Asymmetric Common-Value Auctions with Applications to Private-Value Auctions with Resale

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Abstract

We study a model of common-value auctions with two bidders in which bidders' private information are independently and asymmetrically distributed. We provide three sufficient conditions under which we can determine whether a first-price auction generates higher or lower revenue than a second-price auction (for a selected equilibrium). Necessary conditions are given for the revenue-ranking result to hold in general.

We further establish the observational equivalence between an independent private-value (IPV) auction model with resale and a model of common-value auction, when the resale mechanism satisfies a sure-trade property and the common value is the transaction price. Using this observational equivalence and the revenue-ranking result for the common-value auctions, we provide an alternative proof of the revenue-ranking result of Hafalir and Krishna (2007) in the IPV auctions with resale. The revenue ranking holds when the offer-maker is fixed or is contingent on the auction outcome. In general, revenue ranking may depend on who has bargaining power in the resale stage. We illustrate that the opposite revenue-ranking may arise (i) when one of the distribution functions does not satisfy the regularity property, or (ii) when the resale mechanism involves repeated offers and delay costs, or (iii) when the Coase Conjecture holds as in Gul, Sonnenschein, and Wilson (1986) and Ausubel and Deneckere (1992).

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1 Introduction

In this paper, we study the e¤ects of asymmetry of the bidders on the revenue in a common-value auction model. Many important spectrum auctions held in countries all over the world and participated by communication companies have raised billions of dollars. These auctions are often considered as common-value auctions and participants of such auctions tend to have information disparities. How such information disparities a¤ect the seller's revenue in various auction formats are important questions that deserve a careful study.

We consider a common-value auction model with two bidders in which bidders' private information are independently and asymmetrically distributed. We provide three su¢ cient conditions under which we can rank the two standard auction formats. The conditions are related to the submodular or supermodular property of the common-value function. The submodular (supermodular) property says that when one bidder's private signal is higher, the other bidder's private signal has less (more) marginal impact on the common value.

Our study of common-value auctions has important implications for asymmetric private-value auctions if resale is allowed[1]. In fact, resale is an important source of common value among the bidders. This idea is quite intuitive. In the survey for their book, Kagel and Levin (2002, page 2) said that "There is a common-value element to most auctions. Bidders for an oil painting may purchase for their own pleasure, a private-value element, but they may also bid for investment and eventual resale, re‡ecting the common-value element". Lebrun (2007) has shown that the equilibrium strategy pro…le of an auction with the monopoly or monopsony resale market is the same as that of a (pure) common-value auction. We will provide a theoretical examination for this intuition in more general resale environments. We use the concept of observational equivalence. The observational equivalence means that the two auctions have the same equilibrium bid distributions. In a simple environment a seller has no way of knowing the di¤erence between the two from the bidding behavior in the auctions, nor can an econometrician from the bidding data. The resale stage is described by a general trade mechanism between a buyer and a seller with two-sided asymmetric information. If the trade mechanism satis…es a sure-trade property, then an independent private-value (IPV) …rst-price auction with resale is observationally equivalent to a …rst-price common-value auction with the common-value de…ned by the trade price in the resale stage.

The sure-trade property was …rst proposed by Hafalir and Krishna (2007), and used to show the symmetry property of the equilibrium bid distributions in the …rst-price auctions with resale. We use a variation of this idea, and show that the condition is su¢ cient for the observational equivalence[2]. The

[1] In government spectrum auctions, there are often restrictions on resale. It is not clear why the restrictions are imposed. Beyond the political and legal reasons, resale may facilitate collusions in the English auction as is shown in Garrat, Troger and Zheng (2007). However, it is often possible to get around the resale restrictions.

[2] In more general models (such as a¢ liated signals), the condition is also su¢ cient for obser-

sure-trade property is a very weak condition. It requires that trade must occur with probability one when the trade surplus is nearly the maximum possible amount, and the transaction price is used to define the common-value. The sure-trade property rules out the no-trade equilibrium in which there cannot be observational equivalence between the auction with resale and the common-value auction. We would expect a trade mechanism with a reasonable degree of efficiency to possess the sure-trade property. To the extent that traders would choose to use a more efficient mechanism, this is a rather mild condition. When there are delay costs in repeated offers, the common-value applicable in equilibrium is the first offer price and later offers are not involved in the equilibrium revenue. We adopt a slightly more restrictive description of the trade mechanism by requiring trade to occur with probability 1 or 0 for any realized pair of valuations.

The concept of observational equivalence has been used in Green and Laffont (1987). Laffont and Vuong (1996) showed that for any fixed number of bidders in a first-price auction, any symmetric affiliated values model is observationally equivalent to some symmetric affiliated private-values model. We show that when bidders anticipate trading activities after the auction, the bidding data is observationally equivalent to a common-value auction in which the common value is defined by the trading prices. Lebrun (2007) has shown the observational equivalence property when the resale market is a monopoly or monopsony market. We show that under the sure-trade property, it holds for very general resale mechanisms. Haile (2001) studied the empirical evidence of the effects of resale in the U.S. forest timber auctions.[3]

The equilibrium bid equivalence of the auction with resale and the common-value auction allows us to apply the ranking results for the common-value auctions to the case of auctions with resale. Hafalir and Krishna (2007) have shown that in auctions with resale with a pair of weak-strong bidders[4], the first-price auction has higher revenue than the second-price auction when valuations are independent, regular and the resale market is a single-offer monopoly or monopsony market. Our approach yields an alternative proof of this result. When it is not a weak-strong pair, the ranking result holds when the offer-maker is fixed or contingent on winning the auction. The offer-making bidder can be chosen by any random process with or without contingency on winning the auctions.

One should be cautious in interpreting the above single-offer result. A single-offer model requires the ability of the offer-maker to commit to his or her offer, and not to reduce prices when the offer is not accepted. Furthermore, the regularity assumption is not a technical assumption as in the case of the optimal auction literature. We give an example showing that the result may fail with-

vational equivalence, even though the Hafalir and Krishna (2007) symmetry property typically fails. The observational equivalence property seems to hold in more general environments than is treated in this paper. This will be explored in a separate paper.

[3] His model of resale is different from our specifications here. In his model, there is no asymmetry among bidders before auctions, and trade occurs after the auction because of information differences after the auction. In our model, bidders are asymmetric before auctions.

[4] Their mothods do allow more general pairs of bidders as shown in an earlier working paper of theirs.

out regularity. With regularity, the bargaining power tends to reside with the weak bidder rather than the strong bidder whoever is the offer-maker. Without regularity, the bargaining power can go either way, and hence the ranking can go in different directions.

To illustrate the effect of bargaining power on the ranking result, it is useful to abstract away from the information problem in the resale stage, as done in the Gupta and Lebrun (1999) model. Assume that all private information is disclosed after the auction and before the resale stage so that there is common knowledge of the valuations of both traders. With complete information in the resale stage, the bargaining power resides with the offer-maker, and we show that in this case, the first-price auction is superior if the winner of the auction makes offers, while the second-price auction is superior if the loser of the auction makes offers. This general picture remains true when there is incomplete information in the resale stage. We obtain necessary conditions for the ranking result to hold in either direction when the two bidders are nearly symmetric.

One important insight from our approach is that the revenue ranking property of auctions with resale depends on the bargaining power of the two bidders in the resale stage. Bargaining power is affected by many factors. As an example of the impact of bargaining power on the ranking result, we shall consider the issue of delay costs. When the seller and the buyer have different delay costs in the bargaining process, the person with a higher delay cost will lose bargaining power. We give a simple example of a two-offer monopoly resale mechanism. The valuations of the bidders are all uniformly distributed (hence regular). The second-price auction is superior when the monopolist has a high delay cost, while the buyer has no delay cost. The result is due to the weakened bargaining power of the auction winner.

Now consider the issue of commitment power. It is well-known that when an offer-maker cannot commit to the first offer after it is rejected, the bargaining power of the offer-maker will be reduced. When the Coase conjecture (1972) holds, the seller loses all bargaining power due to the lack of commitment, and as a result, the second-price auction is superior for a similar reason. The validity of the Coase conjecture has been shown in Gul, Sonnenschein and Wilson (1986)[5], when the uninformed party makes the offers, the bargaining interval converges to zero, and the equilibrium is stationary. If we allow alternating offers, it has also been shown in Ausubel and Deneckere (1992) as a consequence of the Silence Theorem.

We restrict our study to the case with two bidders, as there are well-known difficulties in analyzing the equilibrium bid of first-price auctions with asymmetric distributions when there are more bidders. At this stage, many issues need to be understood first in the bilateral context. Our method however has the potential of making it possible to analyze the problem in more general environment as the observational equivalence theorem seems to be true in general environments. In establishing the ranking result for the common-value auc-

[5] For the literature on the Coasian conjecture and theorems, see Coase (1972), Bulow (1982), Stokey (1981), Cramton (1984), Fudenberg, Levine, and Tirole (1985), Ausubel and Deneckere (1987, 1989a, 1989b, 1992), and Gul, Sonnenschein and Wilson (1992).

tions, we have to deal with an issue of multiple equilibria. It is well-known that there is a continuum of equilibria in second-price common-value auctions (with continuous distributions). For the comparison to make sense, we need to deal with the equilibrium selection issue. The equilibrium we select is motivated by later applications to auctions with resale. It is the one that is reduced to the dominant strategy equilibrium in private-value auctions or the only robust equilibrium in auctions with resale in Hafalir and Krishna (2007) when the common-value auction arises from auctions with resale. We also justify the equilibrium selection by a refinement concept allowing for a small private-value component in valuations. There is a unique second-price auction equilibrium when the private-value component is present. As the private-value component goes to 0 in the limit, we get the selected equilibrium under certain symmetric error conditions.[6]

The rest of the paper is organized as follows. In Section 2, we describe the common-value model and state three conditions regarding the common-value function and the distribution functions. We also derive equilibrium bids and revenues for the first-price and second-price auctions, and discuss the equilibrium selection issue in the second-price auction. In section 3, we provide some intuitive explanations for and formal statements of our main results on revenue ranking. Examples are provided to illustrate the necessity of the conditions for the revenue ranking. In Section 4, after a description of the IPV auctions with resale, we establish the observational equivalence of the common-value auctions and the IPV auctions with resale. We apply our ranking results to the auctions with resale in Section 5. In Section 5.3, we give an example to show the superiority of the second-price auction when the monopolist has weakened bargaining power, and in section 5.4, we show the implications of the Coase theorems in our ranking problem. Section 6 contains all the proofs.

2 The Common-Value Model

After laying out the model and assumptions in section 2.1, we derive the equilibrium revenue formulas for the first-price and second-price auctions in Sections 2.2 and 2.3. Equilibrium selection issue is discussed in Section 2.3.

2.1 Model and Assumptions

We consider the following pure common-value auction model. There are two risk neutral bidders in an auction for a single object. There is a common valuation for the object, and each bidder only receives partial information about the

[6] A different selection of equilibrium has been adopted by Parreiras (2006) in an environment with affiliated signals. Mares (2006) provides another equilibrium selection that maximizes the revenue for the seller among all equilibria.

common value. Let $s_i, i = 1, 2$ be the private signal received by bidder i. We assume that s_1, s_2 are independently distributed with cumulative distribution function $F_i(s_i)$ and support $[0, a_i]$ for signal s_i. We assume that $F_i(s_i)$ is strictly increasing and continuously differentiable[7] with the density function $f_i > 0$ everywhere. The common value is given by $V = w(s_1, s_2)$. Assume that w is strictly increasing in each s_i and continuously differentiable on the two regions $H_1 = \{(s_1, s_2) : s_1 \leq s_2\}, H_2 = \{(s_1, s_2) : s_1 \geq s_2\}$, while allowing kinks on the diagonal $s_1 = s_2$. This includes two important cases $w = \max\{s_1, s_2\}$ and $w = \min\{s_1, s_2\}$.

We now relabel the signals by $t_i = F_i(s_i)$. Let $v_i(t_i) = F_i^{-1}(t_i)$. The common-value function can be written as $V = w(v_1(t_1), v_2(t_2))$. Signal t_i is uniformly distributed over $[0, 1]$. Note that v_i is also strictly increasing and continuously differentiable. We have $v_1(0) = v_2(0) = 0$, and $v_1(1) = a_1, v_2(1) = a_2$, and we let $a = \max(a_1, a_2)$. The range of the function V is $[0, w(a_1, a_2)]$. In some of our discussions in this paper, we will consider a weak-strong pair of bidders in the sense that bidder 2 is a stronger bidder than bidder 1 if $v_1(t) \leq v_2(t)$ for all t.[8]

The common-value function w is symmetric if $w(s_1, s_2) = w(s_2, s_1)$ for all s_1 and s_2. The symmetry means that the common valuation does not depend on who receives which signal as long as the collection of individual beliefs are the same. In certain cases such as the case of independent signals, there may be a universal way of updating the information. No personal element is involved in the updating and re-valuation. The valuation depends on the collection of the signals alone, and differences in valuation are only due to the differences in the information received. In this situation, we have symmetry. However, in later applications to the auctions with resale, the common value defined need not be symmetric. Therefore we will not assume symmetry in the following presentation. In many places, symmetry does make the discussion easier to understand. Another useful property we make is

$$w(s, s) = s \text{ for all } s. \tag{1}$$

This property is always satisfied when we apply our results to the resale case.

Function F_i is called regular if the following virtual value function is strictly increasing in s:

$$s - \frac{1 - F_i(s)}{f_i(s)},$$

which implies that for any $y \in (0, a_i)$, the following conditional virtual value is strictly increasing in s:

$$s - \frac{F_i(y) - F_i(s)}{f_i(s)}.$$

[7] Allowing the distributions F_i to have kinks would not invalidate the revenue formulas and the ranking results of the paper. We also allow F_i to have infinite derivatives at 0 (such as power functions) in some of our examples.

[8] Here we only require that F_2 is dominated by F_1 in the sense of the first order stochastic dominance. Note that this concept is weaker than that of Maskin and Riley (2000a), in which conditional stochastic dominance is imposed.

The regularity condition can also be stated in terms of $v_i(t)$. The virtual value is given by

$$J(t) = v_i(t) - (1 - t)v_i'(t).$$

Hence the regularity condition is simply the increasing property of $J(t)$. It is equivalent to the concavity of $(1 - t)v_i(t)$ since

$$-\frac{d^2}{dt^2}[(1 - t)v_i(t)] = \frac{d}{dt}[v(t) - (1 - t)v'(t)] = J'(t) > 0.$$

For any $\tau \in (0, 1)$, the conditional virtual value is given by

$$v_i(t) - (\tau - t)v_i'(t).$$

The common-value function $w(s_1, s_2)$ is submodular if, for all (s_1, s_2) and (s_1', s_2'), $s_1 \le s_1', s_2 \le s_2'$, the following holds

$$w(s_1, s_2) + w(s_1', s_2') \le w(s_1, s_2') + w(s_1', s_2). \tag{2}$$

Given an increasing and concave function ϕ, $w(s_1, s_2) = \phi(s_1 + s_2)$ is both symmetric and submodular. If the inequality in (2) is reversed, we say that the function is supermodular. The maximum function $w = \max\{s_1, s_2\}$ is submodular while the minimum function $w = \min\{s_1, s_2\}$ is supermodular.

One condition of w will be useful for our revenue ranking and can be stated as follows:

Condition (C): for all s_1, s_2, we have

$$w(s_1, s_2) \ge \frac{w(s_1, s_1) + w(s_2, s_2)}{2}. \tag{3}$$

Note that in (C), we do not necessarily impose symmetry. When w is symmetric, the submodular property implies (C). However, when w is not symmetric, condition (C) does not follow from submodularity. For example, $w(s_1, s_2) = \frac{2}{3}s_1 + \frac{1}{3}s_2$ is submodular but does not satisfy condition (C). When (1) holds, condition (C) can be written as

$$w(s_1, s_2) \ge \frac{s_1 + s_2}{2}. \tag{4}$$

It is often the case that condition (C) need not be satisfied for all pairs (s_1, s_2). For a weak-strong pair, the ranking result only requires condition (C) on H_1. Condition (C) cannot hold for all (s_1, s_2) when w is of the form $w(s_1, s_2) = rs_1 + (1 - r)s_2$. Condition (C) holds for all pairs when w is of the form $w(s_1, s_2) = \max\{rs_1 + (1 - r)s_2, (1 - r)s_1 + rs_2\}$, and in this case, we have a kink on the diagonal.

We also provide another condition on w along with one of the distribution functions. Let $w_i(s_i, s_j)$ be the partial derivative with respect to s_i. When (1) holds, define

$$H^{s_j}(s_i) = 2w_i(s_i, s_j) - \frac{1 - F_j(s_i)}{1 - F_j(s_j)}.$$

For our ranking result, it will be sufficient if the following single-crossing condition is satisfied:

Condition (R): For some j, and $i \neq j$, we have $H^{s_j}(s_i) > 0$ if $s_i > s_j$ and $H^{s_j}(s_i) < 0$ if $s_i < s_j$.

Note that when $s_i = s_j$, we have $H^{s_j}(s_i) = 0$. For the ranking results, it is often the case that half of the requirements are needed. For example, if it is weak-strong pair, we only need the condition for $s_i < s_j$. The opposite of condition (R) is the following:

Condition (S): For some j, and $i \neq j$, we have $H^{s_j}(s_i) < 0$ if $s_i > s_j$ and $H^{s_j}(s_i) > 0$ if $s_i < s_j$.

More generally (when (1) need not be true), given a bidder j's signal s_j, define the following function $H^{s_j}(s_i)$ as follows:

$$H^{s_j}(s_i) = \frac{2w_i(s_i, s_j)}{w_1(s_i, s_i) + w_2(s_i, s_i)} - \frac{1 - F_j(s_i)}{1 - F_j(s_j)}.$$

Conditions (R) and (S) with this general definition are sufficient conditions for the revenue ranking results later. As we shall explain in section 3.1, the two conditions imply that the difference of the revenues of the first-price and second-price auctions either increases or decreases as the asymmetry declines.

The following lemma clarifies the relationship between the submodular property and condition (R) and (C). When w is symmetric, condition (C) is an easy consequence of the submodular property. The following lemma says that condition (R) is also a consequence of the submodular property for symmetric w.

Lemma 1 Assume that w is symmetric. Then condition (R) is satisfied for all F_j when w is submodular.

Note that symmetry and supermodularity does not imply condition (S), as the following example shows.

Example A. Let $w(s_1, s_2) = (s_1 + s_2)^2$, and $F_2(s_2) = s_2$ be the uniform distribution on $[0, 1]$. The function w is symmetric and supermodular, but condition (S) fails for F_2. To see this, we have $w_1(s_1, s_2) = 2(s_1 + s_2)$, hence

$$\frac{w_1(s_1, s_2)}{w_1(s_1, s_1)} - \frac{1 - s_1}{1 - s_2} = \frac{s_1 + s_2}{2s_1} - \frac{1 - s_1}{1 - s_2}.$$

Take the partial derivative with respect to s_2 and evaluate at s_1, we have

$$\frac{1}{2s_1} - \frac{1}{1 - s_1} < 0,$$

when $s_1 > \frac{1}{3}$. Thus condition (S) fails near $(\frac{1}{3}, \frac{1}{3})$.

The function $w = \max\{s_1, s_2\}$ satisfies condition (R), while $w = \min\{s_1, s_2\}$ satisfies condition (S). It should be emphasized however that when w is not symmetric, conditions (C) and (R) tend to be different from the submodularity property.

We will use the above notations for a common-value model to express an asymmetric private-value model. This is useful for later applications to the model of asymmetric private-value auctions with resale. This representation is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004). In Section 4.2 of Milgrom (2004), he discusses two advantages: (i) it easily generates predictions about bid distributions for use in empirical work; (ii) it unifies analysis of models with discrete or continuous valuation distributions. We will add another point: with this representation, it is easier to make a simple connection between private auctions with resale and common-value auctions.

In this representation, a bidder is now described by a strictly increasing valuation function $v_i(t_i) : [0,1] \to R$, with the interpretation that $v_i(t_i)$ is the private valuation of bidder i. The word "private" refers to the important property that bidder i's valuation is not affected by the signal t_j of other bidders, while in the common-value model, this is not the case. The function F_i is now the distribution function of the private valuation of bidder i. It will be shown in Section 5.3 that if bidder j's valuation distribution is convex then the optimal (single) offer from bidder i to bidder j in the resale stage satisfies condition (C). Similarly, when F_j is regular then condition (R) is satisfied for F_j and the optimal (single) offer from bidder i to bidder j.

2.2 First-Price Auctions

The existence and uniqueness of the equilibrium in the first-price common-value auctions have been studied in the literature[9]. In this subsection, we derive the equilibrium bid and revenue using the distributional approach.

Let $b_i(t_i)$ be the strictly increasing bidding strategy of bidder i in the first-price auction, and $\phi_i(b)$ be its inverse. The following first order condition is satisfied by the equilibrium bidding strategy

$$\frac{d \ln \phi_i(b)}{db} = \frac{1}{w(v_1(\phi_1(b), v_2(\phi_2(b))) - b} \text{ for } i = 1, 2. \tag{5}$$

[9] The existence of a non-decreasing equilibrium in the common value model is established in Athey (2001). The existence of a strictly increasing equilibrium has been shown in Rodriguez (2000). The uniqueness of equilibrium of the first price auction of the common value model can be found in Lizzeri and Persico (1998) and Rodriguez (2000).

with the boundary conditions $\phi_i(0) = 0$. The ordinary differential equation system with the boundary conditions determine the equilibrium inverse functions.

In the pure common-value model, it is well-known that in equilibrium, the winning probabilities of the two bidders are the same when they bid the same amount.[10] The symmetric property of the winning probabilities is exactly the property that both bidders have identical bidding strategies (as functions of t). In other words, we have $b_1(t) = b_2(t)$. Note that there is asymmetry in the signals as v_1, v_2 are different, and bidding strategies as functions of v_i are not symmetric. However, bidding strategies in terms of t are symmetric.

When signals are independent, the symmetry property of the equilibrium bidding strategy gives us very simple formulas for the bidding strategy and the revenue. The following result for the equilibrium strategy in the first-price common-value auction has been established in the literature (for instance Parreiras (2006)). For the case of independent signals, we give a simple statement and proof based on the symmetry.[11]

Proposition 2 The equilibrium bidding strategy in the first-price common-value auction is symmetric and is given by

$$b(t) = \frac{1}{t}\int_0^t w(v_1(r), v_2(r))dr$$

with the revenue given by

$$R^F = 2\int_0^1 (1 - t)w(v_1(t), v_2(t))dt.$$

When the bidders form a weak-strong pair, we can applying Proposition 1 to two special cases. For the maximum function $w = \max\{s_1, s_2\}$, we have the revenue formula

$$R^F_{\max} = 2\int_0^1 (1 - t)v_2(t)dt.$$

For the minimum function $w = \min\{s_1, s_2\}$, we have the revenue formula

$$R^F_{\min} = 2\int_0^1 (1 - t)v_1(t)dt.$$

When w is separable, we have the following revenue formula for the first-price auctions. A discrete version of this result was given by Hörner and Jamison (2007, supplement).

[10] This can be found in Engelbrecht-Wiggans, Milgrom, and Weber (1983) for the Wilson track model and more generally in Parreiras (2006) and Quint (2007). This property also holds in first-price auctions with resale in Hafalir and Krishna (2007).

[11] We want to thank Jeremy Bulow for pointing out that the bidding formula can also be obtained from the theorem in Milgrom and Weber (1982) by using symmetric signals but asymmetric common value functions.

Corollary 3 If the common-value function w is $w(s_1, s_2) = \frac{s_1+s_2}{2}$, then the revenue of the first-price auction is

$$R^F = \frac{1}{2}\int_0^1 (1-t)^2 dv_1(t) + \frac{1}{2}\int_0^1 (1-t)^2 dv_2(t).$$

2.3 Second-Price Auctions

It is well-known that in the second-price pure common-value auction, there is a continuum of equilibria (see Milgrom (1981)). In fact, for any increasing function h, the following is an equilibrium in the second-price auction (see Milgrom (2004), Theorem 5.4.8).

$$B_1(s_1) = w(s_1, h^{-1}(s_1)), B_2(s_2) = w(h(s_2), s_2).$$

The equilibrium as a function of t can be expressed as

$$b_1(t_1) = w(v_1(t_1), h^{-1}(v_1(t_1))), b_2(t_2) = w(h(v_2(t_2)), v_2(t_2)).$$

When we rank the revenues of the first-price and second-price auctions, we need to specify which equilibrium in the second-price auction is selected for the comparison.

We select the equilibrium with $h(s) = s$, that is,

$$B_i(s_i) = w(s_i, s_i), i = 1, 2$$

or

$$b_i(t_i) = w(v_i(t_i), v_i(t_i)), i = 1, 2. \quad (6)$$

Note that the selected equilibrium as functions of signals s_i is symmetric across the two bidders. The revenue from the second price auction for the selected equilibrium can be derived as follows.

Proposition 4 The revenue of the selected second-price auction equilibrium (6) is

$$R^S = \int_0^a (1 - F_1(x))(1 - F_2(x))dw(x, x),$$

where $a = \max(a_1, a_2)$.

Note that there is an important property associated with the selected equilibrium and revenue in the second price auction. That is, the selected equilibrium and revenue depend on $w(s_1, s_2)$ only through the diagonal $s_1 = s_2$ and are not affected by the value of w off diagonal $s_1 \neq s_2$. In particular, suppose

$w(s,s) = s$, then the selected equilibrium bid is just $b_i(t_i) = v_i(t_i)$ and the revenue is given by

$$R^S = \int_0^a (1 - F_1(x))(1 - F_2(x))dx.$$

This is identical to the equilibrium revenue of the second price auction in an independent private-value model.

In addition to the purpose of applications of our results to auctions with resale, there is another justification for the selected equilibrium above. In practice, it is rare to have a pure common-value model. Instead, there might be a small private component in the valuation of the bidders. Assume that both bidders have the same small portion of the value derived from private-value considerations, while the major portion of the valuation is common. We show that in the limit the unique second-price auction equilibrium converges to the selected equilibrium above.

To formalize this idea, assume that a small part of v_1 is a private component, meaning that when bidder 1 knows t_2, the updated valuation is given by

$$\varepsilon v_1(t_1) + (1 - \varepsilon)w(v_1(t_1), v_2(t_2)).$$

Similarly, when bidder 2 updates the valuation, it is given by

$$\varepsilon v_2(t_2) + (1 - \varepsilon)w(v_1(t_1), v_2(t_2)).$$

We call this an almost common-value model. We have the following result on equilibrium refinement.

Proposition 5 In a model of the almost common value with a small (ε) private-value component, the equilibrium in the second-price auction is unique. As $\varepsilon \to 0$, the equilibrium converges to the selected equilibrium defined in (6).[12]

We now compare our equilibrium selection with that of Parreiras (2006).[13] His selection is $h(s) = v_1(v_2^{-1}(s))$, or

$$b(t) = w(v_1(t), (v_2(t)).$$

This equilibrium as a function of t is symmetric across two bidders, while our equilibrium as a function of s is symmetric across two bidders. The two selections are identical when bidders are symmetric.

[12] In this result, we use the same size ε for both bidders. If we allow $\varepsilon_1, \varepsilon_2$ to be different, the result remains true if the ration goes to 1. If the ratio does not go to one, we may get other equilibria in the limit. In this sense, the refinement concept has some limitations.

[13] By comparison, Parreiras (2006) selected an equilibrium based on a refinement concept through hybrid auctions. The second price auction equilibrium he selected is based on the limit of the hybrid auction when the weight on the first price is close to 0 (corresponding to the second price auction in the limit). It is a refinement idea through the perturbation in auction formats. Our refinement idea is through the perturbation in auction environments (the small private value components).

It can be shown that when the signals are independent, Parreiras (2006)'s selection has the same revenue as the …rst-price auction equilibrium.

Proposition 6 *The equilibrium selected by Parreiras (2006) in the second price auction is*

$$b(t) = w(v_1(t), v_2(t)),$$

yielding the revenue in the second price auction equal to that of the …rst-price auction.

In an a¢ liated common-value model, Parreiras (2006) has shown that his selected second-price auction equilibrium revenue-dominates the …rst-price auction equilibrium. The Parreiras (2006) result implies that the ranking result of Milgrom and Weber (1982) is extended to the case when bidders are asymmetric and that the e¤ect of a¢ liation still favors the second price auction over the …rst price auction. In this paper, we focus on the e¤ect of asymmetry on the ranking of the two auctions in absence of a¢ liation.

3 Revenue Ranking in Common-Value Auctions

From now, on we shall study the revenue ranking problem with the equilibrium selection described in the last section. We are interested in ranking the revenues from two commonly used auctions: …rst-price and second-price auctions.

We give a simple proof of the ranking result when w is symmetric, and separable (therefore also submodular and supermodular) in section 3.1. We also give an intuitive explanation of the conditions (C), (R) and (S) needed for our results[14]. In section 3.2, we present our main ranking results.

3.1 Intuition

Let R^F, R^S denote the revenue of the …rst-price and second-price auction respectively. It is useful to give a simple proof of the ranking result when the

[14] Hausch (1987) and Banerjee (2003) have a reverse ranking result in a common-value model with discrete signals which are independent conditional on the true value. The ranking result in Hausch (1987) holds under a restrictive information condition, without which the ranking may be di¤erent. The ranking result in Banerjee (2003) has a binary information structure. Both choose the same second-price auction equilibrium as ours for their ranking results. Their papers fall under the a¢ liated-signal model of Perreiras (2006), but Perrsiras selects a di¤erent second-price auction equilibrium for the ranking result.

common-value function is of the form $w(s_1, s_2) = \frac{s_1+s_2}{2}$. For simplicity, assume that the support of F_i is $[0,1]$. By Corollary 3, we have

$$\begin{aligned} R^F &= \frac{1}{2}\int_0^1 (1-t)^2 dv_1 + \frac{1}{2}\int_0^1 (1-t)^2 dv_2 \\ &= \frac{1}{2}\int_0^1 (1-F_1(x))^2 dx + \frac{1}{2}\int_0^1 (1-F_2(x))^2 dx \\ &> \int_0^1 (1-F_1(x))(1-F_2(x))dx = R^S, \end{aligned}$$

where the strict inequality holds as long as $F_1(x) \neq F_2(x)$ for a subset of $[0,1]$ with non-zero measure. Therefore, in this case the first-price auction generates higher revenue than the second-price auction. Note that the ranking result is a simple consequence of the revenue formulas and the inequality $A^2 + B^2 \geq 2AB$.

When w is symmetric, both conditions (C) and (R) are weaker than the submodular property. There is a useful intuition why the submodular property leads to the ranking result $R^F > R^S$. The revenue R^S utilizes the w function on the diagonal while R^F uses w off the diagonal. For the simple linear (and submodular) case, we have $R^F > R^S$. As w function becomes strictly submodular, its value off the diagonal tends to be relatively larger than the value on the diagonal. Therefore, $R^F > R^S$ continues to hold for submodular w.

When w satisfies $w(s,s) = s$ (this is always the case in the resale context), condition (C) says that the common-value is above the average of s_1, s_2. Assume that $s_1 < s_2$, and we have a weak-strong pair. We can think of the two common values $\max\{s_1,s_2\} = s_2, \min\{s_1,s_2\} = s_1$ as two extreme cases of $w(s_1,s_2) = (1-r)s_1 + rs_2$. When $r = 0$, it is $\min\{s_1,s_2\}$, and $r = 1$ corresponds to $\max\{s_1,s_2\}$. The ranking result for $\min\{s_1,s_2\}$ is opposite that of $\max\{s_1,s_2\}$. For the minimum case, the revenue of the first-price auction is

$$R^F_{\min} = 2\int_0^1 (1-t)v_1(t)dt = \int_0^1 (1-F_1(x))^2 dx.$$

It is as if the two bidders are symmetric with the valuation distribution F_1 so that the first-price auction revenue is equal to the second-price auction revenue. Clearly, we have

$$R^S = \int_0^1 (1-F_1(x))(1-F_2(x))dx > \int_0^1 (1-F_1(x))^2 dx = R^F_{\min}.$$

For the maximum case, we have the opposite result, as

$$R^S = \int_0^1 (1-F_1(x))(1-F_2(x))dx < \int_0^1 (1-F_2(x))^2 dx = R^F_{\max}$$

and we have

$$R^F_{\max} > R^S > R^F_{\min}.$$

It turns out that when $r \geq 0.5$, we have the ranking result $R^F > R^S$. Note that F^S is strictly increasing in r, and R^S is independent of r. Therefore at some $r^* < 0.5$, we have $R^F = R^S$. For $r < r^*$, we have $R^F < R^S$, and for $r > r^*$, we have $R^F > R^S$.

Condition (C) is particularly attractive because it requires no assumptions on the underlying distributions $F_i, i = 1, 2$. Therefore the ranking result applies to all specifications on the individual signals. However, when applied to the auctions with resale, the optimal pricing function need not satisfy this condition.

The proof for the ranking result using condition (C) is not too different from the arguments shown for the case $w(s_1, s_2) = \frac{s_1+s_2}{2}$. When w is not separable, we need condition (C) to complete the arguments. The proofs for the ranking result using condition (R) or (S) are quite different.

There is an important meaning for conditions (R) and (S). Consider the case when $w(s, s) = s$ is satisfied, and it is a weak-strong pair. The two conditions tell us whether R^F increase slower or faster than R^S as the distributions become more symmetric. Condition (R) for bidder $j = 2$ requires that

$$w_1(s_1, s_2) < \frac{1}{2}\frac{1 - F_2(s_1)}{1 - F_2(s_2)} \text{ when } s_1 < s_2.$$

Assume that we move bidder 1 toward bidder 2, so that $v_1(t)$ approaches $v_2(t)$ pointwise. Then

$$\int_0^1 2(1 - t) w_1(v_1(t), v_2(t)) dt$$

is the rate of increase of the first-price auction revenue R^F. We can rewrite

$$R^S = \int_0^1 (1 - t)(1 - F_2(v_1(t))) dt.$$

Using integration by parts, we have

$$R^S = \int_0^1 \left[\int_0^{v_1(t)} (1 - F_2(v)) dv \right] dt.$$

Hence

$$\int_0^1 (1 - F_2(v_1(t))) dt$$

is the rate of increase of the second-price auction revenue. Therefore $R^F - R^S$ decreases if

$$2(1 - t) w_1(v_1(t), v_2(t)) < (1 - F_2(v_1(t)))$$

or

$$2(1 - F_2(v_2(t)) w_1(v_1(t), v_2(t)) < (1 - F_2(v_1(t)))$$

which is exactly the condition (R). In the limit, the revenue equivalence applies, and therefore condition (R) insures that the difference decreases to 0. This means

$R^F > R^S$. Similarly, condition (S) implies that the difference increases to 0, and we have $R^F > R^S$.

One interesting case that should be mentioned is the Wilson (1968) drainage track model. In this model, one bidder observes the true value of the object, while the other bidder is uninformed or observes signals that are not informative, in the sense that the true value of the object only depends on the observed value of the informed bidder. In the Wilson drainage track model conditions (C) fails, and condition (S) applies. This gives us the ranking result of Milgrom and Weber (1982) as a special case.

It is useful to give some intuition as to why the symmetry property of the equilibrium bidding strategy in Proposition 2 has strong implications for revenue comparisons. In private-value auctions, it is well-known (see Maskin and Riley (2000a)) that the weak bidder contributes more revenue to the seller in the first-price auction than in the second-price auction. For the strong bidder, it is just the reverse. This reversion is the source of the ambiguity in ranking the first-price and second-price private-value auctions. When the strong bidder uses "low ball" strategies, the revenue of the second-price auction can be higher than that of the first-price auction. For common-value auctions, the symmetry in the bidding strategy means that the weak and strong bidders contribute the same revenue to the seller. In other words, our conditions combined with the symmetry property will make the low ball strategies less effective.

3.2 Main Ranking Results

The first result we offer is based on condition (C) of the common-value function w. When condition (C) holds, the ranking holds without detailed knowledge of the valuation distributions $F_i, i = 1, 2$.

Theorem 7 Suppose w satisfies condition (C), and $v_1(t)) \neq v_2(t)$ for a subset of $[0, 1]$ of non-zero measure. Then $R^F > R^S$. For a weak-strong pair, the results holds if condition (C) holds for $s_i \leq s_j$.

The common-value function $w(s_1, s_2) = \max\{s_1, s_2\}$ satisfies condition (C), and the ranking result always applies. When $w(s_1, s_2) = \min\{s_1, s_2\}$, the ranking is always reversed. Before we state this result, we want to note that the revenue equivalence holds when bidders are symmetric ($v_1(t) = v_2(t) = v(t)$ for all t). This is known in the literature, and can be proved easily by our revenue formulas. We have

$$R^F = \int_0^1 2(1 - t)w(v(t), v(t))dt = \int_0^1 (1 - t)^2 dw(v(t), v(t))$$
$$= \int_0^a (1 - F(x))^2 dw(x, x) = R^S.$$

We state this as a proposition.

Proposition 8 Assume that $v_1(t)) = v_2(t)$ for all t, then we have $R^F = R^S$.

In view of the importance of the maximum and minimum value functions, we have the following simple result which has been shown in the last section when we have a weak-strong pair.

Proposition 9 Assume that $v_1(t)) \neq v_2(t)$ for a subset of [0,1] of non-zero measure. (i) If $w(s_1, s_2) = \max\{s_1, s_2\}$, then $R^F > R^S$; (ii) If $w(s_1, s_2) = \min\{s_1, s_2\}$, then $R^F < R^S$.

Our second result is based on condition (R) or (S) which use properties of one of the valuation distributions.

Theorem 10 Assume that condition (R) holds for w and some bidder F_j, and $v_1(t) \neq v_2(t)$ with strict inequality for a subset of $[0, 1]$ of non-zero measure. Then $R^F > R^S$. Similarly, if condition (S) holds for some bidder j, we have $R^F < R^S$.

Remark: To apply the result, it is not necessary that condition (R) holds for all ranges of (s_i, s_j). Let O be the origin $(0,0)$, $D = (\min(a_1, a_2), \min(a_1, a_2))$, and $E = (v_1(1), v_2(1))$. Let H be the region bounded by the two line segments OD, DE and the curve

$$\{(v_i(t), v_j(t)) : 0 \leq t \leq 1\},$$

then it is sufficient that condition (R) holds in the interior of this region. The same applies to condition (S).

We will show later that condition (R) applies when the common-value function is derived from the resale market with regular valuation distributions. A typical example for which condition (S) applies is when

$$w(s_1 + s_2) = rs_1 + (1 - r)s_2, r > 0.5.$$

For instance, let $r = \frac{2}{3}$. Let $v_1(t) \leq v_2(t) = t$. We have $F_2(x) = x$. To apply condition (S), we need

$$\frac{2}{3} \geq \frac{1}{2}\frac{1 - v_1(t)}{1 - t}$$

or

$$v_1(t) \geq \frac{4}{3}t - \frac{1}{3}. \tag{7}$$

Thus when () holds, condition (S) applies, and we get the result $R^F < R^S$. The following example shows that condition (R) may fail for well-known supermodular functions, and the ranking is $R^F < R^S$.

Example B. Let $w(s_1, s_2) = (s_1 + s_2)^4$. This is a symmetric supermodular function. Let the two bidders be $v_1(t_1) = t_1^2, v_2(t_2) = t_2,$ for t_1, t_2 in $[0, 1]$. Condition (C) fails when $s_1 = 0, s_2 = 1$. We have $F_1(x) = \sqrt{x}, F_2(x) = x$. To check the validity of conditions (R), we have

$$\frac{w_1(x,y)}{w_1(x,x)} - \frac{1 - F_2(x)}{1 - F_2(y)} = \frac{1}{8}(1 + \frac{y}{x})^3 - \frac{1 - x}{1 - y} \tag{8}$$

Take the partial derivative of (8) with respect to x, and evaluate at $x = y$, we have

$$-\frac{3}{y} + \frac{1}{1 - y} < 0 \text{ if and only if } y < \frac{3}{4}.$$

hence condition (R) is violated around (y, y) if $y < \frac{3}{4}$. The revenue of the first-price auction is

$$R^F = 2 \int_0^1 (1 - t)(t + t^2)^4 dt = 0.60476$$

and the revenue of the second-price auction is

$$R^S = 64 \int_0^1 (1 - \sqrt{x})(1 - x)x^3 dx = 0.61414$$

We have $R^F < R^S$.

Condition (R) is in fact a necessary condition for the ranking $R^F > R^S$, if the auction is nearly symmetric. This is illustrated by the following example. In this example, the two distributions F_1, F_2 differ only in some small interval $[0, \delta]$. When s_i is in this interval, condition (R) is violated. The ranking is reversed.

Example C. The common-value is given by $w(s_1, s_2) = (\frac{\sqrt{s_1} + \sqrt{s_2}}{2})^2$. Let the two bidders be given by

$$\begin{aligned} v_1(t) &= 0.9t + t^2 \text{ for } t \leq 0.1 \\ &= t \text{ for } t \geq 0.1, \end{aligned}$$

and $v_2(t) = t$ for all t. The two bidders have the same valuation distribution above $t \geq 0.1$, but for $t \leq 0.1$, bidder two is slightly stronger. To find F_1, solve $x = 0.9t + t^2$, and we have

$$\begin{aligned} F_1(x) &= \frac{-0.9 + \sqrt{0.9^2 + 4x}}{2} \text{ for } x \leq 0.1 \\ &= x \text{ for } x \in [0.1, 1]. \end{aligned}$$

We have the following revenues

$$\begin{aligned} R^F &= 2\int_0^{0.1}(1-t)\left(\frac{\sqrt{t}+\sqrt{0.9t+t^2}}{2}\right)^2 dt + 2\int_{0.1}^{1}(1-t)t\,dt \\ &= 0.333\,173\,97, \end{aligned}$$

and

$$\begin{aligned} R^S = & \int_0^{0.1000}(1-x)\left(1-\frac{-0.9+\sqrt{0.9^2+4x}}{2}\right)dx \\ & + \int_{0.1000}^{1}(1-x)^2 dx = 0.333\,174\,83 > R^F. \end{aligned}$$

Note that in this example, we have the partial derivative $w_2 = \frac{1}{4}(1+\sqrt{\frac{s_1}{s_2}})$. Since w_2 is increasing in x, it is not submodular. We also have $w(s,s) = s$, and w does not satisfy condition (C). Next we want to show that w does not satisfy condition (R). For condition (R) to hold, it must be the case that for all $s_1 < s_2$,

$$w_2 = \frac{1}{4}\left(1+\sqrt{\frac{s_1}{s_2}}\right) > \frac{1}{2}\frac{1-F_2(s_2)}{1-F_2(s_1)} = \frac{1}{2}\frac{1-s_2}{1-s_1}. \tag{9}$$

We claim that (9) is false around some neighborhood of $(x,x), x < 0.2$. To see this, it is sufficient to show that the second partial derivative of the left-hand side of (9) is smaller, when we evaluate at $(x,x), x < 0.2$, i.e.

$$w_{22} = \frac{-1}{8x} < \frac{-1}{2(1-x)},$$

which is exactly the condition $x < 0.2$. We conclude that condition (R) is violated around the point $(x,x), x < 0.2$.

The idea in the above example can be generalized to the following necessary condition for the ranking result. It simply says that the function H^{s_j} in condition (R) has a non-negative derivative at (s_j, s_j) for the ranking $R^F > R^S$ to be true. Note that in condition (R), there is no restriction on the other bidder's distribution F_i. The necessary condition can be stated as a necessary condition for $R^F \geq R^S$ to hold for all F_i. More strongly, the necessary condition has to hold when this ranking holds for all F_i close to F_j.

Theorem 11 Fix F_j, w. Assume that w is symmetric and continuously differentiable up to the second order. If $R^F \geq R^S$ for all F_i, then we must have

$$w_{ii}(s,s) + \frac{1}{2}\frac{f_j(s)}{1-F_j(s)}\frac{dw(s,s)}{ds} - \frac{1}{2}\frac{d^2w(s,s)}{ds^2} \geq 0 \text{ for all } s \text{ in } [0, a_j].$$

When $w(s,s) = s$, the condition becomes

$$w_{ii}(s,s) + \frac{1}{2}\frac{f_j(s)}{1 - F_j(s)} \geq 0 \text{ for all } s \text{ in } [0, a_j]. \tag{10}$$

Similarly the necessary condition for $R^F \leq R^S$ for all F_i is that the inequality in (10) is reversed.

The necessary condition by itself is not sufficient for the ranking result. For example, the minimum function $w(s_1, s_2) = \min\{s_1, s_2\}$ satisfies the necessary condition, but the ranking is $R^S > R^F$. Note also that when w is linear and $w(s,s) = s$, the necessary condition has no bite.

4 Observational Equivalence

We give a description of the auctions with resale model and discuss the information assumptions in section 4.1. In section 4.2, we prove an equivalence theorem with a general description of the resale market in the language of mechanism design.

4.1 Auctions with Resale

The first-price auction with resale is a two-stage game. The bidders first participate in a standard sealed-bid first-price auction. In the second stage, either the winner or the loser of the auction may offer to sell or buy the object from the other bidder. The resale market may be in the form of a double auction in which simultaneous offers are made by both the buyer and the seller. At the end of the auction and before the resale stage, some information about the submitted bids may be available. The disclosed bid information in general changes the beliefs of the valuation of the other bidder. This may further change the outcome of the resale market. We shall adopt the simplest formulation in which no bid information is disclosed[15]. We call this the minimal information case. It should be noted that there is valuation updating even if there is no disclosure of bid information, as information about the identity of the winner alone leads to updating of the beliefs. We will consider only strictly monotone equilibrium in auctions with resale in this paper[16].

[15] Although the equivalence result may be established in a broader context with disclosure of different bid information, it is sufficient to restrict ourselves to the resale market with no disclosure of bid information in this paper. We shall deal with a more genereal formulation of the observational equivalence result in a later paper.

[16] Lebrun (2007) shows how the analysis can be carried out when there is full disclosure of bid information. He considered mixed strategy equilibrium. He showed that a mixed strategy equilibrium with full disclosure of all bids is observationally equivalent to an equilibrium with no disclosure of bid information.

If the winning bid is announced, while the lower bid is not (as is often the case in real-world auctions), and the winning bidder makes the offer in the resale stage, the bid information has no impact in the equilibrium behavior. If all bids are announced in between the auction stage and the resale stage, it can be shown that there is no strictly monotone equilibrium (For a proof of this, see Krishna (2002, Chapter 4). In this case, it will be necessary to consider mixed strategy equilibrium bidding strategies.

If the winner of the auction makes a take-it-or-leave-it offer to the loser, we call it the (single-offer) monopoly resale mechanism. If the loser of the auction makes a take-it-or-leave-it offer to the winner, we call it the (single-offer) monopsony resale mechanism. The offer-maker can be fixed before the auction, or contingent on winning or losing the auction. More generally, there can be simultaneous offers by both, or repeated offers with delay costs in a sequential bargaining model of resale.

In the second-price auction with resale, the game differs only in the first stage, in which the first-price auction is replaced by the second-price auction. In a second-price auction with resale, the winner of the auction knows the losing bid if the payment is made, as the losing bid is the price he pays in the auction. To conceal this information, the payment can be deferred after the resale game. There is in fact a continuum of equilibria (see Blume and Heidhues (2004)) in the second-price auction with resale. It is an equilibrium for both bidders to bid their valuation (see Proposition 2 in Hafalir and Krishna (2007)), and this is an efficient equilibrium. The efficiency means that there is no need for resale after the auction, so that the revenue is the same with or without resale. When there is no resale, the "bid-your-value" strategies constitute a weakly dominant equilibrium strategy. With resale, it is no longer weakly dominant. However it is robust in the sense of Borgers and McQuade (2007), and is the only robust equilibrium (see the supplement to Hafalir and Krishna (2007)). This is the equilibrium used in the revenue ranking of the auctions with resale, as well as common-value auctions. Since there is no resale transaction in the bilateral trade mechanisms, the second-price auction revenue does not depend on the different trade mechanisms in the second stage.

The auction with resale is not a common-value auction when there is incomplete information at the resale stage. Let $b_i(v_i)$ be the equilibrium bidding strategy of bidder i, and $\phi_i(b)$ its inverse function (mapping bids to valuations) in the first-price auction with resale. Let x_i be the valuation of the winner of the auction bidding b. Bidder i will make offers to sell to bidder j only if $x_j = \phi_j(b) > x_i$. Assume that this is the case, and bidder j has a regular valuation distribution F_j, then the optimal monopoly price $p(x_i, x_j)$ is the unique solution of the following equation in p determined by the first order condition in maximization:

$$p - \frac{F_j(x_j) - F_j(p)}{f_j(p)} = x_i. \tag{11}$$

We have $p(x, x) = x$, and $x_j > p(x_i, x_j) > x_i$ when $x_i < x_j$.

In the monopsony resale mechanisms after the auction, let x_i be the valuation of the loser of the auction bidding b. Bidder i will make offers to buy from bidder j only if $x_j = \phi_j(b) < x_i$. Assume that this is the case, and bidder j has a regular valuation distribution F_j. The optimal monopsony price r maximizes

$$(F_j(r) - F_j(x_j))(x_i - r),$$

with the first order condition given by

$$r - \frac{F_j(x_j) - F_j(r)}{f_j(r)} = x_i. \qquad (12)$$

Note that (12) is exactly the same as (11). We can in fact have a unified treatment if we think of bidder i as the offer-maker and bidder j as the offer-receiver. There is a unique solution to this equation when $x_j \le x_i$, and let $r(x_j, x_i)$ be the optimal offer satisfying (12). We can extend the definition to the region $x_j > x_i$, just as for the function p. We have $r(x, x) = x, x_j < r(x_j, x_i) < x_i$ when $x_j < x_i$.

For weak-strong pairs, the weak bidder always finds it desirable to make selling-offers to the strong bidder after winning the auction, but has no reason to make buying-offers after losing the auction. For the strong bidder, it is just the opposite. When it is not a weak-strong pair, a bidder may not want to make selling offers after winning the auction, but may want to make buying-offers after losing the auction. If we allow a bidder i to make offers whether he or she is a winner or not, we give the bargaining power to bidder i. If, on the other hand, we only allow the winner of the auction to make selling-offers (announcing the winning bid), we call this contingent bargaining power, as the bargaining power depends on winning the auction. Either kind of bargaining power arrangement will be allowed. We can also imagine a (commonly known) random process of assigning an offer-maker (deciding which bid to announce) with or without contingency on winning the object. For instance, Hafalir and Krishna (2007) consider a resale mechanism in which an independent exogenous random process determines who makes the offer: with probability q, the winner of the auction makes a take-it-or-leave-it offer to the loser, and with probability $1 - q$, the loser of the auction makes a take-it-or-leave-it offer to the winner.

4.2 An Equivalence Theorem

The idea that resale opportunities generate elements of common value in an auction is quite intuitive. In this section, we will show that for a general bilateral trade mechanism satisfying a sure-trade property, a first-price auction with this resale mechanism is observationally equivalent to a first-price common-value auction derived from the equilibrium in the auction with resale game. The auctions with resale is a two-stage game, while the common-value auction is a one-stage game. When we say that the two auctions are observationally equivalent, we mean that the equilibrium bidding strategy profile is the same for both auctions. The auctioneer cannot tell the difference between the auction with resale and the common-value auction from the bidding behavior, and the expected revenue from the two auctions are identical. The two auctions are obviously quite different, but when we compare the equilibrium bidding strategies in the first stage, there is no difference in the way the bidders behave.

The property needed for this result is a variation of the sure-trade property proposed in Hafalir and Krishna (2007). It says that if the difference in the seller's value x and the buyer's value y is the highest possible, then trade takes place with probability 1. The sure-trade property is defined through a direct mechanism corresponding to the resale process which is often described by indirect mechanisms such as bilateral bargaining.

The bidding strategy in the first stage affects the updating of belief in the second stage. Assume that the buyer j's belief about the valuation distribution $F_i(v_i)$ of the seller i has the support $[0, a_i]$. Let $F_i|_x$ be the conditional distribution of F_i over the support $[x, a_i]$. Let the seller's belief of the buyer's valuation distribution $F_j(v_j)$ have support $[0, a_j]$, and $F_j|_y$ is the conditional distribution of F_j over the support $[0, y]$. Let $b_i, b_j, \phi_i = b_i^{-1}, \phi_j = b_j^{-1}$ be bidding strategies and their inverse functions in the first stage satisfying $b_i(0) = 0 = b_j(0)$, and $b_i(a_i) = b_j(a_j) = b^*$. The different types of sellers and buyers are matched according their bid amounts. We can define the matching by $v_j = h(v_i) = \phi_j(b_i(v_i))$ so that bidder types are matched if they bid the same in the first stage. When bidder i with valuation v_i wins the auction, she updates her belief about bidder $j's$ valuations. The updated belief is described by $F_j|_{h(v_i)}$. Therefore different types of bidder one have different updated beliefs. Similarly, when bidder j loses the auction, his updated belief about bidder i is described by $F_i|_{h^{-1}(v_j)}$. Because of the difference in updated beliefs among different types of bidders, the resale game after the auction here differs from the standard bilateral bargaining model. In the standard bilateral bargaining, the beliefs of different types of players are the same. This will make the equilibrium behavior in the second stage resale game R different from the standard bargaining models. Assume that there is a Bayesian equilibrium e in the bilateral trade mechanism R after resale. We apply the revelation principle to define a direct trade mechanism M such that truthful-reporting is incentive compatible and individually rational and yields the same payoffs as the equilibrium payoffs in e for each types of the buyer and the seller in the resale game R. We shall assume that in the direct trade mechanism M, trade takes place with probability 1 or 0, given the reported valuations v_i, v_j[17]. The outcome of the resale game can then be described by a pricing function $p(v_i, v_j)$ and a closed subset Q so that $p(v_i, v_j)$ is defined in Q. The interpretation is that when the reported valuations are $(v_i, v_j) \in Q$, seller i sells the object to buyer j at the price $p(v_i, v_j)$. There is no trade when $(v_i, v_j) \notin Q$.

The sure-trade property can be defined through the indirect trade mechanism M as follows: We say that the resale game R satisfies the sure-trade property

[17] In Hafalir and Krishna (2007)'s formulation, a more general description is adopted in which trade may take place with a probability lower than one. However, trade occurs with probability one when the trade surplus is the maximum possible amount.

if $[v_i, h(v_i)]$ is an interior point of Q (relative to the set $[0, a_i] \times [0, a_j]$) for each $v_i > 0$. Note that when the reported pair is $[v_i, h(v_i)]$, the seller's valuation is the lowest possible according to the belief of the buyer, and the buyer's valuation is the highest possible according the belief of the seller. It represents the case of maximum possible trade surplus. The sure-trade property simply says that trade will take place (with probability 1) when the reported valuations indicate nearly the most desirable opportunity for trade.

To illustrate the relationship between the general bilateral trade R and the direct mechanism M, assume that R is the monopoly market in which the seller makes a take-it-or-leave-it offer. Assume that the seller with valuation v_1 has the belief that the buyer's valuation distribution is $F_2|_{h(v_1)}$, with $h(v_1) > v_1$, when $v_1 > 0$. Assume that there is a uniquely determined optimal offer (equilibrium) price $P(v_1)$ of the seller. In the associated direct trade mechanism M, the pricing function $p(v_1, v_2)$ is defined as follows: let $Q = \{(v_1, v_2) : v_2 \geq P(v_1)\}$, then for $(v_1, v_2) \in Q$, define

$$p(v_1, v_2) = P(v_1).$$

Hence trade occurs if and only if $v_2 \geq P(v_1)$, and the trading price is the optimal offer price. The sure-trade property must be satisfied in this case, as we know $P(v_1) < h(v_1), v_1 > 0$ and by continuity $(v_1, h(v_1)), v_1 > 0$ is an interior point of Q.

Similarly, in a monopsony resale mechanism with a take-it-or-leave-it offer by the buyer, the buyer chooses an optimal monopsony price higher than the lowest possible valuation of the seller. The offer is accepted when the seller has the lowest valuation, hence the sure-trade property also holds, and $p(h^{-1}(v_2), v_2)$ is the optimal monopsony price.

Now we show how the multiple-offer bargaining can be represented by the direct trade mechanism. Consider a bargaining model with two rounds of offers by the seller. The seller with valuation v_1 has the belief $F_2|_{h(v_1)}$ and makes an offer P_1 in the first period. This offer is either accepted or rejected, with the threshold of acceptance represented by Z, $i.e.$ a buyer accepts the first offer if and only if his or her valuation is above Z. If the first offer is accepted, the game ends. If it is not accepted, the seller makes a second offer P_2 which is a take-it-or-leave-it offer. An equilibrium analysis of this model is provided in section 5.4. Let $P_1(v_1), P_2(v_1), Z(v_1)$ denote the equilibrium first-period, second-period prices and threshold level in this bargaining problem. The equilibrium prices in the bargaining model can be used to define the pricing function in the associated direct trade mechanism. The direct trade mechanism is described as follows. Given the reported valuations (v_1, v_2), there is no trade if $v_2 < P_2(v_1)$. Trade occurs (with probability one) with the transaction price $p(v_1, v_2) = P_1(v_1)$ if $v_2 \geq Z(v_1)$, and the transaction price $p(v_i, v_j) = \delta P_2(v_2)$ if $P_2(v_1) \leq v_2 < Z(v_1)$. The set Q is

$$Q = \{(v_1, v_2) : v_2 \geq Z(v_1) \text{ or } P_2(v_1) \leq v_2 < Z(v_1)\}$$

The sure-trade property is satisfied because we must have $Z(v_1) < h(v_1)$, and we have $p(v_1, h(v_1)) = P_1(v_1)$. The sure-trade property holds in a monopoly resale mechanism with many rounds of offers from the seller, if the equilibrium first offer is lower than the highest valuation of the buyer. This is true if the monopolist has a strictly positive payoff in the equilibrium.

We now give a simple resale game with simultaneous offers to illustrate the intuition of the observational equivalence. The first stage is a first-price auction. The resale market is a double auction with simultaneous offers. In the double auction, transaction takes place if and only if $p_s \le p_b$, and the transaction price is given by

$$p = \frac{p_s + p_b}{2}.$$

Assume that $F_1(x) = x, F_2(x) = \frac{x}{2}$, so that $v_1(t) = t, v_2(t) = 2t$. Let the inverse bidding strategy in the first-price auction be ϕ_1, ϕ_2 and assume that $\phi_2(b) = 2\phi_1(b)$. To find an equilibrium with linear strategies in the double auction, let $p_s(v_1) = c_1v_1 + d_1, p_b(v_2) = c_2v_2 + d_2$. Bidder one with valuation v_1 chooses $p \le 2c_2v_1 + d_2$ to maximize

$$\frac{1}{2}\int_{\frac{p-d_2}{c_2}}^{2v_1} \left[\frac{p + c_2v_2 + d_2}{2} - v_1\right] dv_2$$

with the derivative with respect to p given by

$$\begin{aligned} & \frac{1}{2}(-\frac{p - v_1}{c_2} + \frac{1}{2}\int_{\frac{p-d_2}{c_2}}^{2v_1} dv_2) \\ = \; & \frac{1}{2c_2}(-\frac{3}{2}p + (1 + c_2)v_1 + \frac{1}{2}d_2) \end{aligned}$$

which is decreasing in p. Therefore the payoff function is concave. The first-order condition is given by

$$p_s = \frac{2}{3}(1 + c_2)v_1 + \frac{1}{3}d_2.$$

For the bidder two with valuation v_2, the price offer $p \ge \frac{v_2}{2}c_1 + d_1$ maximizes

$$\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} \left[v_2 - \frac{c_1v_1 + d_1 + p}{2}\right] dv_1.$$

The first-order condition for the optimal offer is

$$\frac{v_2 - p}{c_1} - \frac{1}{2}\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} dv_1 = 0$$

or

$$v_2 - p - \frac{c_1}{2}(\frac{p - d_1}{c_1} - \frac{v_2}{2}) = 0,$$

and we have the optimal offer of the buyer

$$p_b = \frac{4 + c_1}{6} v_2 + \frac{1}{3} d_1.$$

To be an equilibrium, we must have

$$d_1 = \frac{1}{3} d_2, d_2 = \frac{1}{3} d_1$$

$$c_1 = \frac{2}{3}(1 + c_2), c_2 = \frac{4 + c_1}{6}$$

To solve the equations, we have

$$d_1 = d_2 = 0.$$

We also have

$$c_1 = \frac{5}{4}, c_2 = \frac{7}{8}.$$

The linear equilibrium in the resale game is then given by

$$p_s(v_1) = \frac{5}{4} v_1, p_b(v_2) = \frac{7}{8} v_2.$$

The direct mechanism corresponding to this resale game has the pricing function

$$p(v_1, v_2) = \frac{1}{2}(\frac{5}{4} v_1 + \frac{7}{8} v_2) = \frac{5}{8} v_1 + \frac{7}{16} v_2$$

when $v_2 \geq \frac{8}{7}\frac{5}{4} v_1 = \frac{10}{7} v_1$. Here $Q = \{(v_1, v_2) : v_2 \geq \frac{10}{7} v_1\}$. Trade occurs with probability one if and only if $(v_1, v_2) \in Q$, and there is no trade outside Q.

We can now define the common-value function corresponding to the resale game as follows. For $(s_1, s_2) \in Q$, we have

$$w(s_1, s_2) = \frac{5}{8} s_1 + \frac{7}{16} s_2$$

and outside Q, we define

$$w(s_1, s_2) = \min\{s_1, s_2\}.$$

Now consider the determination of the equilibrium bidding strategy in the first stage of the IPV auction with resale. Let ϕ_1, ϕ_2 be the inverse bidding functions.

When bidder one with valuation v_1 offers the bid b, the payoff is

$$\frac{1}{2} \int_{\frac{10}{7} v_1}^{\phi_2(b)} p(v_1, v_2) dv_2 + \frac{1}{2} \int_0^{\frac{10}{7} v_1} v_1 dv_2 - \frac{1}{2} \phi_2(b) b$$

When bidder two with valuation v_2 offers the bid $b,$ the payoff is

$$
\begin{aligned}
&(v_2 - b)\phi_1(b) + \int_{\phi_1(b)}^{\frac{7}{10}v_2} (v_2 - p(v_1, v_2))dv_1 \\
&= \frac{7}{10}v_2^2 - \int_{\phi_1(b)}^{\frac{7}{10}v_2} p(v_1, v_2)dv_1 - b\phi_1(b)
\end{aligned}
$$

In the common-value model, let φ_1, φ_2 be the inverse bidding functions. When bidder one with signal s_1 bids $b,$ the payoff is

$$
\frac{1}{2}\int_{\frac{10}{7}s_1}^{\varphi_2(b)} w(s_1, s_2)ds_2 + \frac{1}{2}\int_0^{\frac{10}{7}s_1} \min(s_1, s_2)ds_2 - \frac{1}{2}\varphi_2(b)b
$$

When bidder two with signal s_2 bid $b,$ the payoff is

$$
\begin{aligned}
&\int_0^{\varphi_1(b)} (w(s_1, s_2) - b)ds_1 = \int_0^{\varphi_1(b)} w(s_1, s_2)ds_1 - b\varphi_1(b) \\
&= \int_0^{\frac{7}{10}v_2} w(s_1, s_2)ds_1 - \int_{\varphi_1(b)}^{\frac{7}{10}v_2} w(s_1, s_2)ds_1 - b\varphi_1(b).
\end{aligned}
$$

The difference between the payoff functions in the two different auctions is a constant term which is independent of $b.$ Therefore, the optimal bidding strategy in the two auctions must be the same for each $v_1 = s_1$ and each $v_2 = s_2$.

The equilibrium bidding strategy according to section 2.2 is

$$
b_i(t) = \frac{1}{t}\int_0^t w(r, 2r)dr = \frac{1}{t}\int_0^t 1.5rdr = \frac{3}{4}t, i = 1, 2.
$$

We have

$$
b_1(v_1) = \frac{3}{4}v_1, b_2(v_2) = \frac{3}{8}v_2,
$$

and

$$
\phi_1(b) = \varphi_1(b) = \frac{4}{3}b, \phi_2(b) = \varphi_2(b) = \frac{8}{3}b.
$$

.

To state the equivalence result, we need to define a common-value model with a common-value function $w(s_1, s_2)$ defined by the resale game after the auction. The common-value function we define is also determined by the equilibrium bidding strategy of the auctions with resale model. Let the strictly monotone equilibrium bidding functions of the bidders be $b_i(v_i), i = 1, 2$. Let $h(v_i) = b_j^{-1}(b_i(v_i))$. When bidder i with valuation v_i wins the auction, she believes that bidder j valuation is $F_j|_{h(v_i)}$. She also knows that bidder j with valuation v_j believes that bidder i valuation distribution is $F_i|_{h^{-1}(v_j)}$. In the meantime, bidder j with valuation v_j also knows that bidder i with valuation v_i has the belief described by $F_j|_{h(v_i)}$. Define the inverse bidding function $\phi_i(b) = b_i^{-1}(b), i = 1, 2$.

Given the private valuations F_i, F_j in the IPV model with resale, and the equilibrium inverse bidding strategies ϕ_i, ϕ_j, we define a common-value model with the signal distributions $F_i(s_i)$ and the common-value function

$$w(s_1, s_2) = p(s_2, s_1) \text{ for } (s_1, s_2) \in Q.$$

where the function p is the pricing function of the resale game after the auction. For (s_1, s_2) outside Q, the definition of the common-value is somewhat arbitrary, and for convenience we adopt the definition $w(s_1, s_2) = \min(s_1, s_2)$ for $(s_1, s_2) \notin Q$[18].

We now state the observational equivalence result.

Theorem 12 Let there be an IPV first-price auction with resale and two bidders. The resale is described by a general resale mechanism R. Assume that there is no disclosure of bid information in between the auction stage and the resale stage, and the resale mechanism satisfies the sure-trade property. Assume that there is a strictly monotone equilibrium bidding strategy profile $b_i(t) = b_j(t)$ in the auction with resale. Then there is common-value first-price auction with the same signal distributions and a common-value function defined by the pricing function of the resale game R whenever trade occurs, such that $b_i(t) = b_j(t)$ is also an equilibrium of the common-value auction, and we have observational equivalence between the IPV auction with resale and the common-value auction.

We now give an example of a direct trade mechanism with bilateral uncertainty that does not satisfy the assumptions and the sure-trade property. In the example above, we know that we have an incentive compatible and individually rational trade mechanism in which trade takes place if and only of $v_2 \geq \frac{10}{7} v_1$, and the transaction price is

$$p(v_1, v_2) = \frac{5}{8} v_1 + \frac{7}{16} v_2.$$

If we redefine the trade mechanism as follows: trade takes place with probability 0.5 if and only if $v_2 \geq \frac{10}{7} v_1$. Otherwise, there is no trade. The trading price $p(v_1, v_2)$ is unchanged. Then all the incentive and participation constraints are satisfied. The new trade mechanism does not satisfy the sure-trade property.

In the standard bargaining model, incentive efficiency implies the sure-trade property as is shown in the following proposition.

[18] With this definition, the common-value function can become discontinuous on the boundary of Q. This can be fixed by allowing such functions in the common-value model without affecting our revenue results in section 3. Alternatively, we can extend the definition on Q coninuously or differentiably without affecting the optimality of equilibrium..

Proposition 13 If (i) there is a positive probability of gains from trade, (ii) the trade mechanism is incentive efficient, and (iii) the valuation distributions are regular, then the sure-trade property is satisfied.

The sure-trade property is much weaker than incentive efficiency, and incentive efficiency is not a necessary condition for the property to hold. For instance, the monopoly market is not incentive efficient, but satisfies the sure-trade property. In fact, the property should hold for any sequential bilateral bargaining equilibrium with one-sided asymmetric information in which the uninformed party makes offers under rather general conditions. One may ask to what extent any incentive compatible and individually rational direct mechanism can be implemented by such sequential offers. This question has been studied in Ausubel and Denechere (1989b, 1993) in standard bargaining models.

5 Applications to Auctions with Resale

In section 5.1, we give the intuition of the ranking results in auctions with resale, under the assumption of complete information in the resale stage. In section 5.2, we give the ranking results when the offer-maker can commit to one single offer in the resale market. In section 5.3, we analyze the relationship between bargaining power, delay costs, and the ranking property for the case of a two-offer model. We give an example to show that when the auction-winner has little bargaining power (due to high cost of delay), the second-price auction is superior. Section 5.4 deals with the implication of the Coase Theorems which have to do with weakened bargaining power due to the lack of commitment in sequential offers.

In this section, we assume that the valuations are private, so that $F_i(v_i)$ is the c.d.f. of the private valuation of bidder i.

5.1 Complete information in the resale stage

To understand the effects of resale on the ranking of revenues of the first-price and second–price auctions, we shall first assume away the issue of incomplete information in the resale stage. We do this by assuming that after the auction in the first stage, the private valuation is fully disclosed[19] so that the realized valuations of both bidders are common knowledge. Although this is not a realistic assumption, the insight we gain from this case is very useful. This is also the case considered in Gupta and Lebrun (1999).

Suppose that we have a weak-strong pair. First we assume that the weak bidder (one) makes offers in the resale stage after the auction. In other words,

[19] There is an importance difference between the full disclosure of bids and full disclosure of private information. In the former case, there is no strictly monotone equilibrium, but in the second case there is.

we give bargaining power to the weak bidder. This auction is observationally equivalent to a common-value auction in which the common-value is given by

$$w(x, y) = \max\{x, y\}.$$

Our ranking result for the common-value auction then says that the first-price auction is superior. If the strong bidder makes offers in the resale stage, then the auction is observationally equivalent to a common-value auction with

$$w(x, y) = \min\{x, y\},$$

and the second-price auction is superior. Thus the ranking of the two auction formats depends on who has the bargaining power. This insight is essentially true with incomplete information in the resale stage as well.

Now assume that the two bidders are not necessarily a weak-strong pair. To make the discussions simpler, assume that both has the same support in valuations. Bidder one is weaker in the region $t \in [0, c]$, and stronger in the region $t \in [c, 1]$. Equivalently, bidder one is weaker in the valuation interval $[0, v_1(c) = v_2(c)]$ and stronger in the valuation interval $[v_1(c) = v_2(c), v_1(1) = v_2(1)]$. If we assume that the winner of the auction makes offers (the monopoly market in Hafalir and Krishna (2007)), then the auction is observationally equivalent to the common-value auction $\max\{x, y\}$, and the first-price auction is superior. If we assume that the loser of the auction makes offers (monopsony market), then it is observationally equivalent to the common-value auction $\min\{x, y\}$, and the second-price auction is superior. Thus the ranking of the two auction formats depends on whether it is a monopoly or monopsony resale market; or equivalently it depends on whether the winner or the loser of the auction has bargaining power.

If we always let bidder one make offers whether he or she is the winner or the loser of the auction, then the ranking depends on how likely bidder one is the winner of the auction. If c is very close to 1, then the first-price auction is superior. If c is very close to 0, then the second price auction is superior. In other words, if it is known who has bargaining power independent of who is the winner of the auction, then the ranking is ambiguous.

When there is incomplete information, the main outlines of the results are the same. However, when there is a single offer in the resale market (i.e. the offer-maker has the commitment power) and the valuation distributions are regular, then we have a simpler picture. The first-price auction is always superior whether the offer-maker is fixed or is contingent on the auction outcome. The main reason is that pricing function derived from the monopoly or monopsony resale market always satisfies condition (R) when the offer-receiver has a regular valuation distribution. One way to interpret this is that the bargaining power always resides with the weak bidder whoever makes the offer, when regularity holds for both distributions.

If the distribution functions are not regular, then the bargaining power may shift depending on who makes the o¤er as in the case of the complete information case above. This will be explored in the following sections.

Bargaining power is a¤ected by (i) who makes o¤ers, (ii) di¤erence in delay costs, (iii) ability or inability to commit to o¤ers. Section 5.2 deals with the case of full commitment. Section 5.3 is concerned with delay costs. Section 5.4 explores the consequence of the total lack of commitment.

5.2 Bargaining Power and Commitment

In this section, we assume that the resale market is either a monopoly market or monopsony market. The assumption of full commitment means that the o¤er is a take-it-or-leave-it o¤er. There are no more o¤ers even if the o¤er is rejected. This is the case considered in Hafalir and Krishna (2007).

In the resale context, condition (R) can be interpreted as a condition on the monopoly pricing behavior when the resale mechanism is a monopoly market. In the single-period monopoly-pricing problem, essentially we have provided an upper bound on how monopoly price varies with marginal cost. Assume that a monopolist with marginal cost c faces a demand curve $D(p)$. Suppose $p + \frac{k+D(p)}{D'(p)}$ is increasing in p for a parameter $k > 0$. Then

$$\frac{dp^*}{dc} \le \frac{1}{2} + \frac{D(p^*)}{2k} \le \frac{1}{2} + \frac{D(c)}{2k}.$$

This is essentially our condition (R) in the case of monopoly pricing. In our model, we let $k = 1 - F_j(x_j)$, and $D(p) = F_j(x_j) - F_j(p)$. The assumption on demand is the regularity condition.

Let bidder one be the weak bidder and bidder two the strong bidder. If we change the o¤er-maker from bidder one to bidder two, Lebrun (2007) has shown that R^F in fact becomes smaller. The reason for this is that bidder one faces a strong buyer. If bidder one makes o¤ers, we expect her to have higher bargaining power which is further strengthened by the higher valuation of bidder two. If bidder two makes o¤ers, we expect bidder one to have lower bargaining power which is further weakened by the lower valuation of bidder one. The lower bargaining power depresses the optimal o¤er price and therefore lowers R^F.

To see the consequence of bargaining power on the ranking result, consider the pricing function $p(x, y)$ derived from the resale market. This function is the optimal monopoly price when the seller has valuation x and believes that the buyer has the maximum valuation y. Let π denotes an index of bargaining power of the weak bidder. A higher bargaining power of the weak bidder can be represented by $p(x, y; \pi)$ which is increasing in π. The revenue formula for R^F implies that it is increasing in π, while R^S is independent of π. Hence R^F is an increasing function of the bargaining power of the weak bidder. When we say that the weak bidder has no bargaining power, it is represented by $p(x, y; \pi)$ being very close to $\min(x, y)$; while full bargaining power is represented by $p(x, y; \pi)$ being very close to $\max(x, y)$. In the former case, we have the ranking $R^F < R^S$, and in the second case $R^F > R^S$.

Before we apply the ranking results for common-value auctions to auctions with resale, it is useful to have some characterization of the pricing function arising from resale. We do not have a sharp characterization yet. We do have some useful properties. We say that p is quasi-convex (quasi-concave) if the level curves are concave (convex) to the origin.

In discussing the monopoly or monopsony markets, we shall use the notation i for the bidder who makes offers, and j for the bidder who accepts or rejects the offers. For a weak-strong pair, i can either be the weak bidder or the strong bidder. For our analysis , it does not matter who is the strong or weak bidder, but it does matter who makes offers. Let $p_i(x_i, x_j)$ be the partial derivative of the pricing function with respect to the valuation of offer-making bidder i. Bidder i could be either the winning bidder who makes a monopoly offer or a losing bidder who makes a monopsony offer.

Lemma 14 If the pricing function $p(x, y)$ is derived from a (single-offer) monopoly or monopsony resale market, then

$$p(x, x) = x, p_1(x, x) = p_2(x, x) = \frac{1}{2}. \tag{13}$$

where p_i is the partial derivative with respect to variable i. Furthermore, p is quasi-convex (quasi-concave) if and only if the underlying valuation distribution function is convex (concave).

We need to know whether conditions (C) or (R) are satisfied for the pricing function. The following lemma says that if bidder i makes a take-it-or-leave-it offer to bidder j in the resale market, then the optimal offer price satisfies condition (C) if bidder j has convex valuation distributions. The optimal offer price is the optimal monopoly (monopsony) price when bidder i wins (loses) the object in the auction.

Lemma 15 If the offer-receiver has a convex valuation distribution F_j, then the optimal offer price function satisfies condition (C).

Condition (C) does not necessarily hold when valuation distributions are regular. However, condition (R) holds for F_j and the pricing function when F_j is regular. This is our next lemma. Regularity is somewhat weaker than convexity.

Lemma 16 *If the offer-receiver has a regular valuation distribution F_j, then the pricing function and F_j satisfies condition (R).*

We now state a general ranking result for auctions with resale. Unlike the weak-strong pairs of Hafalir and Krishna (2007), we prove the result more generally.

Theorem 17 *Assume that $v_i(t) \neq v_j(t)$ with strict inequality for a subset of $[0, 1]$ of non-zero measure. We have $R^F > R^S$ if one of the bidder is chosen to make offers in the resale market, and the other bidder has a regular valuation distribution. The choice of the offer-making bidder is fixed before the auction, or randomly determined independently of whether the bidder wins the auction or not.*

Another approach is to give contingent bargaining power to bidders, such as allowing a bidder to make offers only when he or she wins the auction. The ranking result holds if both bidders have regular valuation distributions.

Corollary 18 *Assume that $v_i(t) \neq v_j(t)$ with strict inequality for a subset of $[0, 1]$ of non-zero measure. We have $R^F > R^S$ if both bidders have regular valuation distributions, and a bidder only makes offers contingent on winning (or losing) the object.*

We have the following necessary condition for the revenue ranking result in auctions with resale. It is a consequence of the necessary condition for the ranking result in common-value auctions.

Theorem 19 *Fix the valuation distribution of the offer-receiver F_j. In the resale game, bidder i makes a single offer to bidder j. If $R^F \geq R^S$ for all F_i, then the following condition holds for F_j:*

$$4 + \frac{(1 - F_j(x)) f_j'(x)}{f_j^2(x)} \geq 0.$$

Now we give an example of the reversal of revenue ranking when the distribution function of the offer-receiver is not regular.

Example D. There is a weak-strong pair, and the resale market is the monopoly market. Let the valuation distribution of the strong bidder be $F_s(x) =$

$x^{\frac{1}{2}}$ with the support $[0, 1]$. For $n > 2$, let the weak bidder be defined by[20]

$$\begin{aligned} F_w(x) &= 0.02^{\frac{1}{2} - \frac{1}{n}} x^{\frac{1}{n}}, x \le 0.02; \\ &= x^{0.5}, 0.02 \le x \le 1. \end{aligned}$$

We have $v_s(t) = t^2$, and

$$\begin{aligned} v_w(t) &= 0.02^{1 - \frac{n}{2}} t^n, t \le 0.02^{0.5}; \\ &= t^2, 0.02^{0.5} \le t \le 1. \end{aligned}$$

The resale market is a monopoly. The virtual value of F_s is

$$J(x) = x - \frac{1 - x^{0.5}}{0.5 x^{-0.5}} = 3x - 2x^{0.5},$$

which is not increasing in x as

$$J'(x) = 3 - x^{-0.5} < -4 \text{ when } x < 0.02.$$

Therefore the regularity condition is not satisfied. However we shall see that the optimal monopoly price is uniquely determined. Given $v_w = x$, and the maximum valuation $v_s = y > v_w$ of the strong bidder, the optimal resale price maximizes

$$R(p) = (F_s(y) - F_s(p))p + F_s(p)x = y^{0.5}p - p^{1.5} + p^{0.5}x.$$

The objective function is strictly concave in p. Hence there is a unique optimal price given by the solution of the first order condition

$$y^{0.5} - 1.5p^{0.5} + 0.5p^{-0.5}x = 0.$$

The unique solution is given by

$$p(x, y) = \left(\frac{\sqrt{y} + \sqrt{y + 3x}}{3}\right)^2.$$

This is a supermodular function. We have the first-price auction revenue

$$\begin{aligned} R^F &= 2 \int_0^1 (1 - t) p(v_w(t), v_s(t)) dt \\ &= 2 \int_0^{\sqrt{0.02}} (1 - t) p(0.02^{1 - \frac{n}{2}} t^n, t^2) dt + 2 \int_{\sqrt{0.02}}^1 (1 - t) t^2 dt. \end{aligned}$$

[20] Although the density function of F_i has infinite derivative at 0, and there is a kink in F_w at $x = 0.02$, the example can be slightly modified to produce an example satisfying all the smooth conditions we assume for F_w, F_s. and the ranking is still reversed.

When $n = 4$, we have

$$\begin{aligned} R^F &= 2\int_0^{\sqrt{0.02}} (1 - t)\left(\frac{t + \sqrt{t^2 + 150t^4}}{3}\right)^2 dt + 2\int_{\sqrt{0.02}}^1 (1 - t)t^2 dt \\ &= 0.166\,305\,4, \end{aligned}$$

and

$$\begin{aligned} R^S &= \int_0^1 (1 - F_w(x))(1 - F_s(x))dx \\ &= \int_0^{0.02} (1 - 0.02^{0.25}x^{0.25})(1 - x^{0.5})dx + \int_{0.02}^1 (1 - x^{0.5})^2 dx \\ &= 0.166\,318\,11 > R^F, \end{aligned}$$

hence the ranking is reversed. When $n = 6$, we have

$$\begin{aligned} R^F &= 2\int_0^{\sqrt{0.02}} (1 - t)\left(\frac{t + \sqrt{t^2 + 3(2500t^6)}}{3}\right)^2 dt + 2\int_{\sqrt{0.02}}^1 (1 - t)t^2 dt \\ &= 0.166\,143\,44, \end{aligned}$$

and

$$\begin{aligned} R^S &= \int_0^{0.02} (1 - 0.02^{\frac{1}{3}}x^{\frac{1}{6}})(1 - x^{0.5})dx + \int_{0.02}^1 (1 - x^{0.5})^2 dx \\ &= 0.166\,167\,92 > R^F. \end{aligned}$$

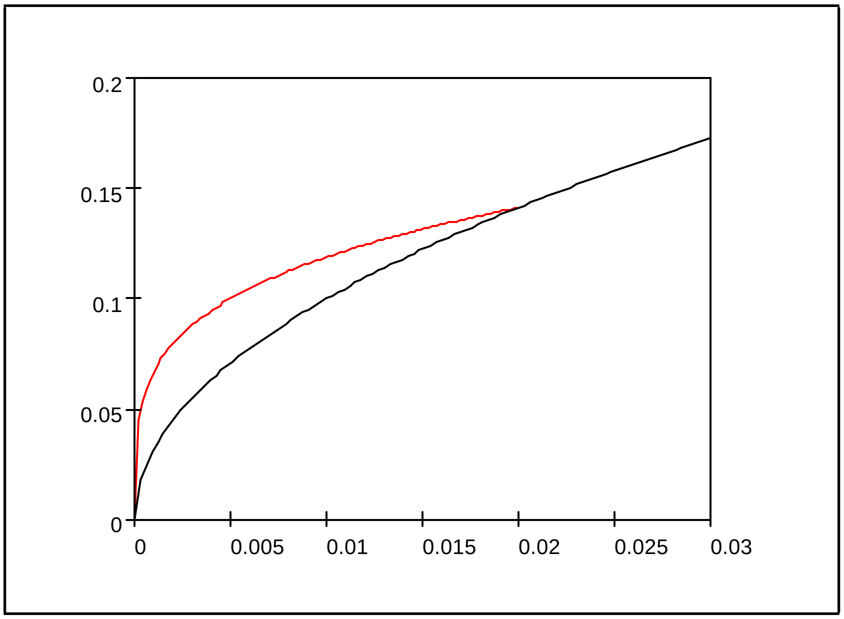
The revenue ranking is reversed with an even greater difference. In the limit, the difference is the greatest, with

$$R^F = 2\int_0^{\sqrt{0.02}} (1 - t)\frac{4}{9}t^2 dt + 2\int_{\sqrt{0.02}}^1 (1 - t)t^2 dt = 0.165\,73,$$

and

$$\begin{aligned} R^S &= \int_0^{0.02} (1 - 0.02)(1 - x^{0.5})dx + \int_{0.02}^1 (1 - x^{0.5})^2 dx \\ &= 0.167\,99 > R^F. \end{aligned}$$

Graph for Example D: The upper curve refers to the graph of $F_1(x)$, and the lower curve refers to the graph of $F_2(x)$. The two curves coincide with each other when $x \geq$ 0.02.



5.3 Bargaining Power and Delay Costs

When there is only one offer (which is equivalent to a commitment equilibrium in the bargaining literature) in the resale mechanism, the regularity assumption insures that the bidders derive sufficient benefits from resale so that the general ranking is possible. If we allow repeated offers with no commitment, it is well-known (Sobel and Takahashi (1983), Fudenberg and Tirole (1983)) that high delay costs weaken the bargaining power of the monopolist. The weakened bargaining power may lead to low trade prices when the auction winner makes offers to the loser. We show by an example that the opposite ranking can occur when the bargaining power is substantially reduced in bargaining with repeated offers.

The bargaining problem with repeated offers from one-side to the other with delay costs is similar to that of Sobel and Takahashi (1983). However, there is a main difference: the seller may have different no-zero costs (or valuations) and different types of the seller have different beliefs about the buyer's valuations. The delay costs are expressed by discount factors δ_1, δ_2 for bidder one, two respectively. Our example assumes that bidder one has low δ_1 (close to 0), and bidder two has high δ_2 (close to 1).

Consider the weak-strong pair of bidders $v_1(t) = t, v_2(t) = 1.5t$ over $[0, 1]$. There are only two rounds of offers. For the example, we adopt the notations x, y for x_i, x_j respectively. We have $F_1(x) = x, F_2(y) = \frac{2}{3}y$. In equilibrium, bidder one with valuation x believes that bidder two valuation distribution is $F_2|_{1.5x}$, after she wins the auction. We let $y = 1.5x$. Given the first price offer p_1, bidder two has a threshold of acceptance z. The offer will be accepted if and only if bidder two's valuation is higher than z. When bidder two rejects the offer, the equilibrium period two offer is given by $p_2(x, z) = \frac{x+z}{2}$. The following equation determines the equilibrium z

$$z - p_1 = \delta_2(z - \frac{z + x}{2}),$$

and we have

$$z = \frac{p_1 - 0.5\delta_2 x}{1 - 0.5\delta_2}.$$

The optimal first offer p_1 maximizes the profit function

$$\frac{2}{3}(y - z)(p_1 - x) + \frac{2}{3}\delta_1(z - p_2)(p_2 - x) = \frac{2}{3}(y - z)(p_1 - x) + \frac{2}{3}\frac{\delta_1}{4}(z - x)^2$$

$$= \frac{2}{3}(y - \frac{p_1 - 0.5\delta_2 x}{1 - 0.5\delta_2})(p_1 - x) + \frac{2}{3}\frac{\delta_1}{4}(\frac{p_1 - x}{1 - 0.5\delta_2})^2.$$

The first order condition for p_1 is

$$y - \frac{2p_1 - (1 + 0.5\delta_2)x}{1 - 0.5\delta_2} + \frac{\delta_1}{2(1 - 0.5\delta_2)^2}(p_1 - x) = 0,$$

and we get the optimal first period offer

$$p_1(x, y, \delta_1, \delta_2) = \frac{(1 - 0.5\delta_2)^2}{2 - \delta_2 - 0.5\delta_1} y + \frac{1 - 0.5\delta_1 - 0.25\delta_2^2}{2 - \delta_2 - 0.5\delta_1} x.$$

where $y = 1.5x$.

Since the first price auction revenue R^F with resale is increasing in p_1, and p_1 is increasing in δ_1, and decreasing in δ_2, we know that R^F is increasing in δ_1 and decreasing in δ_2. Therefore we know that a higher delay cost (or lower bargaining power) for bidder one hurts the revenue in the first price auction, while the opposite is true for bidder two. When $\delta_1 = 0, \delta_2 = 1$, we have the lowest revenue in the first price auction. In this case, we have $w(x, y) = \frac{1}{4}y + \frac{3}{4}x = 1.125x$, hence

$$R^F = \int_0^1 2(1 - t)1.125t\,dt = 0.375,$$

which is lower than the revenue from the second price auction

$$\int_0^1 (1 - x)(1 - \frac{2}{3}x)dx = 0.38889.$$

Thus we have an example in which the opposite ranking holds when the monopolist has low bargaining power due to a high delay cost while the buyer has no delay cost.

5.4 Bargaining Power and Lack of Commitment

When both bidders are very patient, the opposite ranking can also occur. The Coase (1972) conjecture in fact says that the monopolist may lose all bargaining power if the buyer anticipates lower prices in future offers. This has been formalized in Gul, Sonnenschein and Wilson (1986)[21]. In their model, the monopolist makes offers in increasingly short intervals. Assuming stationarity in the equilibrium, they show that all prices including the first offer goes to the marginal cost of the monopolist. In our model, the marginal cost of the monopolist is his or her own valuation for the object. This means that the first offer price, which is the common-value in the associated trade mechanism in equilibrium, will converge to $\min\{x, y\}$. By Proposition 9, the ranking must be reversed when the Coase conjecture holds.

In Gul, Sonnenschein and Wilson (1986), only the monopolist makes offers to the buyer. When alternating offers are allowed, Ausubel and Deneckere (1992) show that the Coase conjecture also holds under the same conditions. The reason is that when the informed party makes offers, only non-serious offers will be made. In fact, the informed party prefers to reveal information only passively by accepting or rejecting offers. This is called the Silence Theorem. The Silence Theorem gives a justification to the model of repeated offers from

[21] Fudenberg, Levine, and Tirole (1985) have a Coase Theorem in the "gap" case in an infinite horizon model of bargaining when the discount rate is close to 1. Our model does not allow the "gap" case.

the uninformed party to the informed party. Again we have the opposite ranking in the model of alternating offers when Coase conjecture holds.

In the literature on Coase conjecture, the seller's cost is usually fixed, and equal to 0. In our resale model, the seller's cost can be any number within the range $[0, a_1]$. To show how the Coase Theorem can be adapted for any cost of the seller with heterogeneous beliefs due to updating, we illustrate with the finite horizon model of Sobel and Takahashi (1983). We show that for any given discount factor $\delta_1 < 1$ of the seller, the Coase conjecture holds as $\delta_2 \to 1$, and the number of periods goes to infinity. We focus on the linear case of Sobel and Takahashi (1983).

Assume that bidder one and two have uniform IPV distributions over the intervals $[0, a_1], [0, a_2]$ respectively and $a_1 < a_2$. After the first-price auction in stage one, the winning bid is announced. In stage two, the winner of the auction makes no commitment offers (except the last one which is a take-it-or-leave-it offer) to the loser for n periods. In this case, only bidder one will make offers after winning the auction. First we derive the unique perfect Bayesian equilibrium of this finite-offer game and show that the revenue ranking is reversed. Let the seller has the valuation x and in equilibrium she believes that the buyer's valuation is uniformly distributed over $[0, y], y = \frac{a_2}{a_1}x$. We denote this bargaining game by $L_n(x, y)$.

Proposition 20 The first period offer of the bargaining game $L_n(x, y)$ in the resale stage with n periods of offers is given by

$$p = c_n y + (1 - c_n)x$$

where c_n is defined recursively by

$$c_1 = \frac{1}{2}, c_k = \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}}.$$

Fix $\delta_1 < 1$, and let $\delta_2 \to 1$, we have

$$c_k \to \frac{c_{k-1}}{2 - \delta_1} \text{ for all } k.$$

Since $c_1 = \frac{1}{2}$, we have $c_n = \frac{1}{2(2-\delta_1)^{n-1}} \to 0$, as $n \to \infty$. Therefore the first period offer p converges to $x = \min\{x, y\}$ as $n \to \infty$. By Proposition 9, the revenue ranking is reversed if $\delta_1 < 1$ is fixed, δ_2 is close to 1, and the number of offer periods n is sufficiently large. In this example, Coase Theorem holds as long as the buyer is sufficiently patient, and the number of bargaining period is sufficiently large.

6 Proofs

Proof of Lemma 1:

When w is symmetric, condition (R) says that for $s_i < s_j$,

$$\frac{w_i(s_i, s_j)}{w_i(s_i, s_i)} < \frac{1 - F_j(s_i)}{1 - F_j(s_j)} \tag{14}$$

If w is submodular, w_i is decreasing in s_j. Since the right-hand side of 14) is increasing in s_j, and for $s_j = s_i$, we have equality between the two sides. Therefore, for $s_j > s_i$, (14) holds. The arguments for the case $s_i > s_j$ are completely similar.

Proof of Proposition 2:

Let $\sigma(t)$ be the equilibrium bidding strategy for both bidders. Let $w_i = \frac{\partial w}{\partial s_i}$ be the partial derivative with respect to s_i. Bidder one with the signal $s_1 = v_1(t)$ chooses b to maximize

$$U(s_1) = \int_0^{\sigma^{-1}(b)} [w(s_1, v_2(r)) - b]\, dr$$

By the envelope theorem, we have

$$U'(s_1) = \int_0^{F_1(s_1)} w_1(s_1, v_2(r))dr.$$

Hence we have

$$\begin{aligned} U(s_1) &= \int_0^{s_1} \left[\int_0^{F_1(s)} w_1(s, v_2(r))dr \right] ds \\ &= \int_0^{F_1(s_1)} \left[\int_{v_1(r)}^{s_1} w_1(s, v_2(r))ds \right] dr \\ &= \int_0^{F_1(s_1)} [w(s_1, v_2(r)) - w(v_1(r), v_2(r))]\, dr, \end{aligned}$$

hence

$$U(t) = \int_0^t w(v_1(t), v_2(r))dr - \int_0^t w(v_1(r), v_2(r))dr \tag{15}$$

We also have

$$U(t) = \int_0^t [w(v_1(t), v_2(r)) - \sigma(t)]\, dr = \int_0^t w(v_1(t), v_2(r))\, dr - t\sigma(t) \tag{16}$$

Equating (15) and (16), we have

$$t\sigma(t) = \int_0^t w(v_1(r), v_2(r))dr,$$

and

$$\sigma(t) = \frac{1}{t}\int_0^t w(v_1(r), v_2(r))dr.$$

The seller's revenue from each bidder is

$$A = \int_0^1 t\sigma(t)dt = \int_0^1 \left(\int_0^t w(v_1(r), v_2(r))dr \right) dt.$$

Using integration by parts, we have

$$A = \int_0^1 (1 - t)w(v_1(t), v_2(t))dt.$$

Since the equilibrium bidding strategy is symmetric, the revenue from each bidder is the same. Hence the theorem is proved.

Proof of Corollary 3:

By Proposition 2, we have

$$R^F = 2\int_0^1 (1 - t)w(v_1(t), v_2(t))dt = \int_0^1 (1 - t)(v_1(t) + v_2(t))dt.$$

Using integration by parts, we have

$$\int_0^1 (1 - t)v_1(t)dt = \frac{1}{2}\int_0^1 (1 - t)^2 dv_1(t) = \frac{1}{2}\int_0^1 (1 - t)^2 dv_1(t).$$

Similarly,

$$\int_0^1 (1 - t)v_2(t)dt = \frac{1}{2}\int_0^1 (1 - t)^2 dv_2(t),$$

and the theorem is proved.

Proof of Proposition 4:

The selected equilibrium has the following bidding strategy

$$b_i(v) = w(v, v) \text{ for } i = 1, 2.$$

The expected revenue from the second-price auction is given by

$$\begin{aligned} R^{SPA} &= \int_0^a w(x, x)d[1 - (1 - F_1(x))(1 - F_2(x))] \\ &= -\int_0^a w(s, s)d[(1 - F_1(x))(1 - F_2(x))] \end{aligned}$$

Using integration by parts, we have

$$R^{SPA} = \int_0^a (1 - F_1(x))(1 - F_2(x))dw(x,x),$$

and the proof is complete.

Proof of Proposition 5:

Let $\phi_1(b), \phi_2(b)$ be the inverse bidding functions (mapping bids to signals) of the two bidders in a second-price auction equilibrium with a small private-value component. Let $F_i(x) = v_i^{-1}(x)$. Bidder one with signal t_1 chooses b to maximize

$$\int_0^{\phi_2(b)} [\varepsilon v_1(t_1) + (1-\varepsilon)w(v_1(t_1), v_2(t_2)) - b_2(t_2)]dt_2.$$

The first order condition is

$$[\varepsilon v_1(t_1) + (1-\varepsilon)w(v_1(t_1), v_2(\phi_2(b))) - b]\,\phi_2'(b) = 0.$$

Since $t_1 = \phi_1(b)$, we have

$$\varepsilon v_1(\phi_1(b)) + (1-\varepsilon)w(v_1(\phi_1(b)), v_2(\phi_2(b))) - b = 0. \tag{17}$$

A similarly argument for bidder two gives us

$$\varepsilon v_2(\phi_2(b)) + (1-\varepsilon)v(v_1(\phi_1(b)), v_2(\phi_2(b))) - b = 0. \tag{18}$$

Combine the two equations (17),(18), we get

$$v_1(\phi_1(b)) = v_2(\phi_2(b)).$$

From (17), we have

$$\varepsilon\phi_1(b) + (1-\varepsilon)w(v_1(\phi_1(b)), v_1(\phi_1(b))) - b = 0, \tag{19}$$

which can be rewritten as

$$b_1(t_1) = \varepsilon t_1 + (1-\varepsilon)w\,(v_1(t_1), v_1(t_1))\,.$$

Hence the equilibrium bidding strategy is unique. As $\varepsilon \to 0$, (19) in the limit we have

$$b_1(t_1) = w\,(v_1(t_1), v_1(t_1))\,, \tag{20}$$

and similarly

$$b_2(t_2) = w(v_2(t_2), v_2(t_2)).$$

Our proof is complete.

Proof of Proposition 6:

The revenue of the second-price auction equilibrium is

$$R^{SPA} = \int_0^s \int_0^t \min(b(s), b(t))dsdt = 2\int_0^1 \left(\int_0^s b(t)dt \right) ds = 2\int_0^1 \left(\int_0^s w(v_1(t), v_2(t))dt \right) ds.$$

Using integration by parts, we have

$$\begin{aligned} R^{SPA} &= 2\int_0^1 w(v_1(s), v_2(s))ds - \int_0^1 sw(v_1(s), v_2(s))ds \\ &= 2\int_0^1 (1 - s)w(v_1(s), v_2(s))ds, \end{aligned}$$

which is the same as the revenue in the first-price auction by Proposition 2.

Proof of Theorem 7:

From Proposition 4, we have

$$R^S < \frac{1}{2}\int_0^a [(1 - F_1(x))^2 + (1 - F_2(x))^2]dw(x, x).$$

Using arguments similar to the proof of Corollary 3, we have

$$\begin{aligned} &\frac{1}{2}\int_0^a [(1 - F_1(x))^2 + (1 - F_2(x))^2]dw(x, x) \\ = &\int_0^{v_1(1)} (1 - F_1(x))w(x, x)dF_1(x) + \int_0^{v_2(1)} (1 - F_2(x))w(x, x)dF_2(x) \\ = &\int_0^1 (1 - t)w(v_1(t), v_1(t))dt + \int_0^1 (1 - t)w(v_2(t), v_2(t))dt \\ = &\int_0^1 (1 - t)\left[w(v_1(t), v_1(t)) + w(v_2(t), v_2(t))\right] dt. \end{aligned}$$

Condition (C) now implies that

$$R^S < 2\int_0^1 (1 - t)w(v_1(t), v_2(t))dt = R^F,$$

and the theorem is proved.

Proof of Proposition 9:

Let F_i, F_j be the corresponding distributions. Let

$$F(x) = \max\{F_1(x), F_2(x)\}.$$

Let $v(t) = F^{-1}(t)$. Then we have $\min\{v_1(t), v_2(t)\} = v(t)$. By Proposition 8, we have

$$R^F = \int_0^1 2(1 - t)v(t)\,dt = \int_0^a (1 - F(x))^2 dx.$$

Hence

$$R^F = \int_0^a (1 - F(x))^2 dx < \int_0^a (1 - F_1(x))(1 - F_2(x)dx.$$

The result for the maximum function follows from Theorem 7.

Proof of Theorem 10:

Let $t = F_i(v_i), h(x) = w(x, x)$. The difference of the revenue is

$$R^F - R^S = \int_0^1 2(1 - t)w(v_i(t), v_j(t))dt - \int_0^1 (1 - F_i(x))(1 - F_j(x))\,dh(x)$$

$$= \int_0^1 2(1 - t)w(v_i(t), v_j(t))dt - \int_0^1 (1 - t)(1 - F_j(v_i(t)))h'(v_i)v_i'(t)dt.$$

Using Integration by parts, we have

$$\int_0^1 (1 - t)(1 - F_j(v_i(t)))]h'v_i'dt$$

$$= \int_0^1 (1 - t)d\left[\int_0^{v_i(t)} (1 - F_j(v))h'(v)dv\right]$$

$$= \int_0^1 \left[\int_0^{v_i(t)} (1 - F_j(v))h'(v)dv\right] dt. \tag{21}$$

Hence we have

$$R^F - R^S = \int_0^1 2(1 - t)w(v_i(t), v_j(t))dt - \int_0^1 \left[\int_0^{v_i(t)} (1 - F_j(v))h'(v)dv\right] dt.$$

Let $p(k, t) = v_j(t) + k(v_i(t) - v_j(t)), 0 \le k \le 1$, and

$$D(k) = \int_0^1 2(1 - t)w(p(k, t), v_j(t))dt - \int_0^1 \left[\int_0^{p(k,t)} (1 - F_j(v))h'(v)dv\right] dt.$$

When condition (R) holds, we want to show that $D'(k) > 0$ on a set of non-zero measure. Since $D(0) = 0$, this proves that $D(1) = R^F - R^S > 0$. We have

$$D'(k) = \int_0^1 2(1 - t)w_i(p(k, t), v_j(t))(v_i(t) - v_j(t))dt$$

$$- \int_0^1 ((1 - F_j(p(k, t)))h'(p(k, t))(v_i(t) - v_j(t))dt$$

$$= \int_0^1 (v_i(t) - v_j(t))[2(1 - t)w_i(p(k,t), v_j(t)) - (1 - F_j(p(k,t)))h'(p(k,t))]dt.$$

Since $v_i(t) > v_j(t)$, if and only if $p(k,t) > v_j(t)$ for $k > 0$, if and only if

$$w_i(p(k,t), v_j(t)) > \frac{1}{2}\frac{1 - F_j(p(k,t))}{1 - F_j(x_j)}h'(p(k,t)) = \frac{1}{2}\frac{1 - F_j(p(k,t))}{1 - t}h'(p(k,t))$$

for $k > 0$, if and only if

$$2(1 - t)w_i(p(k,t), v_j(t)) > (1 - F_j(p(k,t)))h'(p(k,t))$$

for $k > 0$. We conclude that $D'(k) > 0$, for $k > 0$ when $v_i(t) \neq v_j(t)$, and the proof is complete.

The proof for the case of condition (S) is completely similar.

Proof Theorem 11:

When w is symmetric, we have $w_1 = w_2$ at (x,x). Let $h(x) = w(x,x)$, then we have $h'(x) = 2w_i(x,x)$. Let $K^{x_j}(x_i) = 2w_i(x_i,x_j) - \frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_i)}h'(x_i)$. We have $K^{x_j}(x_j) = w_i(x_j,x_j) - \frac{1}{2}h'(x_j) = 0$. Taking the derivative of K^{x_j} at x_j, we get

$$\frac{\partial}{\partial x_i}K^{x_j}(x_j) = w_{ii}(x_j,x_j) + \frac{1}{2}\frac{f_j(x_j)h'(x_j)}{1 - F_j(x_j)} - \frac{1}{2}h''(x_j).$$

Assume that $w_{ii}(x,x) + \frac{1}{2}\frac{f_j(x)h'(x)}{1 - F_j(x)} - \frac{1}{2}h''(x) < 0$ at some point $(x_0, x_0), x_0 \in (0, a_2)$. We have $\frac{\partial}{\partial x_i}K^{x_j}(x_i) < 0$ near x_0. Since $K^{x_j}(x_i) = 0$, we must have $K^{x_j}(x_i) < 0$ for $x_i < x_j, x_i, y_j$ near x_0. This implies that there exists a neighborhood U around x_0 such that

$$w_i(x_i,x_j) < \frac{1}{2}\frac{1 - F_j(x_j)}{1 - F_j(x_j)}h'(x_j) \text{ for } (x_i,x_j) \in U, x_i > x_j.$$

Let $v_j(t_0) = x_0$. There exists a smooth function $k(t)$ such that $s(t) = 1$ outside a neighborhood I of t_0, and $1 + \varepsilon > s(t) > 1$ on I, such that the point $(s(t)v_j(t), v_j(t)) \in U$ for $t \in I$. Now define $v_i(t) = v_j(t)$ outside I, and $v_i(t) = s(t)v_j(t)$ in I. Define $p(k,t) = v_j(t) + k(v_j(t) - v_i(t)), k \in [0,1]$ as in the proof of Theorem 10. From the arguments in that proof, we know that $D(k)$ is a decreasing function of k. Since $D(0) = 0$, we have $D(1) < 0$, and we conclude that for the pair of bidders v_i, v_j so defined, we have $R^{FPA} < R^{SPA}$, violating the assumption of the theorem. This contradiction means that the theorem is proved.

Proof of Theorem 12:

Given an equilibrium bidding strategies $b_i(v_i)$ in the auction with resale and the Bayesian Nash equilibrium e in the resale game after the auction. Let Q be described by $Q = \{(v_1, v_2) : v_2 \geq h(v_1)\}$. Apply the revelation principle to get a pricing function $p(v_1, v_2)$ defined over Q.

Define a common-value model with the signal distributions F_i, F_j. We can define the common-value function corresponding to the resale game as follows. For $(s_1, s_2) \in Q$, let

$$w(s_1, s_2) = p(s_1, s_2)$$

and outside Q, we let

$$w(s_1, s_2) = \min\{s_1, s_2\}.$$

Now consider the determination of the equilibrium bidding strategy in the first stage of the IPV auction with resale. Let ϕ_1, ϕ_2 be the inverse bidding functions.

When bidder one with valuation v_1 offers the bid b, the payoff is

$$\int_{h(v_1)}^{\phi_2(b)} p(v_1, v_2) dF_2(v_2) + \int_0^{h(v_1)} v_1 dF_2(v_2) - F_2(\phi_2(b))b$$

When bidder two with valuation v_2 offers the bid b, the payoff is

$$(v_2 - b)F_1(\phi_1(b)) + \int_{\phi_1(b)}^{h^{-1}(v_2)} (v_2 - p(v_1, v_2)) dF_1(v_1)$$

$$= \int_0^{h^{-1}(v_2)} v_2 dF_1(v_1) - \int_{\phi_1(b)}^{h^{-1}(v_2)} p(v_1, v_2) dF_1(v_1) - bF_1(\phi_1(b))$$

In the common-value model, let φ_1, φ_2 be the inverse bidding functions. When bidder one with signal s_1 bids b, the payoff is

$$\int_{h(s_1)}^{\varphi_2(b)} w(s_1, s_2) dF_2(s_2) + \int_0^{h(s_1)} \min(s_1, s_2) dF_2(s_2) - F_2(\varphi_2(b))b$$

When bidder two with signal s_2 bid b, the payoff is

$$\int_0^{\varphi_1(b)} (w(s_1, s_2) - b) dF_1(s_1) = \int_0^{\varphi_1(b)} w(s_1, s_2) dF_1(s_1) - bF_1(\varphi_1(b))$$

$$= \int_0^{h^{-1}(v_2)} w(s_1, s_2) dF_1(s_1) - \int_{\varphi_1(b)}^{h^{-1}(v_2)} w(s_1, s_2) dF_1(s_1) - bF_1(\varphi_1(b)).$$

The difference between the payoff functions in the two different auctions is a constant term which is independent of b. Therefore, the optimal bidding strategy in the two auctions must be the same for each $v_1 = s_1$ and each $v_2 = s_2$.

Proof of Proposition 13:

By assumption $x < y$. Let trader j be the buyer and trade i be the seller. According to Myerson and Satterthwaite (1983), the incentive efficient mechanism has the property that the probability of trade is 1 if the reported valuations (v_i, v_j) satisfy

$$v_j - \alpha\frac{1 - F_j(v_j)}{f_j(v_j)} > v_i + \alpha\frac{F_i(v_i)}{f_i(v_i)}, \tag{22}$$

where α is the Lagrangian of the participation constraint. When y is the highest valuation, and x is the lowest valuation, (22) becomes

$$y > x.$$

which is true by assumption.

Proof of Lemma 14:

Take the partial derivatives of both sides of the first order condition

$$p(x, y) - \frac{F(y) - F(p(x.y))}{f(p(x, y))} = x.$$

We get

$$2 + \frac{F(y) - F(p)}{f^2} f' = \frac{1}{p_1}, \tag{23}$$

and

$$2 - \frac{f(y)}{f(p)}\frac{1}{p_2} + \frac{F(y) - F(p)}{f^2} f' = 0. \tag{24}$$

From (23), (24), we have

$$\frac{p_1}{p_2} = \frac{f(p)}{f(y)}.$$

The slope of the level curve is given by

$$\left|\frac{dy}{dx}\right| = \frac{p_1}{p_2} = \frac{f(p)}{f(y)}.$$

When y increases (while x decreases) on the level curve keeping p constant, the slope becomes flatter. Hence the level curves of p is quasi-convex if and only if f is increasing. Similarly, for the monopsony pricing function, we have

$$\frac{r_1}{r_2} = \frac{f(x)}{f(r)},$$

and the same result holds for the quasi-convexity of r. We also have $p_1 = p_2, r_1 = r_2$ whenever $x = y$.

Proof of Lemma 15:

Assume that bidder i wins the object and wants to make offers to sell the object to bidder j. The optimal monopoly price $p(x, y)$ satisfies condition (C) if

$$p(x, y) \geq \frac{x + y}{2}.$$

Since $z = p(x, y)$ maximizes the following objective function in variable z

$$K(z) = [F_j(y) - F_j(z)](z - x),$$

it is sufficient to show that

$$K'(\frac{x + y}{2}) > 0,$$

or

$$F_j(y) - F_j(\frac{x + y}{2}) - F_j'(\frac{x + y}{2})(\frac{x + y}{2} - x) > 0.$$

Equivalently, we need to show that

$$\frac{F_j(y) - F_j(\frac{x+y}{2})}{\frac{y-x}{2}} > F_j'(\frac{x + y}{2}). \tag{25}$$

Note that the left-hand side (25) is the slope of the line through the two points $(\frac{x+y}{2}, F_j(\frac{x+y}{2})), (y, F_j(y))$, while the right-hand side is the slope of F_j at $\frac{x+y}{2}$. The convexity of F_j is sufficient for (25) to hold.

If bidder i loses the auction, and wants to make buying offers to bidder j, the arguments are very similar. Since $z = r(x, y)$ maximizes the following objective function in variable z

$$K(z) = (F_j(z) - F_j(x))(y - z),$$

it is sufficient to show that

$$K'(\frac{x + y}{2}) > 0,$$

or

$$F_j'(\frac{x + y}{2})(y - \frac{x + y}{2}) - F_j(\frac{x + y}{2}) + F_j(x) > 0.$$

Equivalently, we need to show that

$$F_j'(\frac{x + y}{2}) > \frac{F_j(\frac{x+y}{2}) - F_j(x)}{\frac{y-x}{2}}. \tag{26}$$

Note that the left-hand side (25) is the slope of the line through the two points $(x, F_j(x)), (\frac{x+y}{2}, F_j(\frac{x+y}{2}))$, while the right-hand side is the slope of F_j at $\frac{x+y}{2}$. The convexity of F_j is sufficient for (25) to hold. The proof is complete.

Proof of Lemma 16:

Let F_j be regular. Fix x_j, and we suppress the variable x_j by letting $p(x_i) = p(x_i, x_j)$. We have the first order condition for the optimal $p(x_i)$:

$$p(x_i) - x_i = \frac{F_j(x_j) - F_j(p(x_i))}{f_j(p(x_i))}, \tag{27}$$

or

$$(p(x_i) - x_i)f_j(p(x_i)) + F_j(p(x_i)) = F_j(x_j). \tag{28}$$

Taking the derivative of (28) with respect to x_i, we have

$$p'(x_i)[2f_j(p(x_i)) + (p(x_i) - x_i)f_j'(p(x_i)] = f_j(p(x_i)),$$

and

$$p'(x_i) = \frac{1}{2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))}} > 0. \tag{29}$$

We need to show

$$p'(x_i) < (>)\frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_j)} \text{ when } j = s(\text{or } w),$$

or

$$\frac{1}{2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))}} < (>)\frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_j)} \text{ when } j = s(\text{or } w).$$

Since $F_j(x_i) < (>)F_j(p(x_i))$ when $j = s(\text{or } w)$, it is sufficient to show

$$\frac{1}{2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))}} < (>)\frac{1}{2}\frac{1 - F_j(p(x_i))}{1 - F_j(x_j)} \text{ when } j = s(\text{or } w),$$

or

$$2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))} > (<)2\frac{1 - F_j(x_j)}{1 - F_j(p(x_i))},$$

which is equivalent to

$$2\frac{F_j(x_j) - F_j(p(x_i))}{1 - F_j(p(x_i))} + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))} > (<)0.$$

Divide both sides by $F_j(x_j) - F_j(p(x_i)) > (<)0$, we need to show

$$\frac{2}{1 - F_j(p(x_i))} + \frac{p(x_i) - x_i}{F_j(x_j) - F_j(p(x_i))}\frac{f_j'(p(x_i))}{f_j(p(x_i))} > 0. \tag{30}$$

Using (27), we know (30) is equivalent to

$$\frac{2}{1 - F_j(p(x_i))} + \frac{f_j'(p(x_i))}{f_j(p(x_i))^2} > 0. \tag{31}$$

From the regularity of F_j, we have, for all p,

$$\frac{d}{dp}\left(p - \frac{1 - F_j(p)}{f_j(p)}\right) > 0, \tag{32}$$

hence

$$2 + \frac{1 - F_j(p)}{f_j(p)^2} f_j'(p) > 0,$$

which implies (31).

Proof of Theorem 17:

Define $w(x,y) = p(x,y)$, or $r(x,y)$ if $x \le y$. The function w can be extended to a continuously differentiable strictly increasing function over all (x,y). The revenue of the auctioneer however depends on the definition of w on the pairs $(x,y), x \le y$.

Apply 16, we know that condition (R) is satisfied for the optimal offer function w. By Theorem 10, the ranking result holds.

If we allow random assignment of the offer-maker, let π be the probability that bidder i makes the offer, and $1 - \pi$ the probability that bidder j makes the offer. Let w^i, w^j be the corresponding pricing function. The common value is now

$$w(x,y) = \pi w^i(x,y) + (1 - \pi) w^j(x,y) \text{ for } x < y.$$

Note that if condition (C) (or (R)) is satisfied by both w^i and w^j, then it is also satisfied by w. Since the revenue is linear in π, the revenue ranking property of this common-value auction follows from those of w^i and w^j.

Proof of Corollary 18:

With contingent bargaining power, the definition of w depends on F_i in some region, and on F_j in others.

Because of Lemma 14, the function w can be extended to all pairs and remains continuously differentiable and strictly increasing. This is because all partial derivatives of p and r on the diagonal (x,x) are identical and equal to $\frac{1}{2}$ regardless of which $F_i, i = 1,2$ is used in the optimal pricing problem. If all distribution functions are regular, the condition (R) is always satisfied in each region. Hence Theorem 10 applies, and the ranking result holds.

Proof of Theorem 19:

From (29), we take the second derivative, and evaluate at (x_j, x_j), we have

$$p''(x_j) = \frac{-(\frac{1}{2} - 1)\frac{f_j'(x_j)}{f_j(x_j)}}{4} = \frac{1}{8}\frac{f_j'(x_j)}{f_j(x_j)}.$$

According to Theorem 11, the necessary condition for the ranking result for all F_i is

$$\frac{1}{8}\frac{f_j'(x_j)}{f_j(x_j)} + \frac{1}{2}\frac{f_j(x_j)}{1 - F_j(x_j)} \geq 0,$$

or

$$\frac{(1 - F_j(x_j))f_j'(x_j)}{f_j^2(x_j)} + 4 \geq 0,$$

and the proof is complete.

Proof of Proposition 20:

Let the number of periods remaining be k, and denote the optimal offer by p_k. The updated belief of the highest valuation z_k of the buyer is the threshold of acceptance in the period before. By backward induction, p_k depends only on x, z_k, and we use the notation $p_k(x, z_k)$. Let $\pi_k(x, z_k)$ be the expected profit function when k periods are remaining. Again by backward induction, z_k depends only on x and z_{k+1}. Given p_k, p_{k-1}, bidder two has a threshold level of acceptance z_{k-1}. Bidder two will accept the offer p_k whenever his or her valuation is above z_{k-1}. Given p_k, p_{k-1}, we can determine z_{k-1} by the condition

$$z_{k-1} - p_k = \delta_2(z_{k-1} - p_{k-1})$$

Thus we have the equation

$$(1 - \delta_2)z_{k-1} + \delta_2 p_{k-1} = p_k \tag{33}$$

If the offer p_k is rejected, the bidder i updates his belief of the valuation of bidder j, and the new highest (lowest) valuation of the buyer (seller) is now z_{k-1}. Let $p_{k-1}(x_i, z_{k-1})$ be the optimal offer with $k - 1$ periods remaining with the updated z_{k-1}. We can rewrite (33) as

$$(1 - \delta_2)z_{k-1} + \delta_2 p_{k-1}(x, z_{k-1}) = p_k \tag{34}$$

If the optimal offer p_{k-1} with $k - 1$ periods remaining has been determined by backward induction and is increasing in z_{k-1}. The left-hand side of (34) is increasing in z_{k-1}, and there is a unique solution denoted by $z_{k-1}(x_i, p_{k-1})$. Thus we know how z_{k-1} is determined once p_k is chosen.

The choice of p_k is determined by the maximization of the profit function of the seller given by

$$[F_2(z_k) - F_2(z_{k-1}(x, p_k))](p_k - x) + \delta_1 \pi_{k-1}(x, z_{k-1}) \tag{35}$$

The first order condition for p_k is

$$F_2(z_k) - F_2(z_{k-1}) - f_2(z_{k-1})(p_k - x)\frac{\partial z_{k-1}}{\partial p_k} + \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}\frac{\partial z_{k-1}}{\partial p_k} = 0.$$

Take the implicit derivative of (33) with respect to p_k, we have

$$(1 - \delta_2)\frac{\partial z_{k-1}}{\partial p_k} + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}\frac{\partial z_{k-1}}{\partial p_k} = 1,$$

or

$$\frac{\partial z_{k-1}}{\partial p_k} = \frac{1}{(1-\delta_2)+\delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}}. \tag{36}$$

Substitute (36) into the first order condition, we have

$$F_2(z_k) - F_2(z_{k-1}) - \frac{f_2(z_{k-1})(p_k - x) - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1-\delta_2)+\delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0.$$

For uniform distributions, we have $f_2 = 1$. Hence we have the first order condition

$$z_k - z_{k-1} - \frac{p_k - x - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1-\delta_2)+\delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0 \tag{37}$$

When $k = 1$, we have

$$p_1(x,y) = \frac{x+y}{2}, \pi_1(x,y) = (\frac{y-x}{4})^2$$

and $p_1(x,z_1) = \frac{x+z_1}{2}, \pi_1(x,z_1) = (\frac{z_1 - x}{2})^2$. Hence

$$\frac{\partial p_1}{\partial z_1} = \frac{1}{2}, \frac{\partial \pi_1}{\partial z_1} = \frac{z_1 - x}{2}.$$

The theorem holds for $k = 1$ with $c_1 = \frac{1}{2}$. More generally, by mathematical induction, assume that the theorem holds for $k-1$, and we have

$$p_{k-1} = c_{k-1} z_{k-1} + (1-c_{k-1})x, \pi_{k-1} = 0.5 c_{k-1}(z_{k-1} - x)^2$$

$$\frac{\partial p_{k-1}}{\partial z_{k-1}} = c_{k-1}, \frac{\partial \pi_{k-1}}{\partial z_{k-1}} = c_{k-1}(z_{k-1} - x).$$

The first order condition (37) for z_{k-1}, p_k is

$$y - z_{k-1} = \frac{(1-\delta_2) z_{k-1} + \delta_2 (c_{k-1} z_{k-1} + (1-c_{k-1})x) - x - \delta_1 c_{k-1}(z_{k-1} - x)}{1-\delta_2+\delta_2 c_{k-1}}$$

or

$$y - z_{k-1} = \frac{(1-\delta_2+\delta_2 c_{k-1} - \delta_1 c_{k-1})(z_{k-1} - x)}{1-\delta_2+\delta_2 c_{k-1}}$$

or

$$\frac{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}}{1-\delta_2+\delta_2 c_{k-1}} z_{k-1} = y + \frac{1-\delta_2+\delta_2 c_{k-1} - \delta_1 c_{k-1}}{1-\delta_2+\delta_2 c_{k-1}} x$$

and we have

$$z_{k-1} = \frac{1-\delta_2+\delta_2 c_{k-1}}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}} y + \frac{1-\delta_2+\delta_2 c_{k-1} - \delta_1 c_{k-1}}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}} x.$$

Let

$$d_{k-1} = \frac{1 - \delta_2 + \delta_2 c_{k-1}}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}},$$

then

$$z_{k-1} = d_{k-1} y + (1 - d_{k-1}) x.$$

We have

$$\begin{aligned} p_k &= (1 - \delta_2) z_{k-1} + \delta_2 p_{k-1} = (1 - \delta_2) z_{k-1} + \delta_2 (c_{k-1} z_{k-1} + (1 - c_{k-1}) x) \\ &= (1 - \delta_2 + \delta_2 c_{k-1}) z_{k-1} + \delta_2 (1 - c_{k-1}) x = c_k y + (1 - c_k) x \end{aligned}$$

where

$$c_k = \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}} = (1 - \delta_2 + \delta_2 c_{k-1}) d_{k-1}.$$

The expected profit can be written as

$$\begin{aligned} \pi_k &= (y - z_{k-1})(p_k - x) + \delta_1 \pi_{k-1} \\ &= c_k (1 - d_{k-1})(y - x)^2 + 0.5 \delta_1 c_{k-1} (z_{k-1} - x)^2 \\ &= (y - x)^2 (c_k - c_k d_{k-1} + 0.5 \delta_1 c_{k-1} d_{k-1}^2) \\ &= (y - x)^2 (c_k - (1 - \delta_2 + \delta_2 c_{k-1}) d_{k-1}^2 + 0.5 \delta_1 c_{k-1} d_{k-1}^2) \\ &= (y - x)^2 (c_k - 0.5 d_{k-1}^2 (2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1})) \\ &= (y - x)^2 \left(c_k - 0.5 \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}}\right) \\ &= (y - x)^2 (c_k - 0.5 c_k) = 0.5 c_k (y - x)^2. \end{aligned}$$

By mathematical induction, the proof is complete.

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