

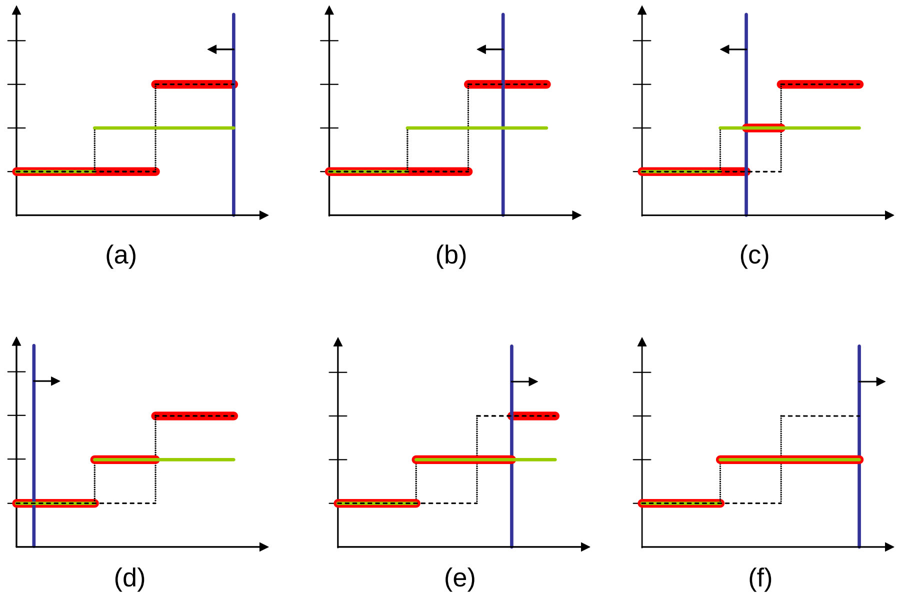

Figure 2.1: The Contraction

insofar as the existence of monotone pure strategy equilibrium is concerned.[13] Second, the action spaces employed in the above illustration are totally ordered, as in Athey (2001). Consequently, if two actions are optimal for some type of player 1, then the maximum of the two actions, being one or the other of them, is also optimal. The optimality of the maximum of two optimal actions is necessary for ensuring that the function $h(\tau, s)$ is itself a monotone best reply. When action spaces are only partially ordered (e.g., when actions are multi-dimensional with the coordinatewise partial order), the maximum of two optimal actions need not even be well-defined, let alone optimal. Therefore, to also cover partially ordered action spaces, we assume in the sequel (see Section 3) that action spaces are semilattices — i.e., that for every pair of actions there is a least upper bound (l.u.b.) — and that the l.u.b.. of two optimal actions is optimal. Stronger versions of both assumptions are employed in McAdams (2003).

3. The Environment

3.1. Partial Orders, Lattices and Semilattices

Let A be a nonempty set partially ordered by $\geq$.[14] For $a, b \in A$, if the set $\{a, b\}$ has a least upper bound (l.u.b..) in A, then it is unique and will be denoted by $a \vee b$, the *join* of a and b. In general, such a bound need not exist. However, if every pair of points in A has

[13] Both Athey (2001) and McAdams (2003) employ single-crossing to help establish the existence of monotone best replies *and* to establish the convexity of the set of monotone best replies. Their single-crossing condition is therefore more restrictive than necessary. See Section 4.1.

[14] Hence, $\geq$ is transitive ($a \geq b$ and $b \geq c$ imply $a \geq c$), reflexive ($a \geq a$), and antisymmetric ($a \geq b$ and $b \geq a$ imply $a = b$).

an l.u.b.. in A, then we shall say that A is a *semilattice.* It is straightforward to show that, in a semilattice, every finite set, $\{a, b, ..., c\}$, has a least upper bound, which we denote by $\vee\{a, b, ..., c\}$ or $a \vee b \vee ... \vee c$.

If the set $\{a, b\}$ has a greatest lower bound (g.l.b.) in A, then it too is unique and it will be denoted by $a \wedge b$, the *meet* of a and b. Once again, in general, such a bound need not exist. If every pair of points in A has both an l.u.b.. in A and a g.l.b. in A, then we shall say that A is a *lattice.*[15]

Clearly, every lattice is a semilattice. However, the converse is not true. For example, employing the coordinatewise partial order on vectors in $\mathbb{R}^m$, the set of vectors whose sum is at least one is a semilattice, but not a lattice.

If A is a metric space, a partial order $\geq$ on A is called *measurable* (*closed*) if $\{(a, b) \in A \times A : b \geq a\}$ is a Borel measurable (closed) subset of $A \times A$. Any two distinct points a, b in A are *strictly ordered* if there are neighborhoods U of a and V of b such that $u \geq v$ for every $u \in U$ and every $v \in V$.

A *metric semilattice* is a semilattice, A, endowed with a metric under which the join operator, $\vee$, is continuous as a function from $A \times A$ into A.[16] Every finite semilattice is a metric semilattice as is every sublattice of $\mathbb{R}^m$ where the join of any two points is their coordinatewise maximum. Note also that because in a semilattice $b \geq a$ if and only if $a \vee b = b$, a partial order in a metric semilattice is necessarily closed.[17]

A semilattice A is *complete* if every nonempty subset S of A has a least upper bound, $\vee S$, in A. A metric semilattice A is *locally complete* if for every $a \in A$ and every neighborhood U of a, there is a neighborhood W of a contained in U such that every nonempty subset S of W has a least upper bound, $\vee S$, contained in U.[18]

Many semilattices are locally complete. For example, local completeness holds trivially in any finite semilattice, and more generally in any compact (Euclidean-) metric semilattice in $\mathbb{R}^K$ endowed with the coordinatewise partial order (see Lemma A.17). On the other hand, infinite-dimensional metric semilattices need not be locally complete even if they are compact.[19] Indeed, it can be shown (see Lemma A.16) that a compact metric semilattice A is locally complete if and only if for every $a \in A$ and every sequence $a_n \to a$, $\lim_m(\vee_{n\geq m} a_n) =$

[15] Defining a semilattice in terms of the join operator, $\vee$, rather than the meet operator, $\wedge$, is entirely a matter of convention.

[16] Product spaces are endowed with the product topology throughout the paper.

[17] But the converse can fail. For example, the set $A = \{(x, y) \in \mathbb{R}^2_+ : x+y = 1\} \cup \{(1, 1)\}$ is a semilattice with the coordinatewise partial order, and this order is closed under the Euclidean metric. But A is not a metric semilattice because whenever $a_n \neq b_n$ and $a_n, b_n \to a$, we have $(1, 1) = \lim(a_n \vee b_n) \neq (\lim a_n) \vee (\lim b_n) = a$.

[18] We have not found a reference to the concept of local completeness in the lattice-theoretic literature.

[19] Whether or not every compact metric semilattice is locally complete was to us an open question until a 2005 visit to The Center for the Study of Rationality at The Hebrew University of Jerusalem. Shortly after we posed the question, Sergiu Hart and Benjamin Weiss settled the matter by graciously providing a subtle and beautiful example of a compact metric semilattice that is not locally complete (see Hart and Weiss (2005)). In contrast, such examples are not difficult to find if compactness is not required. For instance, no $\mathcal{L}_p$ space is locally complete when $p < +\infty$ and endowed with the usual pointwise partial order.

a.[20] A distinct sufficient condition for local completeness is given in Lemma A.18.

3.2. A Class of Bayesian Games

There are N players, $i = 1, 2, ..., N$. Player i's type space is T_i and his action space is A_i, and both are partially ordered. All partial orders, although possibly distinct, will be denoted by $\geq$. Player i's bounded and measurable payoff function is $u_i : A \times T \to \mathbb{R}$, where $A = \times_{i=1}^N A_i$ and $T = \times_{i=1}^N T_i$. The common prior over the players' types is a probability measure μ on $\mathcal{B}(T)$, the Borel subsets of T. Let G denote this Bayesian game.

We shall make use of the following additional assumptions, where μ_i denotes the marginal of μ on T_i. For every player i,

G.1 T_i is a complete separable metric space endowed with a measurable partial order.

G.2 μ_i assigns probability zero to any Borel subset of T_i having no strictly ordered points.[21]

G.3 A_i is a compact locally-complete metric semilattice.

G.4 $u_i(\cdot, t) : A \to \mathbb{R}$ is continuous for every $t \in T$.

Assumptions G.1-G.4 strictly generalize the assumptions in Athey (2001) and McAdams (2003) who assume that each A_i is a compact sublattice of Euclidean space and hence a compact locally-complete metric semilattice, that each $T_i = [0, 1]^{m_i}$ is endowed with the coordinatewise partial order, and that μ is absolutely continuous with respect to Lebesgue measure.[22,23]

Assumption G.1 permits infinite-dimensional type spaces, as can occur for example in share auctions, where a bidder's private information is his downward-sloping marginal value function. Assumption G.1 also permits the partial order on player i's type space to be distinct from the usual coordinatewise partial order when T_i is Euclidean. As we shall see, this flexibility is helpful in providing a new equilibrium existence result for multi-unit auctions with risk averse bidders.

Assumption G.2 implies that each μ_i is atomless,[24] but is not so restrictive as to imply the Milgrom and Weber (1985) assumption that μ is absolutely continuous with respect to the product of its marginals $\mu_1 \times ... \times \mu_n$. For example, when each player's type space is

[20] Hence, compactness and metrizability of a lattice under the order topology (see Birkohff (1967, p.244) is sufficient, but not necessary, for local completeness of the corresponding semilattice.

[21] In an earlier version of this paper, it was assumed that $\mu_i(B) = 0$ if every strict chain in B is countable. I thank Benjamin Weiss for suggesting the current simpler assumption G.2 and also for outlining a proof that the two assumptions are equivalent (see Lemma A.3).

[22] McAdams (2003) assumes, further, that the joint density over types is everywhere strictly positive.

[23] If $T_i = [0, 1]^{m_i}$, then absolute continuity of μ implies G.2. Indeed, if no two members of some Borel subset B of i's type space are strictly ordered, then $B \cap [0, 1]t_i$ contains at most one point for every $t_i \in \mathrm{int}T_i$. Fubini's theorem then implies that B has Lebesgue measure zero, and so $\mu_i(B) = 0$ by absolute continuity.

[24] Singleton sets have no strictly ordered points.

$[0,1]$ with its usual total order, G.2 holds if and only if each μ_i is atomless. In particular, G.2 holds when there are two players, each with unit interval type space, and the types are drawn according to Lebesgue measure conditional on any one of finitely many positively or negatively sloped lines in the unit square. On the other hand, when player i's type space is multidimensional, G.2 imposes more than that μ_i is atomless. For example, if $T_i = [0,1]^2$ has the usual Euclidean metric and the coordinatewise partial order, then G.2 requires μ_i to assign probability zero to any negatively sloped line in T_i.

Remark 1. *A similar argument applies to vertical and horizontal lines. However, it is useful to note that through a judicious choice of the metric, examples in which vertical or horizontal lines receive positive probability can be accommodated. For example, suppose that a player's type is his vector of marginal values,* $(v_1, v_2) \in [0,1]^2$*, in a two-unit auction, and that his marginal distribution is uniform on* $[0,1]^2$ *with probability 1/2 and is uniform on* $[0,1] \times \{0\}$ *with probability 1/2. Thus, the horizontal line* $[0,1] \times \{0\}$ *corresponding to the event* $v_2 = 0$ *receives positive probability, capturing the idea that the bidder may demand at most one unit. As explained in the previous paragraph, under the Euclidean metric and the coordinatewise partial order, this marginal distribution violates G.2.*

Our objective here is to change the metric, but not the partial order, so that G.2 is satisfied.[25] *So, consider instead the metric on* $[0,1]^2$ *that, to any two points* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$*, assigns their Euclidean distance if both points are in* $X = [0,1] \times \{0\}$*, assigns the distance* $\|x - y\| + \left|\frac{1}{x_2} - \frac{1}{y_2}\right|$ *if both points are in* $Y = [0,1] \times (0,1]$*, and assigns the distance one if* $x \in X$ *and* $y \in Y$*. Under this metric,* X *and* Y *are "split apart," yet* $[0,1]^2$ *remains a complete separable metric space, the Borel sets are unchanged, and the coordinatewise partial order remains closed. Hence, G.1 is still satisfied. However, the marginal distribution now satisfies G.2. In particular, distinct points in* $[0,1] \times \{0\}$ *are strictly ordered under the new metric because sets of the form* $(a,b) \times \{0\}$ *are now open. The horizontal line* $[0,1] \times \{0\}$ *is therefore permitted to have positive probability.*

Assumption G.2 helps ensure the compactness of the players' sets of monotone pure strategies (Lemma A.10) in a topology in which ex-ante payoffs are continuous. Indeed, without G.2, a player's type space could be the negative diagonal in $[0,1]^2$ endowed with the coordinatewise partial order.[26] But then every measurable function from types to actions would be monotone because no two distinct types are ordered. Compactness in a useful topology is then effectively precluded. Assumption G.2 therefore plays the same role for

[25] While it is a simple matter to satisfy G.2 by changing the partial order, such changes can preclude the existence of a monotone pure strategy equilibrium under the new partial order. In contrast, putting measurability issues aside, changing the metric alone has no effect on the set of monotone pure strategy equilibria.

[26] This violates G.2 because no two points on the negative diagonal are strictly ordered, yet the negative diagonal receives positive probability – in fact, probability one.

monotone pure strategies as the Milgrom-Weber (1985) absolute-continuity assumption plays for mixed strategies.

As functions from types to actions, best replies for any player i are determined only up to μ_i measure zero sets. This leads us to the following definitions. A *pure strategy* for player i is a function, $s_i : T_i \to A_i$, that is μ_i-a.e. (almost-everywhere) equal to a Borel measurable function, and is *monotone* if $t_i' \geq t_i$ implies $s_i(t_i') \geq s_i(t_i)$ for all $t_i, t_i' \in T_i$. Let S_i denote player i's set of pure strategies and let $S = \times_{i=1}^N S_i$.

A vector of pure strategies, $(\hat{s}_1, ..., \hat{s}_N) \in S$ is an *equilibrium* if for every player i and every pure strategy s_i' for player i,

$$\int_T u_i(\hat{s}(t), t)d\mu(t) \geq \int_T u_i(s_i'(t_i), \hat{s}_{-i}(t_{-i}), t)d\mu(t),$$

where the left-hand side, henceforth denoted by $U_i(\hat{s})$, is player i's payoff given the joint strategy $\hat{s}$, and the right-hand side is his payoff when he employs s_i' and the others employ $\hat{s}_{-i}$.

It will sometimes be helpful to speak of the payoff to player i's type t_i from the action a_i given the strategies of the others, s_{-i}. This payoff, which we will refer to as i's *interim* payoff, is

$$V_i(a_i, t_i, s_{-i}) \equiv \int_T u_i(a_i, s_{-i}(t_{-i}), t)d\mu_i(t_{-i}|t_i),$$

where $\mu_i(\cdot|t_i)$ is a version of the conditional probability on T_{-i} given t_i. A single such version is fixed for each player i once and for all.

4. The Main Result

Call a subset of player i's pure strategies *join-closed* if for any pair of strategies, s_i, s_i', in the subset, the strategy taking the action $s_i(t_i) \vee s_i'(t_i)$ for each $t_i \in T_i$ is also in the subset.[27] We can now state our main result, whose proof is provided in Section 6.

Theorem 4.1. *If G.1-G.4 hold, and each player's set of monotone pure best replies is nonempty and join-closed whenever the others employ monotone pure strategies, then G possesses a monotone pure strategy equilibrium.*

Remark 2. *In any setting in which the action sets are totally ordered (as in Athey (2001)), each player's set of monotone best replies is automatically join-closed.*

Remark 3. *Athey (2001) assumes that the A_i are totally ordered, and McAdams (2003) assumes that each A_i is a sublattice of $\mathbb{R}^k$ with the coordinatewise partial order. This*

[27] Note that when the join operator is continuous, as it is in a metric semilattice, the resulting function is a.e.-measurable, being the composition of a.e.-measurable and continuous functions.

additional structure, which we do not require, is necessary for their Kakutani-Glicksberg-based approach.[28]

A strengthening of Theorem 4.1 can be helpful when one wishes to restrict the players' strategies to subsets of their monotone pure strategies. For example, in a uniform-price auction for m units, a strategy mapping a player's m-vector of marginal values into a vector of m bids is undominated only if his bid for a kth unit is no greater than his marginal value for a kth unit. As formulated, Theorem 4.1 does not permit one to restrict the players' strategies in this way.[29] The next result takes care of this. Its proof is a straightforward extension of the proof of Theorem 4.1, and is provided in Remark 9.

A subset of player i's pure strategies is called *pointwise-limit-closed* if whenever $s_i^1, s_i^2, \ldots$ are each in the set and $s_i^n(t_i) \to_n s_i(t_i)$ for μ_i almost-every $t_i \in T_i$, then s_i is also in the set. A subset of player i's pure strategies is called *piecewise-closed* if whenever s_i and s_i' are in the set, then so is any strategy s_i'' such that for every $t_i \in T_i$ either $s_i''(t_i) = s_i(t_i)$ or $s_i''(t_i) = s_i'(t_i)$.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, if for each player i, C_i is a join-closed, piecewise-closed and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, and the intersection of C_i with i's set of monotone pure best replies is nonempty whenever every other player j employs a monotone pure strategy in his C_j, then G possesses a monotone pure strategy equilibrium in which each player i's pure strategy is in C_i.*

Remark 4. *When player i's action space is a semilattice with a closed partial order (as implied by G.3) and C_i is defined by any collection of weak inequalities, i.e., if $\mathcal{F}_i$ and $\mathcal{G}_i$ are arbitrary collections of measurable functions from T_i into A_i and $C_i = \cap_{f \in \mathcal{F}_i, g \in \mathcal{G}_i} \{s_i \in S_i : g(t_i) \leq s_i(t_i) \leq f(t_i)$ for μ_i a.e. $t_i \in T_i\}$, then C_i is join-closed, piecewise-closed and pointwise-limit-closed.*

It is well-known that within the confines of a lattice, quasisupermodularity and single-crossing conditions on interim payoffs guarantee the existence of monotone best replies and that sets of monotone best replies are lattices and hence join-closed. In the next section, we provide slightly weaker versions of these conditions and, for completeness, show that they too guarantee that the players' sets of monotone best replies are nonempty and join-closed.

[28] Indeed, suppose a player's action set is the semilattice $A = \{(1,0), (1/2,1/2), (0,1), (1,1)\}$ in $\mathbb{R}^2$, with the coordinatewise partial order and note that A is not a sublattice of $\mathbb{R}^2$. It is not difficult to see that this player's set of monotone pure strategies from $[0,1]$ into A, endowed with the metric $d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$, is homeomorphic to three line segments joined at a common endpoint. Consequently, this strategy set is not homeomrphic to a convex set and so neither Kakutani's nor Glicksberg's theorems can be directly applied. On the other hand, this strategy set is an absolute retract (see Lemma A.15), which is sufficient for our approach.

[29] Note that it is not possible to restrict the action space alone to ensure that the player chooses an undominated strategy since the bids that he must be permitted to choose will depend upon his private type, i.e., his vector of marginal values.

4.1. Sufficient Conditions on Interim Payoffs

Suppose that for each player i, A_i is a lattice. We say that player i's interim payoff function V_i is *weakly quasisupermodular* if for all monotone pure strategies s_{-i} of the others, all $a_i, a_i' \in A_i$, and every $t_i \in T_i$

$$V_i(a_i, t_i, s_{-i}) \geq V_i(a_i \wedge a_i', t_i, s_{-i}) \text{ implies } V_i(a_i \vee a_i', t_i, s_{-i}) \geq V_i(a_i', t_i, s_{-i}).$$

This weakens slightly Milgrom and Shannon's (1994) concept of quasisupermodularity by not requiring the second inequality to be strict if the first happens to be strict. McAdams (2003) requires the stronger condition of quasisupermodularity. When actions are totally ordered, as in Athey (2001), interim payoffs are automatically supermodular, and hence both quasisupermodular and weakly quasisupermodular.

A simple way to verify weak quasisupermodularity is to verify supermodularity. For example, it is well-known that V_i is supermodular in actions (hence weakly quasisupermodular) when $A_i = [0, 1]^K$ is endowed with the coordinatewise partial order, and the second cross-partial derivatives of $V_i(a_{i1}, ..., a_{iK}, t_i, s_{-i})$ with respect distinct action coordinates are nonnegative. Thus, complementarities between the coordinates of a player's *own* action vector are natural economic conditions under which weak quasisupermodularity holds.[30]

We say that i's interim payoff function V_i satisfies *weak single-crossing* if for all monotone pure strategies s_{-i} of the others, for all player i action pairs $a_i' \geq a_i$, and for all player i type pairs $t_i' \geq t_i$,

$$V_i(a_i', t_i, s_{-i}) \geq V_i(a_i, t_i, s_{-i})$$

implies

$$V_i(a_i', t_i', s_{-i}) \geq V_i(a_i, t_i', s_{-i}).$$[31]

To ensure that each player's set of monotone best replies is homeomorphic to a convex set, both Athey (2001) and McAdams (2003) assume that V_i satisfies a slightly more stringent single-crossing condition. In particular they each require that, in addition to the above, the second single-crossing inequality is strict whenever the first one is. This more stringent condition can fail in first-price auctions when one bidder's private information provides information about the support of another bidder's private information. Our weaker single-crossing condition nonetheless holds.

The following corollary of Theorem 4.1 states that monotone pure strategy equilibria exist if each V_i is weakly quasisupermodular and satisfies weak single-crossing.

[30] Complementarities between the actions of distinct *players* is not required. This is useful because, for example, many auction games satisfy only own-action complementarity.

[31] For conditions on the joint distribution of types, μ, and the players' payoff functions, $u_i(a, t)$, leading to the weak single-crossing property, see Athey (2001, pp.879-81), McAdams (2003, p.1197) and Van Zandt and Vives (2005).

Corollary 4.3. *If G.1-G.4 hold, if each A_i is a lattice, and if the players' interim payoffs are weakly quasisupermodular and satisfy weak single-crossing, then the hypotheses of Theorem 4.1 are satisfied and so G possesses a monotone pure strategy equilibrium.*

Proof. By Theorem 4.1, it suffices to show that weak quasisupermodularity and weak single-crossing imply that whenever the others employ monotone pure strategies, player i's set of monotone pure best replies is nonempty and join-closed. To see join-closedness, note that if against some monotone pure strategy of the others, actions a_i and a_i' are interim best replies for i when his type is t_i, then weak quasisupermodularity implies that so too is $a_i \vee a_i'$. Since two pure strategies are best replies for i if and only if they specify interim best replies for μ_i-almost every t_i, join-closedness follows. (Because the join operator is continuous in a metric semilattice, the join of two a.e.-measurable functions is a.e.-measurable, being the composition of a.e.-measurable and continuous functions.)

Fix a monotone pure strategy, s_{-i}, for player i's opponents, and let $B_i(t_i)$ denote i's interim best reply actions against s_{-i} when his type is t_i. By G.4, $B_i(t_i)$ is compact and nonempty, and by the argument in the previous paragraph $B_i(t_i)$ is a subsemilattice of A_i. Define $\bar{s}_i : T_i \to A_i$ by setting $\bar{s}_i(t_i) = \vee B_i(t_i)$ for each $t_i \in T_i$. Lemma A.7 together with the compactness and subsemilattice properties of $B_i(t_i)$ imply that, for every t_i, $\bar{s}_i(t_i)$ is well defined and $\bar{s}_i(t_i) \in B_i(t_i)$.

We next show that $\bar{s}_i$ is monotone. Suppose that $t_i' \geq t_i$. Then

$$V_i(\bar{s}_i(t_i), t_i, s_{-i}) \geq V_i(\bar{s}_i(t_i) \wedge \bar{s}_i(t_i'), t_i, s_{-i}), \tag{4.1}$$

since $\bar{s}_i(t_i) \in B_i(t_i)$. By weak single-crossing, (4.1) implies that

$$V_i(\bar{s}_i(t_i), t_i', s_{-i}) \geq V_i(\bar{s}_i(t_i) \wedge \bar{s}_i(t_i'), t_i', s_{-i}). \tag{4.2}$$

Hence, applying weak quasisupermodularity to (4.2) we obtain

$$V_i(\bar{s}_i(t_i') \vee \bar{s}_i(t_i), t_i', s_{-i}) \geq V_i(\bar{s}_i(t_i'), t_i', s_{-i}),$$

from which we conclude that $\bar{s}_i(t_i') \vee \bar{s}_i(t_i) \in B_i(t_i')$. But $\bar{s}_i(t_i') = \vee B_i(t_i')$ is the largest member of $B_i(t_i')$. Hence $\bar{s}_i(t_i') \vee \bar{s}_i(t_i) = \bar{s}_i(t_i')$, implying that $\bar{s}_i(t_i') \geq \bar{s}_i(t_i)$ as desired.

By Lemma A.11, $\bar{s}_i$, being monotone, is μ_i-a.e. equal to a Borel measurable monotone function and so belongs to i's set of pure strategies, S_i. Player i's set of monotone pure best replies is therefore nonempty. ■

Remark 5. *Weak quasisupermodularity is used to ensure both join-closedness and that monotone best replies exist. On the other hand, weak single-crossing is employed only in the proof of the latter.*

Remark 6. *Compact Euclidean sublattices are compact, locally complete, metric semilattices. Hence, Corollary 4.3 generalizes the main results of Athey (2001) and McAdams (2003).*

Corollary 4.3 will often suffice in applications. However, the additional generality provided by Theorem 4.1 is sometimes important. For example, Reny and Zamir (2004) have shown in the context of asymmetric first-price auctions that, when bidders have distinct and finite bid sets, monotone best replies exist even though weak single-crossing fails. Further, since action sets (i.e., real-valued bids) there are totally ordered, best reply sets are necessarily join-closed and so the hypotheses of Theorem 4.1 are satisfied while those of Corollary 4.3 are not. A similar situation arises in the context of a multi-unit auction with risk averse bidders (see the applications below). There, under CARA utility weak quasisupermodularity fails but sets of monotone best replies are nonetheless join-closed and Theorem 4.1 (but not Corollary 4.3) can be applied.

We now turn to several applications of our results.

5. Applications

5.1. Uniform-Price Multi-Unit Auction with Risk Averse Bidders

Consider a uniform-price auction with n bidders and m homogeneous units of a single good for sale. Each bidder i simultaneously submits a bid, $b = (b_1, ..., b_m)$, where $b_{i1} \geq ... \geq b_{im}$ and each b_{ik} is taken from the finite set $\{0, p_1, ..., p_K\} \subset [0, 1]$. Call b_{ik} bidder i's kth unit-bid. The uniform price, p, is the $m + 1$st highest of all nm unit-bids. Each unit-bid above p wins a unit at price p, and any remaining units are awarded to unit-bids equal to p according to a random-bidder-order tie-breaking rule.[32]

Bidder i's private type is his vector of nonincreasing marginal values, so that his type space is $T_i = \{t_i \in [0, 1]^m : t_{i1} \geq ... \geq t_{im}\}$. Bidder i is risk averse with utility function for money $u_i : [-m, m] \to \mathbb{R}$, where $u_i' > 0$, $u_i'' \leq 0$. If bidder i's type is t_i and he wins k units at price p, his payoff is $u_i(t_{i1} + ... + t_{ik} - kp)$. Types are chosen independently across bidders and bidder i's type-vector is chosen according to the density f_i, which need not be positive on all of $[0, 1]^m$.[33]

Multi-unit uniform-price auctions always have trivial equilibria in weakly dominated strategies in which some player always bids very high on all units and all others always bid zero. We wish to establish the existence of monotone pure strategy equilibria that are

[32] The tie-breaking rule is as follows. Bidders are ordered randomly and uniformly. Then, one bidder at a time according to this order, each bidder's *total* remaining demand (i.e., his number of bids equal to p), or as much as possible, is filled at price p per unit until supply is exhausted.

[33] By employing the technique described in Remark 1, it is possible to permit a bidder's total demand to be stochastic in the sense that, for each $k > 1$, his marginal value for a kth and higher unit may be zero with positive probability, as might occur if a bidder's endowment of the good were private information. We will not pursue this further here.

not trivial in this sense. But observe that, because the set of feasible bids is finite, bidding above one's marginal value on some unit need not be weakly dominated. Indeed, it might be a strict best reply for bidder i of type t_i to bid $p_j > t_{ik}$ for a kth unit so long as no feasible bid is in $[t_{ik}, p_j)$. Such a kth unit-bid might permit bidder i to win a kth unit with probability one and earn a surplus rather than tie the other bidders by bidding below t_{ik}, risking losing the unit. On the other hand, in this instance there is never any gain, and there might be a loss, from bidding above p_j on a kth unit.

Call a monotone pure strategy equilibrium *nontrivial* if for each bidder i, for f_i almost-every t_i, and for every k, bidder i's kth unit-bid does not exceed the smallest feasible bid greater than or equal to t_{ik}. As shown by McAdams (2006), under the coordinatewise partial order on type and action spaces, nontrivial monotone pure strategy equilibria need not exist when bidders are risk averse, as we permit here. Nonetheless, we will demonstrate that a nontrivial monotone pure strategy equilibrium does exist under an economically motivated partial order on type spaces that differs from the coordinatewise partial order; we maintain the coordinatewise partial order on action spaces.

Before introducing the new partial order, it is instructive to see what goes wrong with the coordinatewise partial order on type spaces. The heart of the matter is that single-crossing fails. To see why, it is enough to consider the case of two units. Fix monotone pure strategies for the other bidders and consider two bids for bidder i, $\bar{b} = (\bar{b}_1, \bar{b}_2)$ and $\underline{b} = (\underline{b}_1, \underline{b}_2)$, where $\bar{b}_k > \underline{b}_k$ for $k = 1, 2$. Suppose that when bidder i employs the high bid, $\bar{b}$, he is certain to win both units and pay $\bar{p}$ for each, while he is certain to win only one unit when he employs the low bid, $\underline{b}$. Further, suppose that the low bid yields a price for the one unit he wins that is either $\underline{p}$ or $\underline{p}' > \underline{p}$, each being equally likely. Thus, the expected difference in his payoff from employing the high bid versus the low one can be written as,

$$\frac{1}{2}\left[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p}')\right] + \frac{1}{2}\left[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p})\right].$$

Single-crossing requires this difference, when nonnegative, to remain nonnegative when bidder i's type increases according to the coordinatewise partial order, i.e., when t_{i1} and t_{i2} increase. But this can fail when risk aversion is strict because, whenever $t_{i1} + t_{i2} - 2\bar{p} > t_{i1} - \underline{p}'$, the first utility difference above strictly falls when t_{i1} increases. Consequently, the expected difference can become negative if the second utility difference is negative to start with.

The economic intuition for the failure of single-crossing is straightforward. Under risk aversion, the marginal utility of winning a second unit falls when the dollar value of a first unit rises, giving the bidder an incentive to reduce his second unit bid so as to reduce the price paid on the first unit. We now turn to the new partial order, which ensures that a higher type is associated with a higher marginal utility of winning each additional unit.

For each bidder i, let $\alpha_i = \frac{u_i'(-m)}{u_i'(m)} - 1 \geq 0$, and consider the partial order, $\geq_i$, on T_i

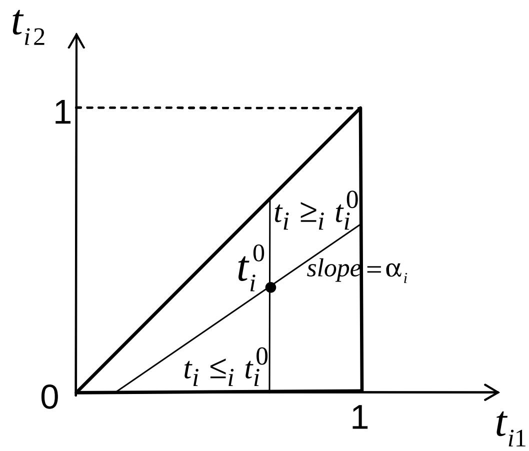

Figure 5.1: Types that are greater than and less than t_i^0 are bounded between two lines through t_i^0, one being vertical, the other having slope α_i.

defined as follows: $t'_i \geq_i t_i$ if,

$$\begin{aligned} &1.\ t'_{i1} \geq t_{i1}, \text{ and} \\ &2.\ t'_{ik} - \alpha_i(t'_{i1} + ... + t'_{ik-1}) \geq t_{ik} - \alpha_i(t_{i1} + ... + t_{ik-1}), \text{ for all } k \in \{2, ..., m\}. \end{aligned} \tag{5.1}$$

Figure 5.1 shows which types are greater than and less than a typical type, t_i^0, when types are two-dimensional, i.e., when $m = 2$.

Under the Euclidean metric on the type space, the partial order $\geq_i$ defined by (5.1) is clearly closed so that G.1 is satisfied. To see that G.2 is satisfied, suppose that $\int_B f_i(t_i)dt_i > 0$ for some Borel subset B of $T_i = [0,1]^m$ Then B must have positive Lebesgue measure in $\mathbb{R}^m$. Consequently, by Fubini's theorem, there exists $z \in \mathbb{R}^m$ (indeed there is a positive Lebesgue measure of such z's) such that the line defined by $z + \mathbb{R}((1+\alpha_i), (1+\alpha_i)^2, ..., (1+\alpha_i)^m)$ intersects B in a set of positive one-dimensional Lebesgue measure on the line. Therefore we may choose two distinct points, t_i and t'_i in B that are on this line. Hence, $t'_i - t_i = \beta((1+\alpha_i), (1+\alpha_i)^2, ..., (1+\alpha_i)^m)$, where we may assume without loss that $\beta > 0$. But then, $t'_{i1} - t_{i1} = \beta(1+\alpha_i) > 0$ and for $k \in \{2, ..., m\}$,

$$\begin{aligned} t'_{ik} - t_{ik} &= \beta(1+\alpha_i)^k \\ &= \beta\{1 + \alpha_i[1 + (1+\alpha_i) + (1+\alpha_i)^2 + ... + (1+\alpha_i)^{k-1}]\} \\ &= \beta(1+\alpha_i) + \alpha_i[\beta(1+\alpha_i) + \beta(1+\alpha_i)^2 + ... + \beta(1+\alpha_i)^{k-1}] \\ &= \beta(1+\alpha_i) + \alpha_i[(t'_{i1} - t_{i1}) + (t'_{i2} - t_{i2}) + ... + (t'_{ik-1} - t_{ik-1})] \\ &> \alpha_i[(t'_{i1} - t_{i1}) + (t'_{i2} - t_{i2}) + ... + (t'_{ik-1} - t_{ik-1})], \end{aligned}$$

from which we conclude that t'_i is strictly greater than t_i (since the strict inequalities will hold for pairwise comparisons of points within sufficiently small balls around t'_i and t_i). This shows that any subset having positive f_i-measure contains at least two strictly ordered points according to the partial order $\geq_i$ defined by (5.1), and so G.2 is satisfied.

Actions spaces, being finite and endowed with the coordinatewise partial order are easily seen to be compact metric semilattices under the Euclidean metric, and Lemma A.17 then implies that action spaces are locally complete. Hence, G.3 holds. Also, G.4 holds because action spaces are finite. Thus, we have so far verified G.1-G.4.

Next, note that action spaces are lattices and that McAdams (2004) shows that each bidder's ex-ante payoff function is modular and hence quasisupermodular. By Corollary 4.3, the hypotheses of Theorem 4.1 will be satisfied if interim expected payoffs satisfy weak single crossing, which we now demonstrate. It is here where the new partial order $\geq_i$ in (5.1) is fruitfully employed.

To verify weak single crossing it suffices to show that ex-post payoffs satisfy increasing differences. So, fix the strategies of the other bidders, a realization of their types, and an ordering of the players for the purposes of tie-breaking if necessary. With these fixed, suppose that the bid, $\bar{b}$, chosen by bidder i of type t_i wins k units at the price $\bar{p}$ per unit, while the coordinatewise-lower bid, $\underline{b}$, wins $j \leq k$ units at the price $\underline{p} \leq \bar{p}$ per unit. The difference in i's ex-post utility from bidding $\bar{b}$ versus $\underline{b}$ is then,

$$u_i(t_{i1} + ... + t_{ik} - k\bar{p}) - u_i(t_{i1} + ... + t_{ij} - j\underline{p}). \tag{5.2}$$

Assuming that $t_i' \geq t_i$ in the sense of (5.1), it suffices to show that (5.2) is weakly greater at t_i' than at t_i. Noting that (5.1) implies that $t_{il}' \geq t_{il}$ for every l, it can be seen that, if $j = k$, then (5.2) is weakly greater at t_i' than at t_i by the concavity of u_i. It therefore remains only to consider the case in which $j < k$, where we have,

$$\begin{aligned}
u_i(t_{i1}' + ... + t_{ik}' - k\bar{p}) - u_i(t_{i1} + ... + t_{ik} - k\bar{p}) &\geq u_i'(m)[(t_{i1}' - t_{i1}) + ... + (t_{ik}' - t_{ik})] \\
&\geq u_i'(m)[(t_{i1}' - t_{i1}) + ... + (t_{ij+1}' - t_{ij+1})] \\
&\geq u_i'(-m)[(t_{i1}' - t_{i1}) + ... + (t_{ij}' - t_{ij})] \\
&\geq u_i(t_{i1}' + ... + t_{ij}' - j\underline{p}) - u_i(t_{i1} + ... + t_{ij} - j\underline{p}),
\end{aligned}$$

where the first and fourth inequalities follow from the concavity of u_i and the third inequality follows because $t_i' \geq t_i$ in the sense of (5.1). We conclude that weak single crossing holds and so the hypotheses of Theorem 4.1 are satisfied.

Finally, for each player i, let C_i denote the subset of his pure strategies such that for f_i almost-every t_i, and for every k, bidder i's kth unit-bid does not exceed $\phi(t_{ik})$, the smallest feasible unit-bid greater than or equal to t_{ik}. By Remark 4, each C_i is join-closed, piecewise-closed and pointwise-limit-closed. Further, because the hypotheses of Theorem 4.1 are satisfied, whenever the others employ monotone pure strategies player i has a monotone best reply, b_i', say. Defining $b_i(t_i)$ to be the coordinatewise minimum of $b_i'(t_i)$ and $(\phi(t_{i1}), ..., \phi(t_{im}))$ for all $t_i \in T_i$ implies that b_i is a monotone best reply contained in C_i. This is because, ex-post, any units won by employing b_i' that are also won by employing b_i are won at a weakly lower price with b_i, and any units won by employing b_i' that are not won by employing b_i cannot be won at a positive surplus. Hence, the hypotheses of Theorem 4.2 are satisfied and we conclude that a nontrivial monotone pure strategy equilibrium exists. We may therefore state the following proposition.

Proposition 5.1. *Consider an independent private value uniform-price multi-unit auction with a random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i's vector of marginal values is decreasing and chosen according to the density f_i, and that each bidder is weakly risk averse.*

Then, there is a pure strategy equilibrium of the auction with the following properties. For each bidder i,

(i) the equilibrium is monotone under the type-space partial order $\geq_i$ defined by (5.1) and under the usual coordinatewise partial order on bids, and

(ii) the equilibrium is nontrivial in the sense that for f_i almost-all of his types, and for every k, bidder i's kth unit-bid does not exceed the smallest feasible unit-bid greater than or equal to his marginal value for a kth unit.

Remark 7. *The partial order defined by (5.1) reduces to the usual coordinatewise partial order under risk neutrality (i.e., when $\alpha_i = 0$), but is distinct from the coordinatewise partial order under strict risk aversion (i.e., when $\alpha_i > 0$), in which case McAdams (2003) does not apply since he employs the coordinatewise partial order.*

Remark 8. *The partial order defined by (5.1) can instead be thought of as a change of variable from t_i to say x_i, where $x_{i1} = t_{i1}$ and $x_{ik} = t_{ik} - \alpha_i(t_{i1} + ... + t_{ik-1})$ for $k > 1$, and where the coordinatewise partial order is applied to the new type space. Our results apply equally well using this change-of-variable technique. In contrast, McAdams (2003) still does not apply because the resulting type space is not the product of intervals, an assumption he maintains together with a strictly positive joint density.*[34] *See Figure 5.2 for the case in which $m = 2$.*

[34] Indeed, starting with the partial order defined by (5.1) there is no change of variable that, when combined with the coordinatewise partial order, is order-preserving and maps to a product of intervals. This is because there is never a smallest element of the type space with the new partial order and there is no largest element when $\alpha_i > 1$, but there is a smallest and a largest element of the product of intervals with the coordinatewise partial order.

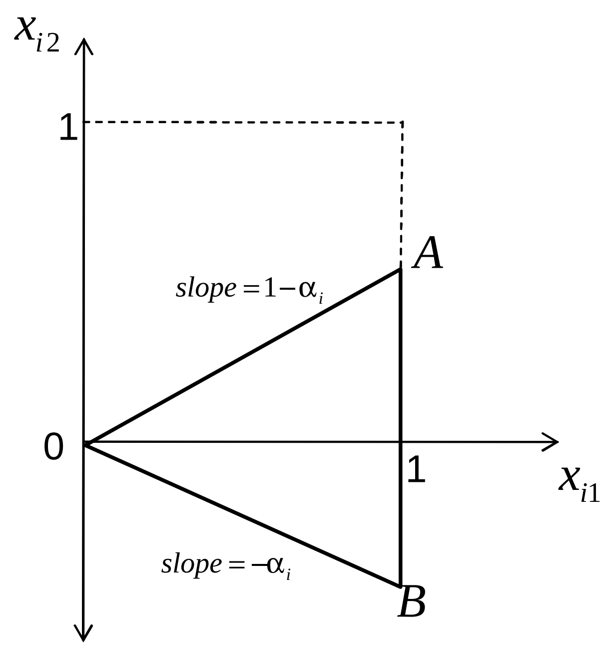

Figure 5.2: After performing the change of variable from t_i to x_i as described in Remark 8 bidder i's new type space is triangle OAB and it is endowed with the coordinatewise partial order. The figure is drawn for the case in which $\alpha_i \in (0, 1)$.

5.2. Discriminatory Multi-Unit Auction with CARA Bidders.

Consider the same setup as in Subsection 5.1 with the exception of two items. First, change the payment rule to one where each bidder pays his kth unit-bid for a kth unit won, and second, assume that each bidder's utility function, u_i, exhibits constant absolute risk aversion.

Even with these two changes single-crossing fails under the coordinatewise partial order on types for the same underlying reason as in a uniform-price auction with risk averse bidders. Nonetheless, just as in the previous section it can be shown here that assumptions G.1-G.4 hold and each bidder i's interim expected payoff function satisfies weak single-crossing under the partial order $\geq_i$, defined in (5.1).[35]

Thus, if it could be shown that interim expected payoffs are quasisupermodular, Corollary 4.3 would imply the existence of a monotone pure strategy equilibrium of the discriminatory auction. However, quasisupermodularity does not hold in discriminatory auctions with strictly risk averse bidders – even CARA bidders.

The intuition for the failure of quasisupermodularity is as follows. Suppose there are two units, and let b_k denote a kth unit-bid. Fixing b_2, suppose that b_1 is chosen to maximize a bidder's expected utility when his type is (t_1, t_2), namely,

$$P_1(b_1)[u(t_1 - b_1) - u(0)] + P_2(b_2)[u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)],$$

[35] This statement remains true with any risk averse utility function. CARA will be employed shortly.

where $P_k(b_k)$ is the probability of winning at least k units.

There are two benefits from increasing b_1. First, the probability, $P_1(b_1)$, of winning at least one unit increases. Second, when risk aversion is strict, the marginal utility, $u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)$, of winning a second unit increases. The cost of increasing b_1 is that the marginal utility, $u(t_1 - b_1) - u(0)$, of winning a first unit decreases. Optimizing the choice of b_1 balances this cost with the two benefits. For simplicity, suppose that the optimal choice of b_1 satisfies $b_1 > t_2$.

But now suppose that b_2 increases. Indeed, suppose that b_2 increases to t_2. Then the marginal utility of winning a second unit vanishes. Consequently, the second benefit from increasing b_1 is no longer present and the optimal choice of b_1 may fall — even with CARA utility.

This illustrates that the change in utility from increasing one's first unit-bid may be positive when one's second unit-bid is low, but negative when one's second unit-bid is high. Thus, the different coordinates of a bidder's bid are not necessarily complementary, and weak quasisupermodularity can fail. We therefore cannot appeal to Corollary 4.3 to establish the existence of a monotone pure strategy equilibrium.

Despite the failure – even with CARA utilities – of both single-crossing with the coordinatewise partial order on types and the failure of weak quasisupermodularity with the coordinatewise partial order on bids, Theorem 4.1 can be used to demonstrate the following.

Proposition 5.2. *Consider an independent private value discriminatory multi-unit auction with a random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i's vector of marginal values is decreasing and chosen according to the density f_i, and that each bidder is weakly risk averse and exhibits constant absolute risk aversion.*

Then, there is a pure strategy equilibrium of the auction that is monotone under the type-space partial order $\geq_i$ defined by (5.1) and under the usual coordinatewise partial order on bids.

PROOF TO BE SUPPLIED

The two applications provided so far demonstrate that it is useful to have flexibility in defining the partial order on the type space since the mathematically natural partial order (in this case the coordinatewise partial order on the original type space) may not be the partial order that corresponds best to the economics of the problem. The next application shows that even when single crossing cannot be established for all coordinates of the type space jointly, it is enough for the existence of a pure strategy equilibrium if single-crossing holds strictly even for a single coordinate of the type space.

5.3. Price Competition.

Consider an n-firm differentiated-product price-competition setting. Firm i chooses price $p_i \in [0,1]$, and receives two pieces of private information – his constant marginal cost, $c_i \in [0,1]$, and information $x_i \in [0,1]$ about the state of demand in each of the n markets. The demand for firm i's product is $D_i(p,x)$ when the vector of prices chosen by all firms is $p \in [0,1]^n$ and when their joint vector of private information about market demand is $x \in [0,1]^n$. Demand functions are assumed to be twice continuously differentiable, and $D_i(p,x) > 0$ whenever $p_i < 1$.

Some products are substitutes, but others need not be. More precisely, the n firms are partitioned into two subsets N_1 and N_2.[36] Products produced by firms within each subset are assumed to be substitutes, so that $D_i(p,x)$ is nondecreasing in p_j whenever i and j are in the same N_k. Marginal costs are affiliated among firms within each N_k and are independent across the two subsets of firms. The joint density of costs is given by the continuously differentiable density $f(c)$ on $[0,1]^n$. Information about market demand may be correlated across firms, but is independent of all marginal costs and has joint density $g(x)$ on $[0,1]^n$. We do not assume that market demands are nondecreasing in x because we wish to permit the possibility that information that increases demand for some products might decrease it for others.

We assume that demands are strictly downward sloping, i.e., that for all i, $\partial D_i(\cdot)/\partial p_i < 0$ and that $\partial D_i(\cdot)/\partial p_i$ is nondecreasing in p_j when i and j are in the same N_k.

Given pure strategies $p_j(c_j, x_j)$ for the others, firm i's interim expected profits are,

$$v_i(p_i, c_i, x_i) = \int (p_i - c_i) D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x) g_i(x_{-i}|x_i) f_i(c_{-i}|c_i) dx_{-i} dc_{-i}. \tag{5.3}$$

Suppose for each firm $j \neq i$ and every x_j that $p_j(c_j, x_j)$ is nondecreasing in c_j. Then,

$$\begin{aligned}
\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} &= -E(\frac{\partial D_i}{\partial p_i}|c_i, x_i) + \frac{\partial}{\partial c_i} E(D_i|c_i, x_i) + (p_i - c_i)\frac{\partial}{\partial c_i} E(\frac{\partial D_i}{\partial p_i}|c_i, x_i) \\
&\geq -E(\frac{\partial D_i}{\partial p_i}|c_i, x_i) \\
&> 0
\end{aligned} \tag{5.4}$$

for all $p_i, c_i, x_i \in [0,1]$ such that $p_i \geq c_i$, where the weak inequality follows because both partial derivatives with respect to c_i on the right-hand side of the first line are nonnegative. For example, consider the expectation in the first partial derivative. If $i \in N_1$, then

$$E(D_i|c_i, x_i) = E\left[E(D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x)|c_i, x_i, (c_j, x_j)_{j \in N_2})|c_i, x_i\right].$$

[36] The extension to any finite number of subsets is straighforward.

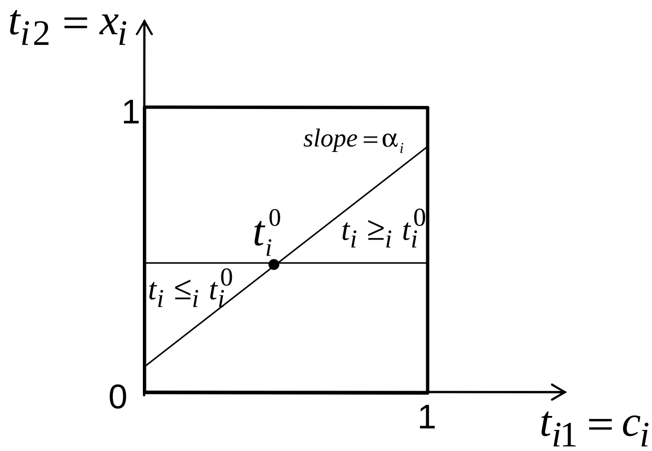

Figure 5.3: Types that are greater than and less than t_i^0 are bounded between two lines through t_i^0, one being horizontal, the other having slope α_i.

The inner expectation is nondecreasing in c_i because the vector of marginal costs for firms in N_1 are affiliated, their prices are nondecreasing in their costs, and their goods are substitutes. That the entire expectation is nondecreasing in c_i now follows from the independence of (c_i, x_i) and $(c_j, x_j)_{j \in N_2}$.

Thus, when $p_i \geq c_i$, single-crossing holds strictly for the marginal cost coordinate of the type space under the usual ordering of the reals. On the other hand, single-crossing need not hold for the market-demand coordinate, x_i, since we have made no assumptions about how x_i affects demand.[37] Nonetheless, we shall now define a partial order on firm i's type space $T_i = [0, 1]^2$ under which a monotone pure strategy equilibrium exists.

Because the cross-partial derivative on the left-hand side of (5.4) is continuous, it is bounded away from zero. Hence, there exists $\alpha_i > 0$ such that

$$\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} + \beta \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial x_i} > 0, \tag{5.5}$$

for all $\beta \in [0, \alpha_i]$ and for all $p_i, c_i, x_i \in [0, 1]$ such that $p_i \geq c_i$. For each player i, define the partial order $\geq_i$ on $T_i = [0, 1]^2$ as follows: $(c_i', x_i') \geq_i (c_i, x_i)$ if $\alpha_i c_i' - x_i' \geq \alpha_i c_i - x_i$ and $x_i' \geq x_i$. Figure 5.3 shows those types greater than and less than a typical type $t_i^0 = (c_i^0, x_i^0)$.

The partial order $\geq_i$ can be shown to satisfy G.1 and G.2 as in Example 5.1. Assumption G.3 holds by Lemma A.17 given the usual partial order over the reals, and G.4 holds by our continuity assumption on demand. Also, because the action space $[0, 1]$ is totally ordered, payoffs are automatically quasisupermodular and therefore sets of monotone best replies are join-closed. So, according to Theorem 4.1, it remains only to show that each firm possess a monotone best reply when the others employ monotone pure strategies.

[37] We cannot simply restrict attention to strategies $p_i(c_i, x_i)$ that are monotone in c_i and jointly measurable in (c_i, x_i) because this set of pure strategies is not compact in a topology rendering ex-ante payoffs continuous.

So, assume that all firms $j \neq i$ employ monotone pure strategies according to $\geq_j$. Therefore, in particular, $p_j(c_j, x_j)$ is nondecreasing in c_j for each x_j. Because firm i's interim payoff function is continuous in his price for each of his types and because his action space, $[0,1]$, is totally ordered and compact, firm i possesses a largest best reply, $\hat{p}_i(c_i, x_i)$, for each (c_i, x_i). We will show that $\hat{p}_i(\cdot)$ is monotone according to $\geq_i$.

Let $\bar{t}_i = (\bar{c}_i, \bar{x}_i)$, $\underline{t_i} = (\underline{c_i}, \underline{x_i})$ and suppose that $\bar{t}_i \geq_i \underline{t_i}$. Hence, $\bar{c}_i \geq \underline{c_i}$ and $\bar{x}_i - \underline{x_i} = \beta(\bar{c}_i - \underline{c_i})$ for some $\beta \in [0, \alpha_i]$. Let $\bar{p}_i = \hat{p}_i(\bar{c}_i, \bar{x}_i)$, $\underline{p_i} = \hat{p}_i(\underline{c_i}, \underline{x_i})$, and $t_i^\lambda = (1-\lambda)\underline{t_i} + \lambda \bar{t}_i$ for $\lambda \in [0,1]$. We wish to show that $\bar{p}_i \geq \underline{p_i}$.

By the fundamental theorem of calculus,

$$v_i(\underline{p_i}, t_i^\lambda) - v_i(p_i', t_i^\lambda) = \int_{p_i'}^{\underline{p_i}} \frac{\partial v_i(p_i, t_i^\lambda)}{\partial p_i} dp_i,$$

so that

$$\begin{aligned}
\frac{\partial \left[v_i(\underline{p_i}, t_i^\lambda) - v_i(p_i', t_i^\lambda) \right]}{\partial \lambda} &= \int_{p_i'}^{\underline{p_i}} \frac{\partial^2 v_i(p_i, t_i^\lambda)}{\partial \lambda \partial p_i} dp_i \\
&= \int_{p_i'}^{\underline{p_i}} \left[\frac{\partial^2 v_i(p_i, t_i^\lambda)}{\partial c_i \partial p_i} (\bar{c}_i - \underline{c_i}) + \frac{\partial^2 v_i(p_i, t_i^\lambda)}{\partial x_i \partial p_i} (\bar{x}_i - \underline{x_i}) \right] dp_i \\
&= (\bar{c}_i - \underline{c_i}) \int_{p_i'}^{\underline{p_i}} \left[\frac{\partial^2 v_i(p_i, t_i^\lambda)}{\partial c_i \partial p_i} + \beta \frac{\partial^2 v_i(p_i, t_i^\lambda)}{\partial x_i \partial p_i} \right] dp_i \\
&\geq 0,
\end{aligned}$$

where the inequality follows by (5.5) if $\underline{p_i} \geq p_i' \geq \bar{c}_i$. Therefore, $v_i(\underline{p_i}, \bar{t}_i) - v_i(p_i', \bar{t}_i) \geq v_i(\underline{p_i}, \underline{t_i}) - v_i(p_i', \underline{t_i}) \geq 0$, where the first inequality follows because $t_i^0 = \underline{t_i}$, $t_i^1 = \bar{t}_i$, and the second because $\underline{p_i}$ is a best reply at $\underline{t_i}$. Therefore, we have shown the following: If $\underline{p_i} \geq \bar{c}_i$, then

$$v_i(\underline{p_i}, \bar{t}_i) - v_i(p_i', \bar{t}_i) \geq 0, \text{ for all } p_i' \in [\bar{c}_i, \underline{p_i}].$$

Hence, if $\underline{p_i} \geq \bar{c}_i$, then $\hat{p}_i(\bar{t}_i) = \bar{p}_i \geq \underline{p_i} = \hat{p}_i(\underline{t_i})$ because $\hat{p}_i(\bar{t}_i)$ is the largest best reply at $\bar{t}_i$ and because no best reply at $\bar{t}_i = (\bar{c}_i, \bar{x}_i)$ is below $\bar{c}_i$. On the other hand, if $\underline{p_i} < \bar{c}_i$, then $\bar{p}_i = \hat{p}_i(\bar{t}_i) \geq \bar{c}_i > \underline{p_i} = \hat{p}_i(\underline{t_i})$, where the first inequality again follows because no best reply at $\bar{t}_i$ is below $\bar{c}_i$. We conclude that $\bar{p}_i \geq \underline{p_i}$, as desired.

Thus, there exists a pure strategy equilibrium in which each firm's price is monotone in (c_i, x_i) according to $\geq_i$. In particular, there is therefore a pure strategy equilibrium in which each firm's price is nondecreasing in his marginal cost, the coordinate in which strict single-crossing holds.

5.4. Finite Type Spaces and Monotone Mixed Strategy Equilibria.

Call a mixed strategy $m_i : T_i \to \Delta(A_i)$ *monotone* if $t_i \geq t_i'$ implies $a_i \geq a_i'$ for all $a_i \in \text{supp} m_i(t_i)$ and all $a_i' \in \text{supp} m_i(t_i')$. Suppose that G.1, G.3 and G.4 hold, but that T is a finite set so that G.2 fails to hold. UPON COMPLETION OF THIS SECTION, it will be shown that, under the hypotheses of Theorem 4.1, except G.2, a monotone mixed strategy equilibrium exists. The novelty here is the ease with which this result is obtained. Proof sketch: Augment player i's type space to $Q_i = T_i \times [0,1]$, where $q_i = (t_i, x_i)$ and the x_i are i.i.d. uniform $[0,1]$ and payoff irrelevant. Employ the lexicographic partial order on Q_i, i.e., $q_i' = (t_i', x_i') \geq (t_i, x_i) = q_i$ if either $t_i' \geq t_i$ and $t_i' \neq t_i$, or $t_i' = t_i$ and $x_i' \geq x_i$. One must check that G.2 now holds. The result then follows immediately since a monotone pure strategy equilibrium of the augmented game induces a monotone mixed strategy equilibrium of the original game. Note that this yields the additional result that, in equilibrium, supp$m_i(t_i)$ is totally ordered for each player i and every type t_i.

6. Proof of Theorem 4.1

Let M_i denote the set of monotone functions from T_i into A_i, and let $M = \times_{i=1}^N M_i$. By Lemma A.11, every element of M_i is equal μ_i almost-everywhere to a Borel measurable monotone function, and so M_i coincides with player i's set of monotone pure strategies. Let $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ denote player i's best-reply correspondence when all players must employ monotone pure strategies. Because, by hypothesis, each player possesses a monotone best reply (among all strategies) when the others employ monotone pure strategies, any fixed point of $\times_{i=1}^n \mathbf{B}_i : M \twoheadrightarrow M$ is a monotone pure strategy equilibrium. The following steps demonstrate that such a fixed point exists.

Without loss, we may assume that the metric d_i on A_i is bounded.[38] Given d_i, define a metric δ_i on M_i as follows:[39]

$$\delta_i(s_i, s_i') = \int_{T_i} d_i(s_i(t_i), s_i'(t_i)) d\mu_i(t_i).$$

This metric does not distinguish between strategies that are equal μ_i almost everywhere, which is natural since, from each player's ex-ante viewpoint, such strategies are payoff equivalent. By Lemmas A.13 and A.15, each (M_i, δ_i) is a compact absolute retract.[40]

We next demonstrate that, given the metric spaces (M_j, δ_j), each player i's payoff function, $U_i : M \to \mathbb{R}$, is continuous under the product topology. To see this, suppose that s^n is a sequence of joint strategies in M, and that $s^n \to s \in M$. By Lemma A.12, for each player i, $s_i^n(t_i) \to s_i(t_i)$ for μ_i almost every $t_i \in T_i$. Consequently, $s^n(t) \to s(t)$ for μ almost every

[38] For any metric, $d(\cdot,\cdot)$, an equivalent bounded metric is $\min(1, d(\cdot,\cdot))$.

[39] Formally, the resulting metric space (M_i, δ_{M_i}) is the space of equivalence classes of functions in M_i that are equal μ_i almost everywhere. Nevertheless, analogous to the standard treatment of $\mathcal{L}_p$ spaces, in the interest of notational simplicity we focus on the elements of the orginal space M_i rather than on the equivalence classes themselves.

[40] One cannot improve upon Lemma A.15 by proving, for example, that M_i, metrized by δ_{M_i}, is homeomorphic to a convex set. It need not be (e.g., see footnote 21). Evidently, the present approach can handle action spaces that the Athey-McAdams approach cannot easily accommodate, if at all.

$t \in T$.[41] Hence, since u_i is bounded, Lebesgue's dominated convergence theorem yields

$$U_i(s^n) = \int_T u_i(s^n(t), t) d\mu(t) \to \int_T u_i(s(t), t) d\mu(t) = U_i(s),$$

establishing the continuity of U_i.

Now, because each player i's payoff function, U_i, is continuous and each M_i is compact, an application of Berge's theorem of the maximum implies that player i's best-reply correspondence, $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$, is nonempty valued and upper-hemicontinuous.

6.1. Contractible Best Reply Sets

Our central observation is that, if monotone best reply sets are join-closed, then they are contractible simply by virtue of the fact that a best reply must be a pointwise best reply almost everywhere.

According to Lemma A.4, for each player i, assumptions G.1 and G.2 imply the existence of a monotone and measurable function $\Phi_i : T_i \to [0,1]$ such that $\mu_i\{t_i \in T_i : \Phi_i(t_i) = c\} = 0$ for every $c \in [0,1]$. Fixing such a function Φ_i permits the construction of a contraction map.

Fix some monotone pure strategy, s_{-i}, for players other than i, and consider player i's set of monotone pure best replies, $\mathbf{B}_i(s_{-i})$. Fix any $s_i^* \in \mathbf{B}_i(s_{-i})$, and define $h : [0,1] \times \mathbf{B}_i(s_{-i}) \to \mathbf{B}_i(s_{-i})$ as follows: For every $t_i \in T_i$,

$$h(\tau, s_i)(t_i) = \begin{cases} s_i(t_i), & \text{if } \Phi(t_i) \leq |1 - 2\tau| \text{ and } \tau < 1/2 \\ s_i^*(t_i), & \text{if } \Phi(t_i) \leq |1 - 2\tau| \text{ and } \tau \geq 1/2 \\ s_i^*(t_i) \vee s_i(t_i), & \text{if } \Phi(t_i) > |1 - 2\tau| \end{cases} \tag{6.1}$$

Note that $h(\tau, s_i)$ is monotone because Φ_i is monotone and $s_i^*(t_i) \vee s_i(t_i) \geq s_i(t_i)$ and $s_i^*(t_i)$ for all t_i. Also, for all $s_i \in \mathbf{B}_i(s_{-i})$, $h(0, s_i) = s_i$ and $h(1, s_i) = s_i^*$, and $h(\tau, s_i)$ is a pointwise best reply μ_i almost everywhere because $s_i(t_i)$, $s_i^*(t_i)$, and, by join-closedness $s_i^*(t_i) \vee s_i(t_i)$, are best replies for μ_i-a.e. t_i. Consequently, $h(\tau, s_i) \in \mathbf{B}_i(s_{-i})$. Therefore, h will be a contraction for $\mathbf{B}_i(s_{-i})$ and $\mathbf{B}_i(s_{-i})$ will be contractible if $h(\tau, s_i)$ is continuous, which is established in Lemma A.14.[42] Thus, we have established the following.

Lemma 6.1. *The correspondence $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is contractible-valued.*

For example, if $T_i = [0,1]^2$ and μ_i is absolutely continuous with respect to Lebesgue measure, we may take $\Phi_i(t_i) = (t_{i1} + t_{i2})/2$.[43] Figure 6.1 provides snapshots of the resulting $h(\tau, s_i)$ as τ moves from zero to one. The axes are the two dimensions of the type vector (t_{i1}, t_{i2}), and the arrow within the figures depicts the direction in which the diagonal line,

[41] This is because $\mu(\times_i Q_i) = \mu(\cap_i (Q_i \times T_{-i})) = 1$ when $\mu(Q_i \times T_{-i}) = \mu_i(Q_i) = 1$ for all i.

[42] Indeed, Lemma A.14 establishes that the right-hand side of (6.1) is continuous in (τ, s_i, s_i^*).

[43] This is not the function that results from our general construction in (A.1). Any monotone function whose level sets have μ_i measure zero will do.

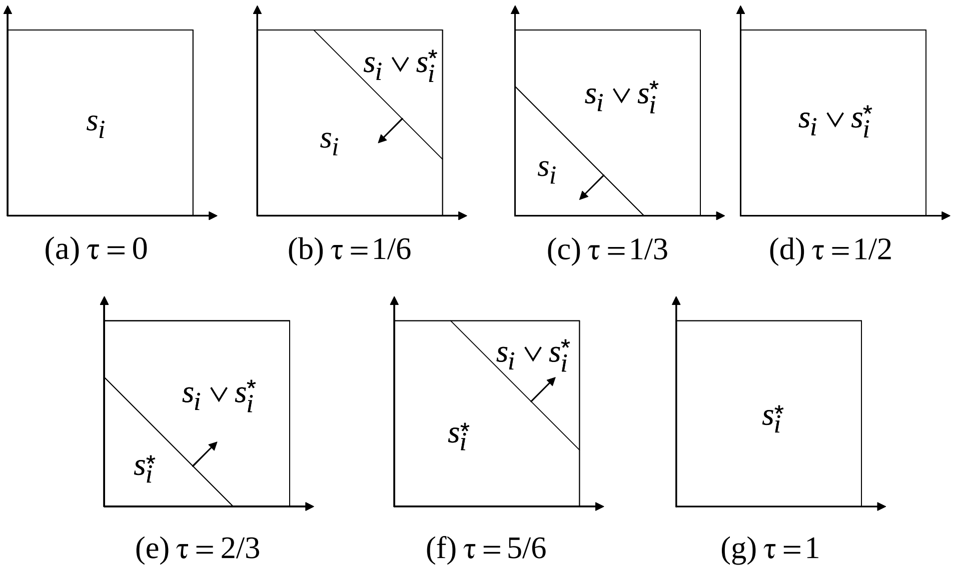

Figure 6.1: $h(\tau, s_i)$ as τ varies from 0 (panel (a)) to 1 (panel (g)) and the domain is the unit square.

$t_{i1} + t_{i2} = 2\,|1 - 2\tau|$, moves as τ increases locally. For example, panel (a) shows that when $\tau = 0$, $h(\tau, s_i)$ is equal to s_i over the entire unit square. On the other hand, panel (f) shows that when $\tau = 5/6$, $h(\tau, s_i)$ is equal to s_i^* below the diagonal line and equal to $s_i \vee s_i^*$ above it.

6.2. Completing the Proof.

The following lemma completes the proof of Theorem 4.1.

Lemma 6.2. *The product of the players' best reply correspondences, $\times_{i=1}^n \mathbf{B}_i : M \twoheadrightarrow M$, possesses a fixed point.*

Proof. We have already argued that $\times_{i=1}^n \mathbf{B}_i : M \twoheadrightarrow M$ is u.h.c., nonempty-valued, and contractible-valued, and that each (M_i, δ_i) is a compact absolute retract. Consequently, under the product topology, M is both compact and, by Borsuk (1966) IV (7.1), an absolute retract. Hence, applying Theorem 2.1 to $\times_{i=1}^n \mathbf{B}_i : M \twoheadrightarrow M$ yields the desired result. ■

Remark 9. *The proof of Theorem 4.2 mimics that of Theorem 4.1, but where each M_i is replaced with $M_i \cap C_i$, and where each correspondence $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is replaced with the correspondence $\mathbf{B}_i^* : M_{-i} \cap C_{-i} \twoheadrightarrow M_i \cap C_i$ defined by $\mathbf{B}_i^*(s_{-i}) = \mathbf{B}_i(s_{-i}) \cap C_i$. The proof goes through because the hypotheses of Theorem 4.2 imply that each $M_i \cap C_i$ is compact, nonempty, join-closed, piecewise-closed, and pointwise-limit-closed (and hence the proof that each $M_i \cap C_i$ is an absolute retract mimics the proof of Lemma A.15), and that each correspondence $\mathbf{B}_i^*$ is upper hemicontinuous, nonempty-valued and contractible-valued (the contraction is once again defined by 6.1). The result then follows from Theorem 2.1.*

A. Appendix

To simplify the notation, we drop the subscript i from T_i, μ_i, and A_i throughout the the appendix. Thus, in this appendix, T, μ, and A should be thought of as the type space, marginal distribution, and action space, respectively, of any one of the players, not as the joint type spaces, joint distribution, and joint action spaces of all the players. Of course, the theorems that follow are correct with either interpretation, but in the main text we apply the theorems below to the players individually rather than jointly and so the former interpretation is the more relevant.

We maintain the following assumptions throughout the appendix.

G.1 T is a complete separable metric space endowed with a measurable partial order.

G.2 μ assigns probability zero to any Borel subset of T having no strictly ordered points.

G.3$'$ A is a compact metric space and a semilattice with a closed partial order.

Assumptions G.1 and G.2 are taken from Section 3.2, while G.3 there is weakened to G.3$'$ here because most of the results in this appendix do not require the join operator on A to be continuous or A to be locally complete. These additional assumptions will be stated explicitly whenever needed. See Lemmas A.14, A.15, and A.17.

A.1. The Partially Ordered Space T

Preliminaries. Because $\geq$ is measurable, Lemma 7.6.1 of Cohn (1980) implies that the sets $\geq(t) = \{t' \in T : t' \geq t)$ and $\leq(t) = \{t' \in T : t \geq t'\}$ are in $\mathcal{B}(T)$ for each $t \in T$. A totally ordered subset of A is called a *chain*. A *strict chain* is a chain in which every pair of distinct points are strictly ordered. Finally, we say that $t \in T$ is in the *order-support* of μ if $\mu(U\cap \geq(t)) > 0$ and $\mu(U\cap \leq(t)) > 0$ for every neighborhood U of t.

Lemma A.1. *There is a Borel measurable subset of the order-support of μ having μ-measure one.*

Proof. Let $\mathcal{A} = \{E \in \mathcal{B}(T \times T) : \mu(E_t)$ is a Borel measurable function of $t \in T\}$, where $E_t = \{t' \in T : (t, t') \in E\}$. Then $\mathcal{A}$ contains all open sets of form $E = U \times V$, since the resulting function $\mu(E_t)$ is lower semicontinuous on T. Suppose that $E^1 \subseteq E^2 \subseteq ...$ is an increasing sequence of sets in $\mathcal{A}$. Then because $(E^2\backslash E^1)_t = E^2_t\backslash E^1_t$ and $(\cup_i E^i)_t = \cup_i E^i_t$, we have $\mu\left[(E^2\backslash E^1)_t\right] = \mu(E^2_t) - \mu(E^1_t)$ and $\mu\left[(\cup_i E^i)_t\right] = \mu(\cup_i E^i_t) = \lim_i \mu(E^i_t)$. Consequently, $E^2\backslash E^1$ and $\cup_i E^i$ are in $\mathcal{A}$. Hence, by Theorem 1.6.1 of Cohn (1980), $\mathcal{A}$ contains $\mathcal{B}(T)\times\mathcal{B}(T)$, the sigma algebra generated by all open sets of the form $U \times V$. But because T is a separable metric space, $\mathcal{B}(T) \times \mathcal{B}(T) = \mathcal{B}(T \times T)$ by Proposition 8.1.5 of Cohn (1980). Hence, $\mathcal{A} = \mathcal{B}(T\times T)$. In particular, because the measurability of $\geq$ implies that $E = (T \times U)\cap\{(t, t') \in T\times T : t' \geq t\}$ is a member of $\mathcal{B}(T\times T)$ for every open subset U of T, we may conclude that $\mu(E_t) = \mu(U\cap \geq(t))$ is a measurable function of $t \in T$ for each open subset U of T.

Let U be any open subset of T, and consider the measurable set $D = \{t \in U : \mu(U\cap \geq(t)) = 0\}$. We next show that $\mu(D) = 0$. Suppose, by way of contradiction, that $\mu(D) > 0$. Because T is a separable metric space, we may assume without loss that D is contained in the support of μ, so that every open set intersecting D has positive μ-measure. By G.2, D contains two strictly ordered points, $t_0 \leq t_1$. Hence, there are disjoint neighborhoods U_0 of t_0 and U_1 of t_1 such that $u_0 \leq u_1$ for every $u_0 \in U_0$ and every $u_1 \in U_1$. In particular, $U_1$is contained in $\geq(t_0)$, so that $U \cap U_1 \subseteq U\cap \geq(t_0)$. The open set $U \cap U_1$

intersects D because both sets contain t_1, and so $\mu(U \cap U_1) > 0$. But then $\mu(U\cap \geq(t_0)) > 0$, contradicting $t_0 \in D$.

Let $\{U_1, U_2, ...\}$ be a countable base for the topology of T and consider the measurable set $S = \cap_i \left[\{t \in U_i : \mu(U_i \cap \geq (t)) > 0\} \cup U_i^c\right]$. The result established in the previous paragraph implies that $\mu(S) = 1$ since, for each i, the set in curly brackets has measure $\mu(U_i)$, and U_i^c has the complementary measure. Now consider any $t \in S$ and any neighborhood U of t. For some i, we have $t \in U_i \subseteq U$, and therefore $\mu(U\cap \geq(t)) \geq \mu(U_i\cap \geq(t)) > 0$, since $t \in S$.

Consequently, for every $t \in S$, $\mu(U\cap \geq(t)) > 0$ for every neighborhood U of t. A similar argument establishes the existence of a measurable set S' such that $\mu(S') = 1$ and every $t \in S'$ satisfies $\mu(U\cap \leq(t)) > 0$ for every neighborhood U of t. Therefore, $S \cap S'$ is a measurable subset of the order-support of μ having μ-measure one. ■

Lemma A.2. *Let C be a chain in T. Then t is an accumulation point of both $C\cap \geq(t)$ and $C\cap \leq(t)$ for all but perhaps countably many $t \in C$.*[44]

Proof. Without loss, we may assume that C is uncountable. Let d denote the metric on T. Suppose first, and by way of contradiction, that there is no $t \in C$ that is an accumulation point of $C\cap \geq(t)$. Then, for every $t \in C$ there exists $\varepsilon_t > 0$ such that $B_{\varepsilon_t}(t)$, the open ball with radius ε_t around t, is disjoint from $[C\cap \geq(t)]\backslash\{t\}$. Consequently, for some fixed $\varepsilon > 0$ there must be uncountably many $t \in C$ such that $B_\varepsilon(t)$ is disjoint from $[C\cap \geq(t)]\backslash\{t\}$. Let C' denote this uncountable subset of C, and consider the collection of open sets $\{B_{\varepsilon/2}(t)\}_{t\in C'}$. The separability of T implies that not all pairs of sets in this collection can be disjoint. Hence, there must be distinct $t, t' \in C'$ such that $B_{\varepsilon/2}(t) \cap B_{\varepsilon/2}(t')$ is nonempty. Then, by the triangle inequality, $d(t, t') < \varepsilon$. However, because C' is a chain, we may assume without loss that $t' \geq t$ and so by the definition of C', $t' \notin B_\varepsilon(t)$, implying that $d(t, t') \geq \varepsilon$, a contradiction. We conclude that some $t \in C$ is an accumulation point of $C\cap \geq(t)$.

But then t is an accumulation of $C\cap \geq(t)$ for all but perhaps countably many $t \in C$ since, otherwise, we could repeat the argument on the uncountable number of remaining points in the chain. Similarly, t is an accumulation point of $C\cap \leq(t)$ for all but perhaps countably many $t \in C$. ■

Lemma A.3. *If $\mu(B) > 0$, then B contains a strict chain with uncountably many elements.*

Proof.[45] Assume that $\mu(B) > 0$. Because T is a complete separable metric space, B contains a compact subset having positive μ-measure. Hence, without loss, we may assume that B is compact. Replacing B if necessary with $B \cap V^c$, where V is the largest open set whose intersection with B has μ-measure zero, we may further assume without loss that $\mu(U \cap B) > 0$ for every open set U intersecting B.[46]

By assumption G.2, B contains two strictly ordered points $t_0 \leq t_1$. Hence, there are disjoint neighborhoods U_0 of t_0 and U_1 of t_1 such that $u_0 \leq u_1$ for every $u_0 \in U_0$ and every $u_1 \in U_1$. Clearly, any two such u_0 and u_1 are strictly ordered. Therefore, by replacing the U_i if necessary with sufficiently small balls around t_0 and t_1, we may assume that u_0 and u_1 are strictly ordered for every $u_0 \in \bar{U}_0$ and every $u_1 \in \bar{U}_1$, where $\bar{U}_i$ denotes the closure of U_i. Because each $U_i \cap B$ is nonempty (t_i is a member), each has positive μ-measure. Hence, for $i = 0, 1$, we may repeat the construction on each $U_i \cap B$, giving rise to strictly ordered points t_{i0} and t_{i1} in $U_i \cap B$ and their strictly ordered closed neighborhoods $\bar{U}_{i0}$ and

[44] Recall that t is an accumulation point of S if every neighborhood of t contains infinitely many points of S.

[45] I am grateful to Benjamin Weiss for outlining the proof given here.

[46] To see that V is well-defined, let $\{U_i\}$ be a countable base for T. Then V is the union of all the U_i satisfying $\mu(U_i \cap B) = 0$.

$\bar{U}_{i1}$, both of which can be chosen to be subsets of $\bar{U}_i$. Continuing in this manner, we obtain a countably infinite collection of open sets U_0, U_1, U_{00}, U_{01}, U_{10}, $U_{11}, \ldots$. The open sets $\{U_s\}_{s \text{ a finite sequence of 0's and 1's}}$ and T form a binary tree with T at its root, where succession is defined by set inclusion, because $U_{i_1 i_2 \ldots i_k} \supseteq U_{i_1 i_2 \ldots i_k i_{k+1}}$ where all the i_j are 0 or 1. Further, each set in $\{U_s\}$ intersects B and, without loss, we may choose them so that their boundaries are mutually disjoint and for each n the radius of $U_{i_1 \ldots i_n}$ is no greater than $1/n$.

For each $\alpha \in [0,1]$, consider its binary expansion (choose one expansion if there are two), $.i_1 i_2 i_3 \ldots$, and the infinite intersection $\bar{U}_{i_1} \cap \bar{U}_{i_1 i_2} \cap \bar{U}_{i_1 i_2 i_3} \cap \ldots$. The sets in the intersection form a decreasing sequence of closed sets whose radii converge to zero. Hence, by the completeness of T, their intersection contains a single point, t_α. Moreover, $t_\alpha \in B$ because each set in the sequence intersects the compact set B. Suppose $\alpha, \beta \in [0,1]$ are distinct. Their binary expansions must therefore differ for the first time at, say, the $n+1$st digit. If their common first n digits are $i_1, \ldots, i_n$ and their $n+1$st digits are j and k for α and β, respectively, then $t_\alpha \in \bar{U}_{i_1 \ldots i_n j}$ and $t_\beta \in \bar{U}_{i_1 \ldots i_n k}$. Hence, because the boundaries of the disjoint open sets $U_{i_1 \ldots i_n j}$ and $U_{i_1 \ldots i_n k}$ do not intersect, t_a and t_β are distinct elements of B. Moreover, by construction, every element of $\bar{U}_{i_1 \ldots i_n j}$ is strictly ordered with every element of $\bar{U}_{i_1 \ldots i_n k}$. Consequently, t_α and t_β are strictly ordered. Thus, $\{t_\alpha : \alpha \in [0,1]\}$ is an uncountable strict chain in B. ■

Lemma A.4. *There is a monotone and measurable function $\Phi : T \to [0,1]$ such that $\mu(\Phi^{-1}(c)) = 0$ for every $c \in [0,1]$.*

Proof. By separability, T admits a countable dense subset, $\{t_1, t_2, \ldots\}$. Define $\Phi : T \to [0,1]$ as follows:

$$\Phi(t) = \sum_{i=1}^{\infty} 2^{-i} \mathbf{1}_{\geq (t_i)}(t). \tag{A.1}$$

Clearly, $\Phi(\cdot)$ is monotone and measurable, being the sum of monotone and measurable functions. It remains only to show that $\mu\{t \in T : \Phi(t) = c\} = 0$ for every $c \in [0,1]$.

By Lemma A.3, it suffices to show that for every $c \in [0,1]$, every strict chain in $\{t \in T : \Phi(t) = c\}$ is countable. In fact, we will show that every such strict chain contains no more than two elements. To see this, suppose, by way of contradiction, that for some $c \in [0,1]$, $\{t \in T : \Phi(t) = c\}$ contains a strict chain with at least three distinct elements, $t \geq t' \geq t''$. Hence, in particular $\Phi(t) = \Phi(t') = \Phi(t'')$, and there are neighborhoods U of t, U' of t' and U'' of t'', such that $u \geq u' \geq u''$ for every $u \in U$, $u' \in U'$ and $u'' \in U''$. Because T is a metric space, we may assume that these open sets are mutually disjoint. The open set U' must contain a member, t_i say, of the dense set $\{t_1, t_2, \ldots\}$. Hence, $t \geq t_i \geq t''$ and $t'' \ngeq t_i$. But then $\Phi(t) \geq \Phi(t'') + 2^{-i} > \Phi(t'')$, a contradiction. ■

A.2. The Semilattice A

The standard proofs of the next two lemmas are omitted.

Lemma A.5. *If a_n, c_n are sequences in A converging to a, and $a_n \leq b_n \leq c_n$ for every n, then b_n converges to a.*

Lemma A.6. *Every nondecreasing sequence and every nonincreasing sequence in A converges.*

Lemma A.7. *A is a complete semilattice.*

Proof. Let S be a subset of A. Because A is a compact metric space, S has a countable dense subset, $\{a_1, a_2, ...\}$. Let $a^* = \lim_n a_1 \vee ... \vee a_n$, where the limit exists by Lemma A.6. Let $b \in A$ be an upper bound for S and let a be an arbitrary element of S. Then, some sequence, a_{n_k}, converges to a. Moreover, $a_{n_k} \leq a_1 \vee a_2 \vee ... \vee a_{n_k} \leq b$ for every k. Taking the limit as $k \to \infty$ yields $a \leq a^* \leq b$. Hence, $a^* = \vee S$. ■

A.3. The Space of Monotone Functions from T into A

In this subsection we will introduce a metric, δ, under which the space $\mathcal{M}$ of monotone functions from T into A will be shown to be a compact metric space. Further, it will be shown that if in addition to the maintained hypotheses G.1, G.2 and G.3′ of this appendix, A is locally complete with a continuous join operator, the metric space $(\mathcal{M}, \delta)$ is an absolute retract. Some preliminary results are required.

Lemma A.8. *If C is a strict chain in T and $f : C \to A$ is monotone, then f is continuous at all but perhaps countably many $t \in C$.*

Proof. If $a = f(t)$ and t is neither the smallest nor the largest element of $f^{-1}(a)$, then there are distinct $t', t'' \in f^{-1}(a)$ such that $t' \leq t \leq t''$. Because $f^{-1}(a)$ is a subset of C, it is a strict chain. Hence, there is a neighborhood U of t such that $t' \leq u \leq t''$ for every $u \in U$. Consequently, if t_k is a sequence in C converging to t, then $t' \leq t_k \leq t''$ and so also $a = f(t') \leq f(t_k) \leq f(t'') = a$ for all k large enough. Hence, $\lim_k f(t_k) = a = f(t)$, and we conclude that f is continuous at t and so at all but at most two points, the smallest and the largest if they exist, in $f^{-1}(a)$. Consequently, if $D \subseteq C$ is the set of discontinuity points of f, then D will be countable if $f(D)$ is countable.

Suppose that $t \in D$. Then, focusing on one of two possibilities, we may assume that C contains a sequence $t_n \to t$ such that $t_n \geq t$ for all n and $f(t_n) \to a \geq f(t) \neq a$, where the latter inequality uses the assumed (see G.3′) closedness of the partial order on A.[47] Because C is a strict chain, if $t' \in C$ is distinct from t and $t' \geq t$, there is a neighborhood U of t such that $t' \geq u$ for every $u \in U$. Hence, for all n sufficiently large, $t' \geq t_n$ and therefore also $f(t') \geq f(t_n)$. Taking the limit in n implies that $f(t') \geq a$ because the partial order on A is closed. From this we may conclude that $f(t)$ is not an accumulation point of $f(C) \cap \geq(f(t))$. To see this, suppose otherwise that there is a sequence $t'_n \in C$ with $f(t) \neq f(t'_n) \geq f(t)$ and $f(t'_n) \to f(t)$. Because C is a strict chain and f is monotone, the first two relations imply $t \neq t'_n \geq t$ and so, as just shown, $f(t'_n) \geq a$ for every n. Taking limits yields $f(t) \geq a$. However, $a \geq f(t)$ then yields $a = f(t)$, a contradiction, establishing that $f(t)$ is not an accumulation point of $f(C) \cap \geq(f(t))$. But then $f(t)$ is not an accumulation point of $f(D) \cap \geq(f(t))$ either. Because $f(t)$ was an arbitrary element of $f(D)$, we have shown that $f(D)$ is a chain such that no $a \in f(D)$ is an accumulation point of $f(D) \cap \geq(a)$. By Lemma A.2, $f(D)$ is countable. ■

Lemma A.9. *If $f : T \to A$ is measurable and monotone, then f is continuous μ almost everywhere.*

Proof. Let D denote the set of discontinuity points of f. Note that D is Borel measurable because its complement, the set of continuity points of f, is $\cap_{i=1}^{\infty} \left(\text{int} f^{-1}(U_i) \cup [f^{-1}(U_i)]^c\right)$, where $\{U_i\}$ is a countable base for A.[48] It suffices to show that $\mu(D) = 0$. Let C be a strict chain in D. By Lemma A.3, it suffices to show that C is countable. Let $f|_C$ be the restriction

[47] The other possibility involves the reverse inequalities.

[48] Every compact metric space has a countable base.

of f to C, and let C' be the set of $t \in C$ that are accumulation points of both $C \cap \geq(t)$ and $C \cap \leq(t)$ and also continuity points of $f|_C$. By Lemmas A.2 and A.8, C' contains all but countably many $t \in C$. Hence, it suffices to show that C' is empty. Suppose by way of contradiction that $t \in C'$. Then C contains sequences t'_n and t''_n converging to t such that $t'_n \leq t \leq t''_n$ and both t'_n and t''_n are distinct from t for all n. Let t_k be an arbitrary sequence in T converging to t such that $f(t_k)$ converges to some $a \in A$. Because C is a strict chain, for each n there is a neighborhood U_n of t such that $t'_n \leq u \leq t''_n$ for every $u \in U_n$. Hence, for each n, $t'_n \leq t_k \leq t''_n$ and therefore $f(t'_n) \leq f(t_k) \leq f(t''_n)$ for all large enough k. Taking the limit first as $k \to \infty$ and then as $n \to \infty$ implies that $f(t) \leq a \leq f(t)$, because t is a continuity point of $f|_C$ and the partial order on A is closed. But then $a = f(t)$ and we conclude, because A is compact, that $t \in C$ is a continuity point of f, contradicting the definition of C. ■

Lemma A.10. *(A Generalized Helley's Theorem). If $f_n : T \to A$ is a sequence of monotone functions – not necessarily measurable – then there is a subsequence, f_{n_k}, and a measurable monotone function, $f : T \to A$, such that $f_{n_k}(t) \to_k f(t)$ for μ almost every $t \in T$.*

Proof. Let $\{t_1, t_2, ...\}$ be a countable dense subset of T. Choose a subsequence, f_{n_k}, of f_n such that, for every i, $\lim_k f_{n_k}(t_i)$ exists. Define $f(t_i) = \lim_k f_{n_k}(t_i)$ for every i, and extend f to all of T by defining $f(t) = \vee\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$.[49] By Lemma A.7, this is well defined because $\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$ is nonempty for each t since it contains any limit point of $f_{n_k}(t)$. Indeed, if $f_{n_{k_j}}(t) \to_j a$, then $a = \lim_j f_{n_{k_j}}(t) \leq \lim_j f_{n_{k_j}}(t_i) = f(t_i)$ holds for every $t_i \geq t$. Further, as required, the extension to T is monotone and leaves the values of f on $\{t_1, t_2, ...\}$ unchanged, where the latter follows because the monotonicity of f on $\{t_1, t_2, ...\}$ implies that $\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t_k\} = \{a \in A : a \leq f(t_k)\}$. To see that f is measurable, note first that $f(t) = \lim_m g_m(t)$, where $g_m(t) = \vee\{a \in A : a \leq f(t_i)$ for all $i = 1, ..., m$ such that $t_i \geq t\}$, and where the limit exists by Lemma A.6. Because $\geq$ is measurable, each g_m is a measurable simple function. Hence, f is measurable, being the pointwise limit of measurable functions.

By Lemmas A.1 and A.9, it suffices now to show that $f_{n_k}(t) \to f(t)$ for all continuity points t of f in the order-support of μ. So, let t be a continuity point of f in the order-support of μ, and suppose that $f_{n_{k_j}}(t) \to a \in A$ for some subsequence n_{k_j} of n_k. By the compactness of A, it suffices to show that $a = f(t)$. Because t is in the order support of μ, both $\mu(U \cap \geq(t))$ and $\mu(U \cap \leq(t))$ are positive for every neighborhood U of t. Hence, by G.2, $U \cap \geq(t)$ and $U \cap \leq(t)$ each contain a pair of strictly ordered points. In particular therefore, we may choose two distinct points $t' \geq t''$ in $U \cap \geq(t)$ and choose an open set U' contained in U and containing t' such that $u' \geq t'' \geq t$ for every $u' \in U'$. Because U' is open, it contains some t_i in the dense set $\{t_1, t_2, ...\}$ and so $t_i \geq t$. Similarly, by considering a pair of strictly ordered points in $U \cap \leq(t)$, we can find t_j in U such that $t_j \leq t$. Since U was an arbitrary open set containing t, this shows that there are sequences t_{i_m} and t_{j_m} each converging to t and contained in $\{t_1, t_2, ...\}$ and such that $t_{j_m} \leq t \leq t_{i_m}$ for every m. Hence, because the f_n are monotone, $f_{n_{k_j}}(t_{j_m}) \leq f_{n_{k_j}}(t) \leq f_{n_{k_j}}(t_{i_m})$ for every j and m. Taking the limit in j gives $f(t_{j_m}) \leq a \leq f(t_{i_m})$, and taking next the limit in m gives $f(t) \leq a \leq f(t)$, because t is a continuity point of f. Hence, $a = f(t)$ as desired. ■

By setting f_n in Lemma A.10 equal to a constant sequence, we obtain the following.

Lemma A.11. *Every monotone function from T into A is μ-almost everywhere equal to a Borel measurable monotone function.*

[49] Note then that $f(t) = \vee A$ if no $t_i \geq t$.

We can now introduce a metric on $\mathcal{M}$, the space of monotone functions from T into A. Denote the metric on A by d and assume without loss that $d(a,b) \leq 1$ for all $a, b \in A$. Define the metric, δ, on $\mathcal{M}$ by

$$\delta(f,g) = \int_T d(f(t), g(t))d\mu(t),$$

which is well-defined by Lemma A.11.

Formally, the resulting metric space $(\mathcal{M}, \delta)$ is the space of equivalence classes of monotone functions that are equal μ almost everywhere. Nevertheless, and analogous to the standard treatment of $\mathcal{L}_p$ spaces, we focus on the elements of the original space $\mathcal{M}$ rather than on the equivalence classes themselves.

Lemma A.12. *In $(\mathcal{M}, \delta)$, f_k converges to f if and only if in (A, d), $f_k(t)$ converges to $f(t)$ for μ almost every $t \in T$.*

Proof. (only if) Suppose that $\delta(f_k, f) \to 0$. Recall that $t \in T$ is in the order-support of μ if for every neighborhood U of t, $\mu(U \cap \geq(t)) > 0$ and $\mu(U \cap \leq(t)) > 0$. By Lemmas A.1 and A.9, it suffices to show that $f_k(t) \to f(t)$ for all continuity points, t, of f in the order-support of μ.

Let t_0 be a continuity point of f in the order-support of μ. Because A is compact, it suffices to show that an arbitrary convergent subsequence, $f_{k_j}(t_0)$, of $f_k(t_0)$ converges to $f(t_0)$. So, suppose that $f_{k_j}(t_0)$ converges to $a \in A$. By Lemma A.10, there exists a further subsequence, $f_{k'_j}$ of f_{k_j} and a monotone measurable function, $g : T \to A$ such that $f_{k'_j}(t) \to g(t)$ for μ a.e. t in T. Because d is bounded, the dominated convergence theorem implies that $\delta(f_{k'_j}, g) \to 0$. But $\delta(f_{k'_j}, f) \to 0$ then implies that $\delta(f, g) = 0$ and so $f_{k'_j}(t) \to f(t)$ for μ a.e. t in T.

Because $f_{k'_j}(t) \to f(t)$ for μ a.e. t in T and because t_0 is in the order-support of μ, for every $\varepsilon > 0$ there exist $t_\varepsilon, t'_\varepsilon$ each within ε of t_0 such that $t_\varepsilon \leq t_0 \leq t'_\varepsilon$ and such that $f_{k'_j}(t_\varepsilon) \to_j f(t_\varepsilon)$ and $f_{k'_j}(t'_\varepsilon) \to_j f(t'_\varepsilon)$. Consequently, $f_{k'_j}(t_\varepsilon) \leq f_{k'_j}(t_0) \leq f_{k'_j}(t'_\varepsilon)$, and taking the limit as $j \to \infty$ yields $f(t_\varepsilon) \leq a \leq f(t'_\varepsilon)$, and taking next the limit as $\varepsilon \to 0$ yields $f(t_0) \leq a \leq f(t_0)$, so that $a = f(t_0)$, as desired.

(if) To complete the proof, suppose that $f_k(t)$ converges to $f(t)$ for μ almost every $t \in T$. Then, because d is bounded, the dominated convergence theorem implies that $\delta(f_k, f) \to 0$. ■

Combining Lemmas A.10 and A.12 we obtain the following.

Lemma A.13. *$(\mathcal{M}, \delta)$ is compact.*

Lemma A.14. *Suppose that the join operator on A is continuous and that $\Phi : T \to [0,1]$ is a monotone and measurable function such that $\mu(\Phi^{-1}(c)) = 0$ for every $c \in [0,1]$. Define $h : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by*

$$h(\tau, f, g)(t) = \begin{cases} f(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau < 1/2 \\ g(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau \geq 1/2 \\ f(t) \vee g(t), & \text{if } \Phi(t) > |1 - 2\tau| \end{cases} \tag{A.2}$$

Then $h : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is continuous.

Proof. Suppose that $(\tau_k, f_k, g_k) \to (\tau, f, g) \in [0,1] \times \mathcal{M} \times \mathcal{M}$. By Lemma A.12, there is a full μ measure subset, D, of T such that $f_k(t) \to f(t)$ and $g_k(t) \to g(t)$ for every $t \in D$. There are three cases: $\tau = 1/2$, $\tau > 1/2$ and $\tau < 1/2$.

Suppose that $\tau < 1/2$. For each $t \in D$ such that $\Phi(t) < |1 - 2\tau|$, we have $\Phi(t) < |1 - 2\tau_k|$ for all k large enough. Hence, $h(\tau_k, f_k, g_k)(t) = f_k(t)$ for all k large enough, and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \to f(t) = h(\tau, f, g)(t)$. Similarly, for each $t \in D$ such that $\Phi(t) > |1 - 2\tau|$, $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(\tau, f, g)(t)$, where the limit follows because $\vee$ is continuous. Because $\mu(\{t \in T : \Phi(t) = |1 - 2\tau|\}) = 0$, if $\tau < 1/2$, $h(\tau_k, f_k, g_k)(t) \to h(\tau, f, g)(t)$ for μ a.e. $t \in T$ and so, by Lemma A.12, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$.

Because the case $\tau > 1/2$ is similar to $\tau < 1/2$, we need only consider the remaining case in which $\tau = 1/2$. In this case, $|1 - 2\tau_k| \to 0$. Consequently, for any $t \in T$ such that $\Phi(t) > 0$, we have $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t)$ for k large enough and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(1/2, f, g)(t)$. Hence, because $\mu(\{t \in T : \Phi(t) = 0\}) = 0$, $h(\tau_k, f_k, g_k)(t) \to h(1/2, f, g)(t)$ for μ a.e. $t \in T$, and so again by Lemma A.12, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$. ∎

Lemma A.15. *If the join operator on A is continuous and A is locally complete, then the metric space $(\mathcal{M}, \delta)$ is an absolute retract.*

Proof. As a matter of notation, for $f, g \in \mathcal{M}$, write $f \leq g$ if $f(t) \leq g(t)$ for μ almost every t in T. Also, for any sequence of monotone functions $f_1, f_2, ...,$ in $\mathcal{M}$, denote by $f_1 \vee f_2 \vee ...$ the monotone function taking the value $\lim_n [f_1(t) \vee f_2(t) \vee ... \vee f_n(t)]$ for each t in T. This is well-defined by Lemma A.6.

By Lemmas A.4 and A.14, the function $h : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ defined by (A.2) is continuous, where the monotone function $\Phi(\cdot)$ appearing in (A.2) is defined by (A.1). Since for any $g \in \mathcal{M}$, $h(\cdot, \cdot, g)$ is a contraction for $\mathcal{M}$, $(\mathcal{M}, \delta)$ is contractible. Hence, by Borsuk (1966, IV (9.1)) and Dugundji (1965), it suffices to show that for each $f' \in \mathcal{M}$ and each neighborhood U of f', there exists a neighborhood V of f' and contained in U such that the sets V^n, $n \geq 1$, defined inductively by $V^1 = h([0,1], V, V)$, $V^{n+1} = h([0,1], V, V^n)$, are all contained in U.[50]

For each V, note that if $g \in V^1$, then $g = h(\tau, f_0, f_1)$ for some $\tau \in [0,1]$ and some $f_0, f_1 \in V$. Hence, by the definition of h, we have $g \leq f_0 \vee f_1$ and either $f_0 \leq g$ or $f_1 \leq g$. We may choose the indices so that $f_0 \leq g \leq f_0 \vee f_1$. Inductively, it can similarly be seen that if $g \in V^n$, then there exist $f_0, f_1, ..., f_n \in V$ such that

$$f_0 \leq g \leq f_0 \vee ... \vee f_n. \tag{A.3}$$

Suppose now, by way of contradiction, that there is no open set V containing $f' \in \mathcal{M}$ and contained in the neighborhood U of f' such that all the V^n as defined above are contained in U. Then, successively for each $k = 1, 2, ..$, taking V to be $B_{1/k}(f')$, the $1/k$ ball around f', there exists n_k such that some $g_k \in V^{n_k}$ is not in U. Hence, by (A.3), there exist $f_0^k, ..., f_{n_k}^k \in V = B_{1/k}(f')$ such that

$$f_0^k \leq g_k \leq f_0^k \vee ... \vee f_{n_k}^k. \tag{A.4}$$

Consider the sequence $f_0^1, ..., f_{n_1}^1, f_0^2, ..., f_{n_2}^2, ...$. Because f_j^k is in $B_{1/k}(f')$, this sequence converges to f'. Let us reindex this sequence as $f_1, f_2, ...$. Hence, $f_j \to f'$.

Because for every n the set $\{f_n, f_{n+1}, ...\}$ contains the set $\{f_0^k, ..., f_{n_k}^k\}$ whenever k is large enough, we have

$$f_0^k \vee ... \vee f_{n_k}^k \leq \vee_{j \geq n} f_j,$$

for every n and all large enough k. Combined with (A.4), this implies that

$$f_0^k \leq g_k \leq \vee_{j \geq n} f_j \tag{A.5}$$

[50] This condition, which is related to the local contractibility of $\mathcal{M}$, can more easily be related to local convexity. For example, if $\mathcal{M}$ is convex, instead of merely contractible, and $h(\alpha, f, g) = \alpha f + (1-\alpha) g$ is the usual convex combination map, the condition follows immediately if $\mathcal{M}$ is, in addition, locally convex.

for every n and all large enough k.

Now, $f_0^k \to f'$ as $k \to \infty$. Hence, by Lemma A.12, $f_0^k(t) \to f'(t)$ for μ a.e. t in T. Consequently, if for μ a.e. t in T, $\vee_{j\geq n} f_j(t) \to f'(t)$ as $n \to \infty$, then (A.5) and Lemma A.5 would imply that for μ a.e. t in T, $g_k(t) \to f'(t)$. Then, Lemma A.12 would imply that $g_k \to f'$ contradicting the fact that no g_k is in U, and completing the proof that $(\mathcal{M}, \delta)$ is an absolute retract.

It therefore remains only to establish that for μ a.e. $t \in T$, $\vee_{j\geq n} f_j(t) \to f'(t)$ as $n \to \infty$. But, by Lemma A.16, because A is locally complete this will follow if $f_j(t) \to_j f'(t)$ for μ a.e. t, which follows from Lemma A.12 because $f_j \to f'$. ■

A.4. Local Completeness

Lemma A.16. *A is locally complete if and only if for every $a \in A$ and every sequence a_n converging to a, $\lim_n(\vee_{k\geq n} a_k) = a$.*

Proof. We first demonstrate the "only if" direction. Suppose that A is locally complete, that U is a neighborhood of $a \in A$, and that $a_n \to a$. By local completeness, there exists a neighborhood W of a contained in U such that every subset of W has a least upper bound in U. In particular, because for n large enough $\{a_n, a_{n+1}, ...\}$ is a subset of W, the least upper bound of $\{a_n, a_{n+1}, ...\}$, namely $\vee_{k\geq n} a_k$, is in U for n large enough. Since U was arbitrary, this implies $\lim_n(\vee_{k\geq n} a_k) = a$.

We now turn to the "if" direction. Fix any $a \in A$, and let $B_{1/n}(a)$ denote the open ball around a with diameter $1/n$. For each n, $\vee B_{1/n}(a)$ is well-defined by Lemma A.7. Moreover, because $\vee B_{1/n}(a)$ is nonincreasing in n, $\lim_n \vee B_{1/n}(a)$ exists by Lemma A.6. We first argue that $\lim_n \vee B_{1/n}(a) = a$. For each n, we may construct, as in the proof of Lemma A.7, a sequence $\{a_{n,m}\}$ of points in $B_{1/n}(a)$ such that $\lim_m(a_{n,1} \vee ... \vee a_{n,m}) = \vee B_{1/n}(a)$. We may therefore choose m_n sufficiently large so that the distance between $a_{n,1} \vee ... \vee a_{n.m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$. Consider now the sequence $\{a_{1,1}, ..., a_{1,m_1}, a_{2,1}, ..., a_{2,m_2}, a_{3,1}, ..., a_{3,m_3}, ...\}$. Because $a_{n,m}$ is in $B_{1/n}(a)$, this sequence converges to a. Consequently, by hypothesis,

$$\lim_n (a_{n,1} \vee ... \vee a_{n,m_n} \vee a_{(n+1),1} \vee ... \vee a_{(n+1),m_{(n+1)}} \vee ...) = a.$$

But because every $a_{k,j}$ in the join in parentheses on the left-hand side above (denote this join by b_n) is in $B_{1/n}(a)$, we have

$$a_{n,1} \vee ... \vee a_{n,m_n} \leq b_n \leq \vee B_{1/n}(a).$$

Therefore, because for every n the distance between $a_{n,1} \vee ... \vee a_{1,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$, Lemma A.5 implies that $\lim_n \vee B_{1/n}(a) = \lim_n b_n$. But since $\lim_n b_n = a$, we have $\lim_n \vee B_{1/n}(a) = a$. Next, for each n, let S_n be an arbitrary nonempty subset of $B_{1/n}(a)$, and choose any $s_n \in S_n$. Then $s_n \leq \vee S_n \leq \vee B_{1/n}(a)$. Because $s_n \in B_{1/n}(a)$, Lemma A.5 implies that $\lim_n \vee S_n = a$. Consequently, for every neighborhood U of a, there exists n large enough such that $\vee S$ (well-defined by Lemma A.7) is in U for every subset S of $B_{1/n}(a)$. Since a was arbitrary, A is locally complete. ■

Lemma A.17. *If A is a subset of $\mathbb{R}^K$ with the coordinatewise partial order and $\vee$ is continuous, then A is locally complete.*

Proof. Suppose that $a_n \to a$. By Lemma A.16, it suffices to show that $\lim_n(\vee_{k\geq n} a_k) = a$. By Lemma A.6, $\lim_n(\vee_{k\geq n} a_k)$ exists and is equal to $\lim_n \lim_m(a_n \vee ... \vee a_m)$ since $a_n \vee ... \vee a_m$ is nondecreasing in m, and $\lim_m(a_n \vee ... \vee a_m)$ is nonincreasing in n. For each dimension

$k = 1, ..., K$, let $a^k_{n,m}$ denote the first among $a_n, a_{n+1}, ..., a_m$ with the largest kth coordinate. Hence, $a_n \vee ... \vee a_m = a^1_{n,m} \vee ... \vee a^K_{n,m}$, where the right-hand side consists of K terms. Because $a_n \to a$, $\lim_m a^k_{n,m}$ exists for each k and n, and $\lim_n \lim_m a^k_{n,m} = a$ for each k. Consequently, $\lim_n \lim_m (a_n \vee ... \vee a_m) = \lim_n \lim_m (a^1_{n,m} \vee ... \vee a^K_{n,m}) = a \vee ... \vee a = a$, as desired. ■

Lemma A.18. *If for all $a \in A$, every neighborhood of a contains a' such that $b' \leq a'$ for all b' close enough to a, then A is locally complete.*

Proof. Suppose that $a_n \to a$. By Lemma A.16, it suffices to show that $\lim_n(\vee_{k \geq n} a_k) = a$. For every n and m, $a_m \leq a_m \vee a_{m+1} \vee ... \vee a_{m+n}$, and so taking the limit first as $n \to \infty$ and then as $m \to \infty$ gives $a \leq \lim_m \vee_{k \geq m} a_k$, where the limit in n exists by Lemma A.6 because the sequence is monotone. Hence, to show that $\text{limsup}_m a_m = a$, it suffices to show that $\lim_m \vee_{k \geq m} a_k \leq a$.

Let U be a neighborhood of a and let a' be chosen as in the statement of the lemma. For m large enough, $a_m \in U$ and so $a_m \leq a'$. Consequently, for m large enough and for all n, $a_m \vee a_{m+1} \vee ... \vee a_{m+n} \leq a'$. Taking the limit first in n and then in m yields $\lim_m \vee_{k \geq m} a_k \leq a'$. Because for every neighborhood U of a this holds for some a' in U, $\lim_m \vee_{k \geq m} a_k \leq a$, as desired. ■

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