The Economics of Contingent Re-Auctions[*]

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March 31, 2008

Preliminary
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Abstract

We consider an auction environment in which the object can be sold with restrictions or without restrictions. Restricting the object generates a direct benefit to the seller, but lowers the buyers' willingness to pay. In this environment, sellers such as the FCC have used a contingent re-auction, whereby they first offer the restricted object with a reserve price, and if the reserve is not met, re-auction the object without restrictions. We characterize the equilibria of a contingent re-auction, and we derive the optimal mechanism and an efficient mechanism, both of which can be implemented through a straighforward auction design. Our results have implications for how contingent re-auctions can be improved.

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1   Introduction

The U.S. Federal Communications Commission (FCC) began its Auction 73 for spectrum licenses in January of 2008. The licenses were offered with substantial restrictions, which the FCC views as in the public interest, but which presumably reduce their value to bidders. In the event the reserve prices for the licenses were not met, the FCC committed to re-auction the licenses without many of the restrictions, but with the same reserve price. The motivation for this auction format, which has been referred to as a "do-over auction" in the press, is straightforward: the FCC believes the contemplated restrictions are in the public interest, but does not want to sacrifice too much revenue in the process of imposing those restrictions. If the restricted licenses can be sold in the first auction for a price greater than the reserve, then the FCC prefers that outcome, but in the absence of bids greater than the reserve, it prefers to sell the licenses without the restriction.

The recent attempted sale of the Italian airline Alitalia followed a similar pattern, although the procedure was not formalized to the same extent as that of the FCC. Alitalia was first put up for sale with a series of restrictions, such as limitations on the ability by the new owner to fire employees. These restrictions were perceived as desirable by the Italian government, but they reduced the value of the airline to bidders. After it became clear that no bidder was interested at the proposed price, the airline was put up for sale again, but with fewer restrictions.[1]

We refer to the procedures used by the FCC and Alitalia described above as a contingent re-auction.[2] The general problem faced in these situations is as follows: A seller has an object for sale. The seller can 'damage' the object, for example by restricting its future use. If the restricted object is sold, the seller receives a benefit

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[1]Alitalia was put on sale on January 2007. A number of bidders initially expressed interest but they all dropped out by July 2007. The bidders cited "restrictive conditions imposed by the government and a lack of access to the airline's books. ... These conditions, when originally expressed ahead of the auction, included maintaining certain staff levels, continued operation of some routes and traffic rights regardless of profitability, preserving Alitalia's identity, and not selling certain Alitalia interests for three years." See Aude Lagorce, "Alitalia Still Hoping for Rescue," *MarketWatch*, September 12, 2007.

After the failure of the first attempt the government has signaled that it may relax some of these rules and try to sell the company again. See, e.g., "Alitalia boss wants state sell-off finalised by Christmas," Yahoo! News, available at

http://news.yahoo.com/s/afp/20070908/bs_afp/italyairlinecompanyalitalia (checked 9/18/2007).

[2]This term has been used in the press to describe FCC Auction 73. See, e.g., Stifel and Nicolaus's Telecom, *Media & Tech Insider*, August 24, 2007.

B in addition to the sale price. If the unrestricted object is sold, the seller receives no benefit beyond the sale price. Bidders value the unrestricted object more than the restricted object and have private values. We refer to this environment as an 'environment with seller-benefitting restrictions.' In such an environment, efficiency requires that the object be restricted if B is larger than the difference between the highest value for the unrestricted object and the highest value for the restricted object, and that it not be restricted otherwise.

In this paper, we contribute to the theoretical understanding of contingent re-auctions and propose alternative mechanisms that offer improvement over the contingent re-auction in certain environments. The analysis has four components. First, we characterize the equilibrium of a contingent re-auction involving either two sequential second-price auctions or two sequential English auctions. Second, we identify an optimal mechanism for an environment with seller-benefiting restrictions when buyers' values for the restricted and unrestricted object do not vary independently, so that the environment is one with one-dimensional types. Third, for the case with two-dimensional types, which in many environments would create substantial analytical difficulties, we are able to identify a mechanism that achieves the efficient outcome, and can be implemented with a straightforward selling procedure (the 'exclusive buyer' mechanism).[3] This efficient mechanism, although it does not maximize the seller's revenue, provides the seller with greater expected revenue than the contingent re-auction in some simulations we have performed. The mechanism is an 'exclusive buyer' mechanism in which buyers compete in a second-price auction for the right to be the single buyer that then chooses between the restricted object for no additional payment and the unrestricted object for an additional payment of B. Fourth, for environments with two-dimensional types, we conjecture that an exclusive buyer mechanism is optimal. In the optimal exclusive buyer mechanism, the second-stage prices differ from those of the efficient exclusive buyer mechanism and are those associated with the optimal mechanism for an environment with a single buyer.

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[3]We assume no allocative externalities, i.e., a buyer not receiving the object has no preference over which other buyer receives the object. In environments with allocative externalities (as well as multidimensional information and informational externalities), Maskin (1992) and Jehiel and Moldovanu (2001) show that every Bayesian Nash equilibrium is (generically) inefficient. In addition, in such environments, Jehiel et al. (2006) call into question the existence of ex-post equilibrium. However, Bikhchandani (2006) shows that these non-existence results rely on the assumption of allocative externalities. In addition, Mezzetti (2004) shows that efficiency can be obtained with two-stage mechanisms in which payments can be conditioned on reports about the agents' allocation payoffs.

Drawing from our theoretical results and numerical simulations, we are able to make some recommendations for how sellers such as the FCC might improve their implementation of contingent re-auctions in the future, and we are able to suggest an alternative mechanism, the exclusive buyer mechanism, which our results show has superior properties.

We do not know of any economics literature that considers contingent re-auctions directly, but there is a related literature on auctions with resale. Horstmann and LaCasse (1997) show that in a common value environment, a seller may choose not to sell an object, even if it receives bids above the announced reserve price, and then to re-auction the item after a delay in order to signal its private information about the value of the object. In contrast, in the environment we consider, the seller has no private information. Cassady (1967), Ashenfelter (1989), and Porter (1995) indicate that goods that are not sold at an initial auction are often offered for sale again later, but in these cases it is the same items that are re-offered, not a modified version as in the cases we consider. McAfee and Vincent (1997) consider a model in which a seller cannot commit not to re-auction an object if the announced reserve price is not met. They show that when the time between auctions goes to zero, the seller's expected revenues converge to that of a static auction with no reserve price, and they characterize the optimal dynamic reserve price policy of the seller. In our model, we assume that the seller can commit to a reserve price in the second auction, but we also consider the case of a zero reserve at the second auction.

There is also a literature on auctions with resale, which focuses on environments in which bidders that win objects at an auction can then resell them after the auction. See, e.g., Gupta and Lebrun (1999), Haile (2000, 2001, 2003), Zheng (2002), Garratt and Tröger (2005), Hafalir and Krishna (2007), Garratt, Tröger, and Zheng (2006), Lebrun (2007), and Pagnozzi (2007). In our model, we assume no resale. Finally, it is well known in the literature on price discrimination that it can be optimal for a seller to offer both damaged and undamaged versions of a product (Deneckere and McAfee, 1996). In our model, the seller can only offer one version of the product, either restricted or unrestricted, so there are no price discrimination motivations for restricting the use of product.

In analyzing the optimal mechanism for an environment with seller-benefitting restrictions, we first characterize the optimal mechanism under the assumption that buyers' values for the restricted and unrestricted objects are related in such a way that

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a buyer's type becomes one-dimensional. In this case, standard mechanism design techniques apply. There is a large related literature, but we rely particularly on the results of Myerson (1981).

However, the more general version of our problem involves multidimensional types—buyers have values for both the unrestricted and restricted versions of the object. As described in Armstrong (1996) and Rochet and Choné (1998), results for mechanism design problems with multidimensional types can be difficult to obtain. A number of papers have contributed to the development of methods for such problems, including Rochet (1985), Matthews and Moore (1987), McAfee and McMillan (1988), Armstrong (1996), and Rochet and Choné (1998), where the last two of these papers focus on the case of a multiproduct monopolist. More recently, Manelli and Vincent (2006 and 2007) make progress on the problem of a multiproduct monopolist, with Manelli and Vincent (2007) characterizing the set of all mechanisms that maximize the seller's expected revenue for some seller's beliefs and Manelli and Vincent (2006) identifying conditions on the seller's beliefs such that posted prices maximize the seller's expected revenue.

Our paper is related to the classic paper by Mussa and Rosen (1978), which considers a model in which a monopolist chooses quality. Their model differs from ours in that there is no benefit to the seller associated with particular qualities and, more importantly, they assume one-dimensional types.

The remainder of this paper proceeds as follows. In Section 2, we describe in more detail the contingent re-auction being used by the FCC. In Section 3, we present a complete information example to illustrate some of the issues involved. In Section 4, we describe our general model with incomplete information and two-dimensional types. In Section 5, we characterize the equilibria of a contingent re-auction in the context of our model. In Section 6, we characterize the optimal mechanism under an assumption that reduces the type space to one dimension. In Section 7, we consider the mechanism design problem with two-dimensional types and propose the exclusive buyer mechanism. As we show, there exists an exclusive buyer mechanism that generates the efficient outcome. We conjecture that a version of the exclusive buyer mechanism is an optimal mechanism, at least among dominant strategy mechanisms that never retain the object. The formal demonstration of this is work in progress. Section 8 provides numerical calculations comparing the various mechanisms. Section 9 concludes with a discussion of implications for the mechanism design in environ-

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ments with seller-benefitting restrictions.

2   Contingent re-auction of spectrum licenses

On January 24, 2008, the FCC began its Auction 73 offering 1,099 spectrum licenses in the 698–806 MHz band, which is referred to as the 700 MHz Band. Given the large number of licenses for sale and the high quality of the spectrum in the 700 MHz Band, this auction was expected to generate significant revenue for the U.S. government.[4] The FCC proposed total reserve prices for the auction of over $10 billion,[5] and to date bids totalling approximately $20 billion have been submitted.

The licenses to be auctioned are defined by their geographic scope and their location in the electromagnetic spectrum. The band plan for the 700 MHz auction defines five blocks of licenses: A, B, C, D, and E. The A-block licenses are 12 MHz licenses defined over 176 medium-sized geographic areas referred to as EAs.[6] The B-block licenses are 12 MHz licenses defined over 734 small geographic areas referred to as CMAs.[7] The C-block licenses are 22 MHz licenses defined over 12 large geographic areas referred to as REAGs,[8] and bidders will also be able to submit bids for the nationwide package of licenses.[9] The D-block is organized as a single 10 MHz nation-

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[4]According to the FCC: "The unique propagation characteristics of this spectrum means that fewer towers will be needed to serve a given license area, as compared to providing service at higher frequencies, and thus large license areas may be served at lower infrastructure costs." Second Report and Order (FCC 07-132), "Service Rules for the 698-746, 747-762 and 777-792 MHz Bands," Released 8/10/2007, at paragraph 154. Available at http://fjallfoss.fcc.gov/edocs_public/attachmatch/FCC-07-132A1.pdf

There will not be any incumbents in the band after the DTV transition is completed on February 17, 2009. See the Digital Television Transition and Public Safety Act of 2005, which is Title II of the Deficit Reduction Act of 2005, Pub. L. No. 109-171, 120 Stat. 4 (2006).

[5]See FCC Public Notice (DA 07-3415), "Auction of 700 MHz Band Licenses Scheduled for January 16, 2008; Comment Sought on Competitive Bidding Procedures for Auction 73," Released 8/17/2007, at paragraph 53. Available at http://fjallfoss.fcc.gov/edocs_public/attachmatch/DA-07-3415A1.pdf

[6]A map showing the EAs, or Economic Areas, is available at http://wireless.fcc.gov/auctions/data/maps/EA_GOM.pdf.

[7]The CMAs are sometimes divided into Metropolitan Statistical Areas (MSAs) and Rural Service Areas (RSAs). A map showing the CMAs, or Cellular Market Areas, is available at http://wireless.fcc.gov/auctions/data/maps/CMA.pdf.

[8]A map showing the REAGs, or Regional Economic Area Groups, is available at http://wireless.fcc.gov/auctions/data/maps/REAG.pdf.

[9]Specifically, the FCC has proposed to allow a package bid on the eight licenses covering the 50 U.S. states, excluding the four licenses covering Puerto Rico, the U.S. Virgin Islands, the Gulf of Mexico, and the U.S. Pacific territories. See FCC Public Notice (DA 07-3415), paragraph 23.

wide license.[10] The E-block licenses are 6 MHz licenses defined over the 176 EAs. Associated with each block will be a reserve price. The FCC has proposed block-specific aggregate reserve prices of: Block A, $1.81 billion; Block B, $1.38 billion; Block C, $4.64 billion; Block D, $1.33 billion; Block E, $0.90 billion,[11] and stated that "Because of the value-enhancing propagation characteristics and relatively unencumbered nature of the 700 MHz Band spectrum, we believe these are conservative estimates."[12]

For blocks A, B, C and E, the FCC ordered that significant performance requirements be attached to the licenses. However, if the reserve price for a block were not met, the FCC ordered that the block be re-auctioned with less stringent requirements, at the same reserve price. As described in the service rules order for the auction,[13] the performance requirements include the use of interim and end-of-term benchmarks, with geographic area benchmarks for licenses based on CMAs and EAs,[14] and population benchmarks for licenses based on REAGs.[15] Failure to meet the performance requirements can result in a reduction in the license term, forfeiture of a license, or the loss of authorization for unserved portions of the license area.[16]

In addition, for the C-block licenses, the FCC "will require licensees to allow customers, device manufacturers, third-party application developers, and others to use or develop the devices and applications of their choice, subject to certain conditions."[17] The FCC views this requirement of open platforms for devices and applications as being for the benefit of consumers.[18]

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[10]The D-block license is subject to conditions relating to a public/private partnership, and will not be the focus of our analysis.

[11]FCC Public Notice (DA 07-3415), paragraph 53.

[12]FCC Public Notice (DA 07-3415), paragraph 54.

[13]Second Report and Order (FCC 07-132), paragraph 153.

[14]Specifically, paragraph 157 states that "licensees must provide signal coverage and offer service to: (1) at least 35 percent of the geographic area of their license within four years of the end of the DTV transition, and (2) at least 70 percent of the geographic area of their license at the end of the license term."

[15]Specifically, paragraph 162 states that "licensees must provide signal coverage and offer service to: (1) at least 40 percent of the population of the license area within four years, and (2) at least 75 percent of the population of the license area by the end of the license term."

[16]Second Report and Order (FCC 07-132), paragraph 153.

[17]Second Report and Order (FCC 07-132), paragraph 195.

[18]Second Report and Order (FCC 07-132), paragraph 195. As stated in paragraph 198, "Although wireless broadband services have great promise, we have become increasingly concerned that certain practices in the wireless industry may constrain consumer access to wireless broadband networks and limit the services and functionalities provided to consumers by these networks." And as stated in paragraph 199, "We are also concerned that wireless service providers appear to have required that

"This auction provides a window of opportunity to have a significant effect on the next phase of mobile wireless technological innovation, and on the evolution of market and institutional arrangements—such as arrangements regarding open platforms for devices and applications to the benefit of consumers—that will go along with that innovation. As a result, in light of the evidence suggesting that wireless service providers are blocking or degrading consumer-chosen hardware and applications without an appropriate justification, we believe that it is appropriate to take a measured step to encourage additional innovation and consumer choice at this critical stage in the evolution of wireless broadband services, by removing some of the barriers that developers and handset/device manufacturers face in bringing new products to market. By fostering greater balance between device manufacturers and wireless service providers in this respect, we intend to spur the development of innovative products and services."[19]

In the event that the reserve price for the A, B, C, or E block were not met, the FCC would offer less restricted licenses "as soon as possible" after the first auction.[20] In particular, the C-block licenses would be offered without the open platform conditions.[21] According to the service order (at paragraph 307), "This will address the possibilities that license conditions adopted today significantly reduce values bidders ascribe to those licenses and/or have unanticipated negative consequences."

In the remainder of this paper, we analyze a contingent re-auction in the context of a theoretical model, and we contrast it with the optimal mechanism in an environment

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equipment manufacturers disable certain capabilities in mobile devices, such as Wi-Fi capabilities. ... Despite these technological possibilities and potential consumer advantages, wireless handsets with Wi-Fi capabilities have been largely unavailable in the United States for reasons that appear unrelated to reasonable network management or technological necessity." Paragraph 200 continues: "We have not found, however, that competition in the CMRS marketplace is ensuring that consumers drive handset and application choices, especially in the emerging wireless broadband market. For example, while it is easy for consumers to differentiate among providers by price, most consumers are unaware when carriers block or degrade applications and of the implications of such actions, thus making it difficult for providers to differentiate themselves on this score. As a result, while many commenters assert that market forces require that wireless providers support handsets and applications that consumers want, there is evidence that wireless service providers nevertheless block or degrade consumer-chosen hardware and applications without an appropriate justification."

[19]Second Report and Order (FCC 07-132), paragraph 201.

[20]Second Report and Order (FCC 07-132), paragraph 307. No definitive resolution is proposed for the D block should its reserve price not be met.

[21]Second Report and Order (FCC 07-132), paragraph 311. As discussed in paragraph 312, the band plan for the reauctioned C-block would also be modified.

with seller-benefitting restrictions, and we discuss implications for the FCC's auction design.

3   Contingent re-auction with complete information

We begin with a simple example to illustrate some of the issues that arise in a contingent re-auction. The owner of a single object can sell it in restricted or unrestricted form. There are n potential buyers. Buyer i has private values l_i and h_i for the restricted and unrestricted object respectively. We assume here that these values are common knowledge, with $h_1 > h_2 > ... > h_n$. We do not want to assume that the ranking of the values for the restricted object are the same as those for the unrestricted object, so we let $l_{(j)}$ denote j-th highest value among $l_1, ..., l_n$. For example, $l_{(1)} \equiv \max \{l_1, ..., l_n\}$. In addition to the sale revenue, the seller receives a benefit of B if and only if the object is sold in restricted form.

Thus the seller's first-best outcome entails selling the restricted object to the bidder with value $l_{(1)}$, for his value, if $B + l_{(1)} > h_1$; and selling the unrestricted object to bidder 1 for h_1 otherwise. For ease of exposition, we ignore ties. Note that this outcome is efficient.

We model the contingent re-auction as follows. The object is first put up for sale in restricted form in a second-price auction with reserve price r. If no bidder bids above r, the object is put up for sale in another second-price auction in unrestricted form and with no reserve price. It is a useful simplification to assume no reserve price at the second auction, and it guarantees that the object is sold, which is realistic in environments in which the seller cannot commit to retain the object.

As a preliminary observation, note that, in any equilibrium with undominated strategies, if the second auction takes place, bidder 1 wins the unrestricted object and earns $h_1 - h_2$. Now consider the first auction. The object remains unsold if the reserve exceeds all bidders' values. If instead $r \leq l_{(1)}$, the outcome depends on whether bidder 1 is also the bidder with the highest value for the restricted object: if $l_1 < l_{(1)} = l_j$ for $j \neq 1$, then bidder j wins the restricted object and pays $\max\{r, l_{(2)}\}$;

if instead $l_{(1)} = l_1$, then bidder 1 bids

$$b_1 = \begin{cases} 0, & \text{if } l_{(2)} < r \text{ and } l_1 - r < h_1 - h_2 \\ l_1, & \text{otherwise.} \end{cases}$$

To see this, note that when $l_{(2)} < r$, bidder 1 has no competition in the first auction, and thus can choose between buying the restricted object for r and forcing the second auction for the unrestricted object by bidding zero. Thus bidder 1 suppresses his bid in the first auction whenever he has the highest value for the restricted object $(l_1 = l_{(1)})$, faces no competition for the unrestricted object $(l_{(2)} < r < l_{(1)})$, and earns more by winning the unrestricted object at price h_2 than by winning the restricted object at price r $(h_1 - h_2 > l_1 - r)$.

Proposition 1 *In the complete information case, the unrestricted object is sold in the second auction if $l_{(1)} < r$ or $l_{(2)} - r < 0 < l_1 - r < h_1 - h_2$. Otherwise, the restricted object is sold in the first auction. The optimal reserve price for the seller is*

$$r = \begin{cases} l_{(1)} - (h_1 - h_2), & \text{if } l_1 = l_{(1)} \text{ and } h_2 \leq B + \max \left\{ l_{(2)}, l_{(1)} - (h_1 - h_2) \right\} \\ l_{(1)}, & \text{if } l_1 \neq l_{(1)} \text{ and } h_2 \leq B + l_{(1)} \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. The first part of the proposition is proven in the text. To determine the optimal reserve price, note that the seller's surplus is h_2 if $l_{(1)} < r$ or $l_{(2)} - r < 0 < l_1 - r < h_1 - h_2$ and $\max\{l_{(2)}, r\} + B$ otherwise. $\square$

An implication of Proposition 1, is that for some parameter values, demand reduction reduces the seller's surplus relative to what it would be if the bidders bid truthfully in each of the two second-price auctions. Under truthful bidding, the optimal reserve price is $l_{(1)}$ if $h_2 \leq B + l_{(1)}$ and ∞ otherwise, so seller surplus is $\max \left\{ B + l_{(1)}, h_2 \right\}$. But if $B + l_{(1)} > h_2$ and $l_1 = l_{(1)}$, then demand reduction in the equilibrium bids results in seller surplus of only $B + \max \left\{ l_{(2)}, l_{(1)} - (h_1 - h_2) \right\}$, which is less than $B + l_{(1)}$.

It is worth pointing out that the demand reduction effect in the contingent re-auction is different in nature from the one described by Ausubel and Cramton (2002), who consider auctions of multiple objects. In their environment, demand reduction

has a collusive flavor, requiring that all bidders reduce their demand and buy fewer objects in order to pay a lower price. In our case, there is only one object for sale and demand reduction occurs only when there is a bidder who is effectively a monopsonist for the restricted object and thus can refuse to buy the restricted object in order to force the seller to re-auction the unrestricted object.

Both the demand reduction discussed above and the presence of a positive reserve price at the first auction can cause the outcome of the contingent re-auction to be inefficient. The seller's only instrument for increasing revenue, the reserve price at the first auction, creates an incentive for demand reduction since there is then a positive probability that the second auction will occur and it can induce an inefficient allocation of the object. Nevertheless, the optimal contingent re-auction may have a positive reserve price.

We explore these issues in a more general model in the remainder of this paper.

4   Environment with incomplete information

We now turn to the general model with privately known values. We maintain the assumption that the seller has a single object that can be offered either in restricted or unrestricted form, with an extra benefit $B > 0$ for the seller if the restricted version is sold. We let $N \equiv \{1, ..., n\}$ denote the set of bidders. For all $i \in N$, bidder i has private values $\theta_i = (l_i, h_i)$ that are obtained as the realization of a random variable $\widetilde{\theta}_i$ having cumulative distribution function $F(l, h)$ and differentiable density $f(l, h)$. The random variables $\{\widetilde{\theta}_i, \ldots, \widetilde{\theta}_n\}$ are independent, and identically distributed, with support

$$\Theta_i \equiv \left\{ (l, h) \in \mathbb{R}_+^2 \mid l \in [\underline{l}, \bar{l}], \ h \in [\underline{h}, \bar{h}], \ l \leq h \right\}.$$

Define $\Theta \equiv \times_{i \in N} \Theta_i$. We let $f(l \mid h)$ denote the conditional density of l given h and let $F(l \mid h)$ denote the cumulative distribution.

5   Equilibria of the contingent re-auction

In this section, we characterize and establish existence of the equilibria of the contingent re-auction in the general model with two-dimensional types.

To model a contingent re-auction, assume the seller first offers the restricted object

for sale in a second-price auction with reserve price r and then, if the reserve is not met, offers the unrestricted object for sale in a second-price auction with no reserve price. As we show, the analysis changes little if instead an English auction is used at each stage. For now, we leave the possibility of a reserve price at the second auction for future research.

We start by observing that in any equilibrium with undominated strategies all bidders bid truthfully in the second auction, i.e., bidder i bids h_i. In light of this fact, we can establish the optimal strategy in the first stage. Essentially, for each value h_i there will be a threshold value $g_i(h_i)$ such that types with $l_i < g(h_i)$ will prefer to suppress their bids and wait for the second auction, while types with $l_i > g_i(h_i)$ will bid above the reserve price in the first auction. The next proposition makes this precise.

Proposition 2 *All the perfect Bayesian equilibria in undominated strategies of the contingent re-auction game have the following structure: for each bidder $i \in N$, there is a nondecreasing function $g_i : [\underline{h}, \bar{h}] \rightarrow [r, \bar{l}]$ such that the bid function in the first auction is given by $b_i(l_i, h_i) = \begin{cases} 0, & \text{if } l_i < g_i(h_i) \\ l_i, & \text{if } l_i \geq g_i(h_i), \end{cases}$ and in the second auction, type (l_i, h_i) bids h_i.*

Proof. See Appendix A.

As an example, for the case with two symmetric bidders drawing their values (l, h) from the uniform distribution on $[0, 1] \times [3, 4]$, one can calculate numerically the functions g_i for various values of the reserve price (this calculation is not trivial—see Appendix E). The functions g_i for six possible reserve prices are shown in Figure 1.

The next result proves that there is no equilibrium in which the bidders bid truthfully in the first auction, where by 'truthfully' we mean bidding l_i whenever $l_i \geq r$.

Proposition 3 *In every equilibrium, there is an open set of types (l_i, h_i) with $l_i > r$ who bid $b_i(l_i, h_i) = 0$ in the first auction.*
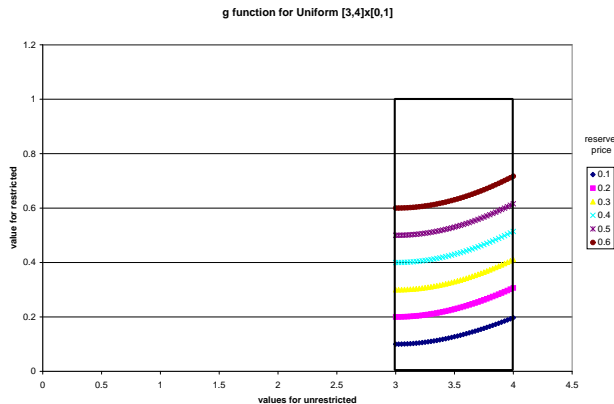
Proof. See Appendix A.

Figure 1: Function g_i for the case of two symmetric bidders drawing values (l_i, h_i) from the uniform distribution on $[0, 1] \times [3, 4]$ for six possible values of the reserve price

The intuition for Proposition 3 is straightforward: in order for a truthful equilibrium to exist, it must be that all bidders with l_i greater than r bid l_i at the first auction. But if l_i is close to r, bidder i's expected surplus from the first auction is small relative to his expected surplus at the second auction. Thus, if l_i is close to r, bidder i prefers to bid zero, and so a truthful bidding equilibrium does not exist.

Proposition 3 implies that if the seller wants the object to be sold for values l_i above a threshold $\hat{l}$, then the reserve price should be set at a level strictly less than $\hat{l}$.

Proposition 3 also implies that in every equilibrium there is 'excessive delay' in the sense that the second auction is reached with a probability strictly higher than $\Pr(\max_{i \in N} l_i \leq r)$.

Corollary 1 *Every equilibrium involves excessive delay due to non-truthful bidding in the first auction.*

The results proved so far show that all equilibria of the contingent re-auction game have a relatively simple structure and that 'excessive delay' is part of every equilibrium. It is worth pointing out that the results of Proposition 3 hold for any reserve price $r > l$. In fact, there is a sort of 'multiplier effect' similar to the one discussed in Brusco and Lopomo (2007). When the reservation price is $r > \underline{l}$ all types with $l_i \in [\underline{l}, r]$ do not bid in the first auction, thus making the probability of reaching the second auction strictly positive. But this implies that types with $l_i = r + \varepsilon$ also

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prefer to delay if ε is sufficiently small, since the expected gain in the first auction is small. In turn, this increases the probability that the second auction is reached, thus potentially convincing other types to delay the bid. What happens when $r \to \underline{l}$ depends on the distribution of types, but using arguments similar to the ones in Brusco and Lopomo (2007) it is possible to produce examples in which

$$\lim_{r \to \underline{l}} \Pr \left(\text{second auction is reached} \mid r \right) > 0.$$

Since at $r = \underline{l}$ there is no delay. The implication is that, under some conditions the imposition of even a minimal reserve price may produce delays.

Up to now, we have not established that an equilibrium exists. We therefore complete now the analysis by showing the existence of a symmetric equilibrium.

Proposition 4 *A symmetric perfect Bayesian equilibrium of the contingent re-auction game exists.*

Proof. See Appendix B.

We now characterize when a contingent re-auction produces a higher expected social welfare, given by the sum of expected revenue and the social benefit B, than a simple one-shot auction either for the restricted or for the unrestricted object. See Section 7 for comparisons with the exclusive buyer mechanism.

5.1 Comparison with auctioning the unrestricted object

As a necessary condition for a contingent re-auction to maximize the seller's expected surplus, we must have

$$E\left[l_{(1)}\right] + B > E\left[h_{(2)}\right]. \tag{1}$$

If condition (1) is not satisfied then the seller prefers to sell the unrestricted object and forfeit the social benefit B because the revenue loss from selling the restricted object is too large. Considering a slightly different condition, if

$$E\left[l_{(2)}\right] + B > E\left[h_{(2)}\right], \tag{2}$$

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then then there is always a contingent re-auction that does better than a single auction of the unrestricted object. To see this, note that when (2) is satisfied, a contingent re-auction with a reserve price of zero has higher expected surplus for the seller than a single auction for the unrestricted object. Thus, if the reserve price is chosen optimally, the contingent re-auction must do better.

Proposition 5 *Comparing a contingent re-auction with a single auction for the unrestricted object, the seller prefers a contingent re-auction if $B > E\left[h_{(2)}\right] - E\left[l_{(2)}\right]$, but not if $B < E\left[h_{(2)}\right] - E\left[l_{(1)}\right]$. If $E\left[h_{(2)}\right] - E\left[l_{(1)}\right] \leq B \leq E\left[h_{(2)}\right] - E\left[l_{(2)}\right]$, the preference depends on the distribution of types.*

5.2   Comparison with auctioning the restricted object

If the restricted object is sold using a single auction with an optimal reserve price r_D, the expected surplus to the seller is

$$S_D\left(r_D\right) = \Pr\left(l_{(1)} > r_D\right)\left(E\left[\max\left\{l_{(2)}, r_D\right\} \mid l_{(1)} > r_D\right] + B\right).$$

For a contingent re-auction with reserve price r_C let $N\left(r_C\right)$ be the event

$$N\left(r_C\right) = \left\{l_i > g_{r_C}\left(h_i\right) \text{ at least one } i\right\},$$

where g_{r_C} is the function describing the symmetric equilibrium when the reserve price is r_C, and $\neg N\left(rc\right)$ the negation of the event. Thus, $\neg N\left(rc\right)$ is the event in which no bids are submitted at the first auction of a contingent re-auction. Then the expected surplus to the seller from a contingent re-auction is

$$S_C\left(r_C\right) = Pr\left(N\left(r_C\right)\right)\left(E\left[s\left(l, h, r_C\right) \mid N\left(r_C\right)\right] + B\right) + \Pr\left(\neg N\left(rc\right)\right)\left(E\left[h_{(2)} \mid \neg N\left(r_C\right)\right]\right),$$

where $s\left(l, h, r_C\right)$ is the revenue in the first auction, which depends on the vectors of l's and h's as well as on the reserve price.

The comparison in this case is complicated because the introduction of the second auction changes equilibrium behavior in the first auction in important ways. On the one hand, if the object is not sold at the first auction, the seller can still secure some revenue in the second. On the other hand, the existence of the second auction makes it more likely that the first auction will fail. The details of the equilibrium,

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as described by the function g, cannot be ignored and, in general, it is impossible to determine which one among $S_D(r_D)$ and $S_C(r_C)$ is higher.

However, we can say that a contingent re-auction does better when the number of bidders is sufficiently large. We know that in this case the equilibrium approaches truthful behavior, with all bidders with $l_i > r_C$ bidding in the first auction. The second effect (bidders are more reluctant to bid in the first auction) therefore vanishes.

Proposition 6 *Comparing a contingent re-auction with a single auction for the restricted object, the seller prefers a contingent re-auction if the number of bidders is sufficiently large.*

6 Optimal mechanism with one-dimensional types

In this section, we derive the optimal mechanism under an assumption that reduces the type space to one dimension. We assume a deterministic mapping from a buyer's value for the unrestricted object to his value for the restricted object so that buyers' types become one dimensional. In Section 7, we consider the case of two-dimensional types.

For the purposes of this section, assume that buyer i's value for the restricted object is a deterministic function of its value for the unrestricted object, $l_i = \lambda_i(h_i)$. This assumption reduces the type space to a single dimension. The type space for buyer i is $\Theta_i \equiv \left\{ (h_i, l_i) \in \mathbb{R}^2 \mid \underline{h} \leq h_i \leq \overline{h},\ l_i = \lambda_i(h_i) \right\}$. As before, let $\Theta \equiv \times_{i \in N} \Theta_i$.

We consider incentive compatible mechanisms in which buyers report their values for the unrestricted object, and the mechanism assigns either the restricted or unrestricted object to a buyer and requires payments.

Let $q_i^H : \Theta \to [0,1]$ be the probability that buyer i wins the unrestricted object as a function of the buyers' reports, and let $q_i^L : \Theta \to [0,1]$ be the probability that buyer i wins the restricted object as a function of the buyers' reports. Let $m_i : \Theta \to \mathbb{R}$ be buyer i's payment as a function of the buyers' reports.

Define $\hat{\Theta}_i \equiv \left\{ h_i \in \mathbb{R} \mid \underline{h} \leq h_i \leq \overline{h} \right\}$ and $\hat{\Theta} \equiv \times_{i \in N} \hat{\Theta}_i$. Buyer i's ex-post expected surplus is

$$u_i(h_i, h_{-i}) = h_i\, q_i^H(h_i, h_{-i}) + \lambda_i(h_i)\, q_i^L(h_i, h_{-i}) - m_i(h_i, h_{-i}), \qquad (3)$$

and the seller's expected surplus is

$$\int_{\hat{\Theta}} \sum_{i=1}^{n} \left(m_i\left(h_i, h_{-i}\right) + B \; q_i^L\left(h_i, h_{-i}\right) \right) dF\left(h\right).$$

In what follows, we use capital letters denote interim expected quantities, i.e., $Q_i^H\left(h_i\right) \equiv \int_{\hat{\Theta}_{-i}} q_i^H\left(h_i, y\right) dF_{-i}\left(y\right)$, $Q_i^L\left(h_i\right) \equiv \int_{\hat{\Theta}_{-i}} q_i^L\left(h_i, y\right) dF_{-i}\left(y\right)$, $M_i\left(h_i\right) \equiv \int_{\hat{\Theta}_{-i}} m_i\left(h_i, y\right) dF_{-i}\left(y\right)$, and $U_i(h_i) \equiv \int_{\hat{\Theta}_{-i}} u_i\left(h_i, y\right) dF_{-i}\left(y\right)$. Note that using (3),

$$U_i(h_i) = h_i \; Q_i^H\left(h_i\right) + \lambda_i\left(h_i\right) \; Q_i^L\left(h_i\right) - M_i\left(h_i\right). \tag{4}$$

Also define

$$\hat{U}_i\left(h_i', h_i\right) \equiv h_i \; Q_i^H\left(h_i'\right) + \lambda_i\left(h_i\right) \; Q_i^L\left(h_i'\right) - M_i\left(h_i'\right).$$

A mechanism $\left(q_i^H, q_i^L, m_i\right)_{i\in N}$ satisfies interim incentive compatibility for buyer i, if $\forall h_i, h_i'$,

$$\hat{U}_i\left(h_i, h_i\right) \geq \hat{U}_i\left(h_i', h_i\right). \tag{5}$$

The following lemma is standard in Mechanism Design.

Lemma 1 *A mechanism $\left(q_i^H, q_i^L, m_i\right)_{i\in N}$ satisfies interim incentive compatibility for buyer i, if and only if it satisfies the following 'envelope' and 'monotonicity' conditions:*

$$U_i(h_i) = U_i\left(\underline{h}\right) + \int_{\underline{h}}^{h_i} \left(Q_i^H\left(x\right) + \lambda_i'\left(x\right) \; Q_i^L\left(x\right) \right) dx. \tag{6}$$

and

$$x \geq x' \quad \Rightarrow \quad Q_i^H\left(x\right) + \lambda_i'\left(x\right) \; Q_i^L\left(x\right) \geq Q_i^H\left(x'\right) + \lambda_i'\left(x'\right) \; Q_i^L\left(x'\right). \tag{7}$$

Given Lemma 1, buyer i's *ex ante* expected surplus can be written as

$$
\begin{aligned}
\int_{\underline{h}}^{\bar{h}} U_i(h_i) dF_i\left(h_i\right) &= U_i\left(\underline{h}\right) + \int_{\underline{h}}^{\bar{h}} \left(\int_{\underline{h}}^{h_i} \left(Q_i^H\left(x\right) + \lambda_i'\left(x\right) Q_i^L\left(x\right) \right) dx \right) dF_i\left(h_i\right) \\
&= U_i\left(\underline{h}\right) + \int_{\underline{h}}^{\bar{h}} \left(1 - F_i\left(h_i\right)\right) \left(Q_i^H\left(h_i\right) + \lambda_i'\left(h_i\right) Q_i^L\left(h_i\right) \right) dh_i,
\end{aligned} \tag{8}
$$

where the first equality uses (6), and the second uses integration by parts.

Defining virtual valuations

$$v_i^H\left(h_i\right) \equiv h_i - \frac{1 - F_i\left(h_i\right)}{f_i\left(h_i\right)}$$

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and

$$v_i^L (h_i) \equiv B + \lambda_i (h_i) - \frac{1 - F_i (h_i)}{f_i (h_i)} \lambda_i' (h_i),$$

we have the following result.

Lemma 2 *The seller's expected benefit generated ex ante by buyer i is*

$$\int_{\underline{h}}^{\bar{h}} \left(v_i^H (h_i) \ Q_i^H (h_i) + v_i^L (h_i) \ Q_i^L (h_i) \right) dF_i (h_i) - U_i (\underline{h}).$$

Proof. See Appendix C.

Finally, summing over all buyers, 'unpacking' the interim Q's, and assuming the lowest type has zero interim expected surplus so that for all $i \in N$, $U_i (\underline{h}) = 0$, we can write the seller's objective function as

$$\int_{\hat{\Theta}} \sum_{i=1}^{n} \left(m_i (h_i, h_{-i}) + B \ q_i^L (h_i, h_{-i}) \right) dF (h)$$
$$= \int_{\hat{\Theta}} \sum_{i=1}^{n} \left(v_i^H (h_i) \ q_i^H (h) + v_i^L (h_i) \ q_i^L (h) \right) dF (h).$$

Maximizing pointwise, we find that the license should be given to the buyer with the highest virtual valuation:

$$q_i^H (h) = \chi_i^H (h) \equiv \begin{cases} 1, & \text{if } v_i^H (h_i) > \max\{0, \max_{j \neq i} v_j^H (h_j), \max_j v_j^L (h_j)\} \\ 0, & \text{otherwise}; \end{cases} \quad (9)$$

$$q_i^L (h) = \chi_i^L (h) \equiv \begin{cases} 1, & \text{if } v_i^L (h_i) > \max\{0, \max_{j \neq i} v_j^L (h_j), \max_j v_j^H (h_j)\} \\ 0, & \text{otherwise}. \end{cases} \quad (10)$$

The functions $\left(\chi_i^H, \chi_i^L \right)$, together with any payment functions m_i that satisfy the envelope condition in (6), i.e.,

$$M_i (h_i) = h_i \ Q_i^H (h_i) + \lambda_i (h_i) \ Q_i^L (h_i) - \int_{\underline{h}_i}^{h_i} \left(Q_i^H (x) + \lambda_i' (x) \ Q_i^L (x) \right) dx, \quad (11)$$

constitute a solution if and only if the function

$$A_i (h_i) \equiv \int_{\Theta_{-i}} \left[\chi_i^H (h) + \lambda_i' (h_i) \chi_i^L (h) \right] dF_{-i} (h_{-i}) \quad (12)$$

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is nondecreasing for all $i \in N$. Otherwise we need to "iron" (see Myerson, 1981). In the next proposition, we establish that under mild conditions on F_i and λ_i there exists a mechanism that is ex-post incentive compatible and ex-post individually rational and that maximizes the seller's surplus among all interim incentive compatible and interim individually rational mechanisms. More specifically, the conditions (13) and (14) given in Proposition 7, which we maintain for the remainder of the paper, guarantee that the ex-post assignment functions χ_i^H and χ_i^L defined in (9) and (10) are such that, for all $i \in N$, $a_i(h_i, h_{-i}) \equiv \chi_i^H(h_i, h_{-i}) + \lambda_i'(h_i) \chi_i^L(h_i, h_{-i})$ is nondecreasing in h_i for all h_{-i}, guaranteeing that $A_i(h_i)$ is nondecreasing. For the purposes of Proposition 7, it is useful to define $\rho_i(h_i) \equiv \frac{1 - F_i(h_i)}{f_i(h_i)}$.

Proposition 7 *Suppose that for each $i \in N$, the distribution F_i is regular, i.e.,*

$$1 - \rho_i'(h_i) \geq 0, \tag{13}$$

and F_i and λ_i satisfy

$$-(1 - \rho_i'(h_i))(1 - \lambda_i'(h_i)) < \rho_i(h_i) \lambda_i''(h_i) < (1 - \rho_i'(h_i)) \lambda_i'(h_i). \tag{14}$$

Then there exist two threshold functions $h_i^0 : \widehat{\Theta}_{-i} \to \widehat{\Theta}_i$ and $h_i^1 : \widehat{\Theta}_{-i} \to \widehat{\Theta}_i$ such that $h_i^0(h_{-i}) \leq h_i^1(h_{-i})$ and $\chi_i^L(h_i, h_{-i}) = 1$ if $h_i < h_i^0(h_{-i})$ and $\chi_i^H(h_i, h_{-i}) = 1$ if $h_i^0(h_{-i}) < h_i < h_i^1(h_{-i})$. Moreover, each function A_i defined in (12) is nondecreasing. Hence the mechanism (χ, m), where

$$\begin{aligned} m_i^*(h_i, h_{-i}) \equiv \ & h_i \chi_i^H(h_i, h_{-i}) + \lambda_i(h_i) \chi_i^L(h_i, h_{-i}) \\ & - \int_{\underline{h}_i}^{h_i} \left(\chi_i^H(z, h_{-i}) + \lambda_i'(z) \chi_i^L(z, h_{-i}) \right) dz, \end{aligned} \tag{15}$$

satisfies ex-post incentive compatibility and ex-post individually rationality and maximizes the seller's surplus among all interim incentive compatible and interim individually rational mechanisms.

Proof. See Appendix C.

The next proposition addresses the efficiency of the optimal mechanism.

Proposition 8 *When the optimal mechanism allocates the object in its unrestricted (respectively restricted) form, it always allocates it to the buyer with the highest value*

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for the unrestricted (respectively restricted) object if and only if buyers are symmetric and $h - \frac{1-F(h)}{f(h)}$ (respectively $\lambda(h) - \frac{1-F(h)}{f(h)}\lambda'(h)$) is increasing in h.

As Proposition 8 shows, the optimal mechanism is not necessarily efficient. However, the corollary below gives an environment in which, conditional on the form of the object allocated, it is always allocated to the highest-valuing buyer.

Corollary 2 *Conditional on the form of the object allocated, it is always allocated to the highest-valuing buyer if buyers are symmetric, $h - \frac{1-F(h)}{f(h)}$ is increasing in h, and $\lambda'(h)$ is a constant.*

As we now show, in some environments the optimal mechanism can be interpreted as a type of second-price auction.

6.1 Example: Symmetric buyers with linear λ

Assume buyers are symmetric with $\lambda(h) = \alpha h$ for some $\alpha \in (0,1)$ and that the conditions of Proposition 7 hold.

In this case, the virtual valuations are $v^H(h) = h - \frac{1-F(h)}{f(h)}$ and $v^L(h) = B + \alpha\left(h - \frac{1-F(h)}{f(h)}\right)$. In the optimal mechanism, the object is sold in its restricted form if and only if $v^L(\max_{i \in N} h_i) > 0$ and $v^L(\max_{i \in N} h_i) > v^H(\max_{i \in N} h_i)$, i.e., if and only if

$$-\frac{B}{\alpha} < \max_{i \in N}\left(h - \frac{1-F(h)}{f(h)}\right) < \frac{B}{1-\alpha}.$$

The object is sold in its unrestricted form if and only if $v^H(\max_{i \in N} h_i) > 0$ and $v^L(\max_{i \in N} h_i) < v^H(\max_{i \in N} h_i)$, i.e., if and only if

$$\max_{i \in N}\left(h - \frac{1-F(h)}{f(h)}\right) > \frac{B}{1-\alpha}.$$

As shown by these expressions, if B is sufficiently large, the object is always sold in its restricted form, and if B is sufficiently small, then it is always sold in its unrestricted form.

Define cutoff values h^r and h^* as follows:

$$h^r \equiv \begin{cases} \underline{h}, & \text{if } \underline{h} - \frac{1}{f(\underline{h})} > -\frac{B}{\alpha} \\ h \text{ such that } h - \frac{1-F(h)}{f(h)} = -\frac{B}{\alpha}, & \text{otherwise} \end{cases}$$

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and

$$
h^* \equiv \begin{cases} \bar{h}, & \text{if } \bar{h} < \frac{B}{1-\alpha} \\ \underline{h}, & \text{if } \underline{h} - \frac{1}{f(\underline{h})} > \frac{B}{1-\alpha} \\ h \text{ such that } h - \frac{1-F(h)}{f(h)} = \frac{B}{1-\alpha}, & \text{otherwise.} \end{cases}
$$

Given our assumption that $h - \frac{1-F(h)}{f(h)}$ increasing in h, cutoff values h^r and h^* are well defined and $h^r < h^*$. Note that h^r and h^* are independent of the number of buyers n.

We can describe the optimal mechanism based on the cutoff values h^* and h^r. The proof follows from the analysis above and (11).

Proposition 9 *Order the buyers so that $h_1 \geq h_2 \geq ... \geq h_n$. In the optimal mechanism, if $h_1 < h^r$, the object is not sold; if $h^r \leq h_1 < h^*$, the object is sold in restricted form; and if $h^* \leq h_1$, the object is sold in unrestricted form. When the object is sold, it is sold to one of the buyers with the highest value, who pays*

$$
m(h_1, h_2) = \begin{cases} \alpha \max\{h^r, h_2\}, & \text{if } h^r < h_1 < h^* \\ \alpha \max\{h^r, h_2\} + (1-\alpha) \max\{h^*, h_2\}, & \text{if } h^* < h_1 \\ 0, & \text{otherwise,} \end{cases}
$$

where $y_i \equiv \max_{j \neq i} h_j$.

Proposition 9 implies that the optimal mechanism can be implemented through a modified second-price auction where bidders submit their values for the unrestricted object. If the highest bid is less than h^r, the object is not sold. If the highest bid is greater than h^r but less than h^*, the high bidder wins the restricted object and pays $\alpha \max\{h^r, y\}$, where y is the second-highest bid. If the highest bid is $h_i > h^*$, the high bidder wins the unrestricted object and pays $\alpha \max\{h^r, y\} + (1-\alpha)h^*$ if the second-highest bid is less than h^* and pays the second-highest bid if the second-highest bid is greater than h^*. Note that a high bidder with value h^* is indifferent between receiving the restricted object, in which case its surplus is $\alpha h^* - \alpha \max\{h^r, y\}$, and the unrestricted object, in which case its surplus is $h^* - \alpha \max\{h^r, y\} - (1-\alpha)h^* = \alpha h^* - \alpha \max\{h^r, y\}$. By the usual second-price logic, it is optimal for buyers to truthfully report their values for the unrestricted object. However, this auction differs from the usual second-price auction in that a bidder's bid can affect the quality of the object the bidder receives.

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One can show that if h^r and h^* have interior values, then the efficiency loss associated with the optimal mechanism is

$$\int_{\underline{h}}^{h^r} (\alpha x + B)\, dF^n(x) + \int_{\frac{B}{1-\alpha}}^{h^*} ((1-\alpha)x - B)\, dF^n(x). \tag{16}$$

The first term in (16) is the efficiency loss associated with not allocated the restricted object when the highest valuation for the restricted object is less than h^r, and the second term is the efficiently loss from allocating the restricted object when efficiency requires that the unrestricted object be allocated.

6.2 Continuation of example: uniform distribution

If we specialize the above example further to the case with $F(x) = x$, i.e., values for the unrestricted object are drawn from the uniform distribution on $[0, 1]$, then $h^r = \max\left\{\frac{\alpha - B}{2\alpha}, 0\right\}$ and $h^* = \min\left\{\frac{B+1-\alpha}{2(1-\alpha)}, 1\right\}$. Thus, if $\alpha = 1$, which implies that firms value the restricted and unrestricted object equally, then $h^* = 1$, so if the object is allocated, it is allocated in its restricted form. If $\alpha = 1$, then as B approaches zero, the second-price auction that implements the optimal mechanism reduces to a second-price auction with reserve price $\frac{1}{2}$ for the restricted object.

For comparison, in this case the reserve price in the optimal contingent re-auction is either zero or one, so the optimal contingent re-auction reduces to either a single auction for the restricted object or a single auction for the unrestricted object. To see this, note that in the contingent re-auction with reserve price r, in the auction for the restricted object buyer i bids zero if $h_i < \frac{2r}{2\alpha - 1}$,[22] so (assuming $\frac{2r}{2\alpha - 1} \in [0, 1]$) the restricted object is allocated with probability $1 - \left(\frac{2r}{2\alpha - 1}\right)^2$ and expected seller surplus is

$$r\int_{\beta r}^{1}\int_{0}^{\beta r} 2dxdy + \int_{\beta r}^{1}\int_{\beta r}^{y} \alpha x 2dxdy + \int_{0}^{\beta r}\int_{0}^{y} x 2dxdy + B\left(1 - (\beta r)^2\right),$$

which one can show is is convex at any internal optimum, implying that the maximum occurs at the boundary.

Figure 2 below is coarse, but it shows h^r and h^* for different values of B, shown across the horizontal axis, and different values of α, as indicated by the legend. Thus, for each value of α, there are two curves, the lower one showing h^r as a function of

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[22]The cutoff is g such that $(\alpha g - r)F^{n-1}(g) = \int_0^g F^{n-1}(y)dy$.

B and the upper one showing h^* as a function of B. If the highest value among the bidders is below the lower curve, the object is not allocated. If it is between the curves, the object is allocated in restricted form. And if it is above the upper curve, the object is allocated in unrestricted form. As shown in the figure, as B increases, h^r decreases and h^* increases so the range of values for which the object is allocated in restricted form increases.
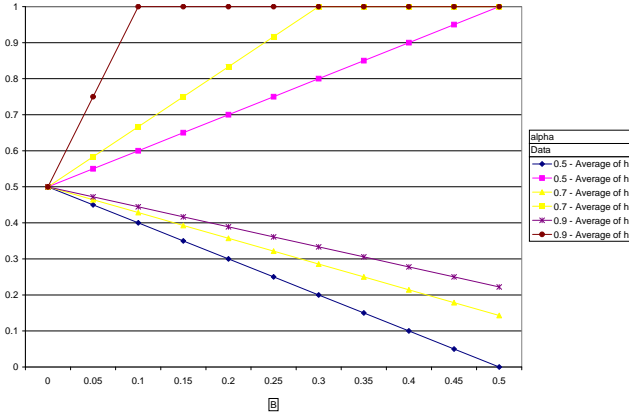

Figure 2: Curves h^r and h^* as a function of B. For high values below the lower curve, the object is not allocated. For high values between the curves, the object is allocated in restricted form. For high values above the upper curve, the object is allocated in unrestricted form.

In this example, a bidder's expected payment is

$$E[M(h_i)] = \frac{n-1}{n(n+1)} + \frac{1-\alpha}{n}(h^*)^n + \frac{\alpha}{n}(h^r)^n - \frac{2(1-\alpha)}{n+1}(h^*)^{n+1} - \frac{2\alpha}{n+1}(h^r)^{n+1}.$$

Figure 3 shows the seller's expected revenue as a function of B, shown on the horizontal axis, and α, as given by the legend. The seller's expected revenue, as a function of B, has a kink corresponding to the point where, in expectation, the seller would allocate the unrestricted rather than the restricted object. As shown in the figure, expected revenue decreases with B because as B increases the optimal mechanism is more likely to allocated the object in restricted form.

Total expected surplus to the seller is $nE[M(h_i)] + B\Pr\left(h^r < \max_{i \in N} h_i < h^*\right)$, which is shown in Figure 4 as a function of B, shown on the horizontal axis, and α, as given by the legend. As shown in the figure, the seller's total expected surplus is
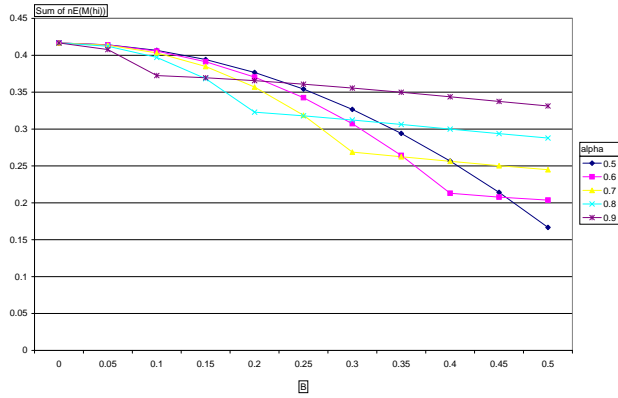
Figure 3: Seller's expected revenue from the optimal mechanism as a function of B
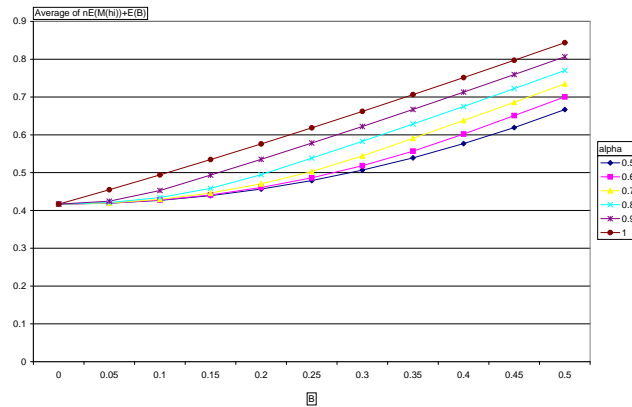
increasing in B.

Figure 4: Total expected surplus to the seller, including expected revenue and the expected benefit from the sale of the restricted object, as a function of B

We summarize the results for this example in the following proposition.

Proposition 10 *If $B > 1 - \alpha$, the optimal mechanism is a second-price auction for the restricted object (with a reserve price if $B < \alpha$). If $B < 1 - \alpha$, the optimal mechanism is the auction described in Proposition* 9. *It is never optimal to auction the unrestricted object (unless $B = 0$ and $\alpha = 1$).*

This last point of Proposition 10 is important because it says that if there is some restriction that is socially valuable, then it is never optimal to auction the unrestricted

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object. The optimal mechanism is always either an auction for the restricted object or the mechanism of Proposition 9, which allows the possibility that the restricted or unrestricted object could be allocated.

To compare the results of the optimal mechanism with the contingent re-auction, we have done some calculations for the case with two bidders, $\alpha = \frac{3}{4}$ and $B = 0.1$. In this case, the optimal reserve price in the contingent re-auction is $r = 0$, so the object is always sold in restricted form. In the optimal mechanism, $h^r = 0.433$ and $h^* = 0.7$. Table 1 provides some preliminary comparisons.

Table 1: Comparisons of mechanisms for two symmetric buyers
with h_i uniform on $[0, 1]$, $l_i = \frac{3}{4} h_i$, and $B = 0.1$

	probability restricted object is sold	probability unrestricted object is sold	probability of no sale	expected seller revenue	expected seller surplus	expected bidder surplus	expected total surplus
optimal contingent re-auction ($r = 0$)	1	0	0	0.250	0.350	0.250	0.600
optimal mechanism s.t. must sell ($h^r = 0$, $h^* = 0.7$)	0.490	0.510	0	0.342	0.391	0.268	0.659
change relative to optimal contingent re-auction				36.6%	11.6%	7.2%	9.8%
optimal mechanism ($h^r = 0.433$, $h^* = 0.7$)	0.302	0.510	0.188	0.401	0.431	0.168	0.599
change relative to optimal contingent re-auction				60.4%	23.2%	-32.9%	-0.2%
most efficient cont re-auction ($r = 1$)	0	1	0	0.333	0.333	0.333	0.667
change relative to optimal contingent re-auction				33.2%	-4.9%	33.2%	11.2%
first-best	0.160	0.840	0				0.672
change relative to optimal contingent re-auction							-81.4%

As shown in Table 1, the expected seller surplus is 23.2% higher in the optimal mechanism versus the optimal contingent re-auction. The improvement is only 11.6% if we require that the seller never retain the object. The optimal mechanism is slightly less efficient that the optimal contingent re-auction (0.2% lower expected total surplus). The table also shows that the optimal contingent re-auction allocates the object in restricted form far more often than in the first-best (with probability 1 versus 0.16). Finally, the table shows that the efficiency of the optimal mechanism subject to the seller never retaining the object and the efficiency of the most efficient contingent re-auction, which has $r = 1$, is close to the first-best level.

In work in progress, we are trying to quantify the impact of assumptions such as those made in this Section that reduce the type space to a single dimension. This

may give us a better understanding of when it is and is not appropriate to focus on one-dimensional environments in generally multi-dimensional problems.

7 Exclusive buyer mechanism

In this section, we describe the exclusive buyer mechanism. In the exclusive buyer mechanism, buyers bid for the right to be the sole buyer to choose between purchasing either the restricted or unrestricted object at fixed incremental prices. We show that there exists an exclusive buyer mechanism that produces the efficient outcome.[23] In addition, we conjecture that there is an exclusive buyer mechanism that is an optimal mechanism, at least within the class of dominant strategy mechanisms that never retain the object. The formal demonstration of this is work in progress.

In work in progress, we are considering the relation between our exclusive buyer mechanism and the optimal mechanism of Myerson (1981), which can also be implemented as an exclusive buyer mechanism.

7.1 Efficient mechanism

The efficient allocation has $q_i^H = 1$ if $h_i > \max\left\{\max_{j \neq i} h_j, \max_j B + l_j\right\}$ and $q_i^L = 1$ if $B + l_i > \max\{\max_j h_j, \max_{j \neq i} B + l_j\}$. This can be implemented by an exclusive buyer mechanism in which there is a second-price auction with no reserve price for the right to face the choice between the restricted object for no incremental payment and the unrestricted object for an incremental payment of B. (Equivalently, given the assumption of private values, an ascending-bid auction, including the simultaneous multiple round format of the FCC, can be used.)

To see this, note that bidder i has value $\max\{l_i, h_i - B\}$ from winning the second-price auction. Thus, the winner of that auction will be the buyer with the maximal value of $\max\{l_i, h_i - B\}$. If buyer i is the winning bidder and $l_i > h_i - B$, then efficiency requires that buyer i receive the object in restricted form. If buyer i is the winning bidder and $l_i < h_i - B$, then efficiency requires that buyer i receive the object in unrestricted form. With an incremental price of B for the unrestricted object in the second stage of the exclusive buyer mechanism, this is the outcome.

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[23]Regarding efficiency in environments with budget balance constraints, see Williams (1999) and Krishna (2002, Chapter 5).

Proposition 11 *The efficient outcome can be implemented with an exclusive buyer mechanism using a second-price auction with no reserve price in the first stage, and in the second stage incremental payment of zero for the restricted object and B for the unrestricted object.*

The efficient mechanism described in Proposition 11 is (ex post) outcome equivalent to a VCG mechanism (see Vickrey, 1961; Clarke, 1971; Groves, 1973; and Green and Laffont, 1977), but it is an indirect mechanism that can be implemented with a simple auction. Implementation of the efficient mechanism requires only that the mechanism designer know B. In addition, this mechanism can accommodate arbitrarily many possible restrictions.

The FCC has experience offering bidding credits (refunds) to certain bidders, so it may be worth noting that the efficient outcome can also be achieved using an exclusive buyer mechanism with a price of $-B$ (i.e., a refund of B) for the restricted object and a price of zero for the unrestricted object. In this case, the minimum bid in the initial auction could be set at B. Furthermore, the FCC's existing auction software should be able to accommodate exclusive buyer mechanisms being offered simultaneously for multiple licenses using their simultaneous multiple round auction format, although with multiple licenses and externalities across licenses the outcome need not be efficient.

7.2 Optimal mechanism

We develop the optimal mechanism in two steps: first, we solve the seller's expected surplus maximization problem for the case of a single buyer, and then we show that auctioning the right to be the (sole) participant in this mechanism is optimal among all interim incentive compatible and interim individually rational mechanisms.

Suppose the seller faces a single buyer. To define the optimal mechanism in the present environment, we require an assignment functions $q^H(\theta)$ and $q^L(\theta)$ and payment function $m(\theta)$ (since we have only one buyer, interim and ex-post quantities coincide, so we use small letters and drop all buyer's subscripts). We assume q^H, q^L, and m are twice continuously differentiable.

Since there is only one object for sale, the assignment function must satisfy the

constraint that the object be sold only once, $\forall\, (h, l) \in \Theta$,

$$q^H(h, l) + q^L(h, l) \leq 1. \tag{17}$$

Incentive compatibility requires, $\forall\, (h, l), (h', l') \in \Theta$,

$$h\, q^H(h', l') + l\, q^L(h', l') - m(h', l') \leq h\, q^H(h, l) + l\, q^L(h, l) - m(h, l), \tag{18}$$

and individual rationality requires, $\forall\, (h, l) \in \Theta$,

$$0 \leq h\, q^H(h, l) + l\, q^L(h, l) - m(h, l). \tag{19}$$

In this environment, the buyer's ex-post expected surplus is

$$u(h, l) = \max_{h', l'} \left\{ h\, q^H(h', l') + l\, q^L(h', l') - m(h', l') \right\}.$$

Lemma 3 *All constraints in (19) are satisfied iff $u(\underline{h}, \underline{l}) \geq 0$.*

Proof. See Appendix D.

In what follows, we set $u(\underline{h}, \underline{l}) = 0$, as this is optimal for the seller. Thus, the seller's problem can be stated as

$$\max_{q^H, q^L, m} \int_\Theta \left(m(h, l) + Bq^L(h, l) \right) dF(h, l),$$

subject to (17) and (18).

As we now show, the optimal one-bidder mechanism is characterized by a threshold δ^* such that bidders with $h - l < \delta^*$ choose the restricted object and pay incremental cost $\underline{l}$ and those with $h - l > \delta^*$ choose the unrestricted object and pay incremental cost δ^*.

We focus on mechanisms in which the object is always sold, where (17) holds with equality. In this case, we have the following substantial simplification.

Lemma 4 *Suppose that $\left(q^H, q^L, m \right)$ satisfies (17) with equality, and (18). Then each of the functions q^H, q^L, and m depend only on the difference $\delta = h - l$.*

Proof. See Appendix D.

We can now prove that the optimal mechanism for the case of a single buyer consists of just two quality-price choices. To do so, we let Φ denote the marginal c.d.f. of δ, and to simplify the analysis we assume Φ is regular, i.e., we assume the function $\psi(\delta) \equiv \delta - \frac{1 - \Phi(\delta)}{\Phi'(\delta)}$ is increasing. The function $\psi + B$ is the virtual valuation.

Proposition 12 *The seller's expected surplus is maximized by* $\left(q^{H*}, q^{L*}, m^*\right)$, *where*
$$q^{H*}(\delta) = 1 - q^{L*}(\delta) = \begin{cases} 1, & \text{if } \delta > \delta^* \\ 0, & \text{if } \delta < \delta^* \end{cases} \quad \text{and} \quad m^*(\delta) = \begin{cases} 0, & \text{if } \delta < \delta^* \\ \delta^*, & \text{if } \delta > \delta^* \end{cases} \quad \text{and } \delta^* \text{ is}$$
defined by $\psi(\delta^*) = B$, *if* $B < 1$, *and* $\delta^* = 1$, *if* $1 \le B$.

Proof. See Appendix D.

Using Proposition 12, note that δ^* is increasing in B so that when the seller's benefit from the restriction is large, the object is more likely to be allocated in restricted form. When B is sufficiently large, the unrestricted object is never offered.

In the first-best allocation, the buyer receives the unrestricted object with probability 1 if $\delta > B$ and the unrestricted object otherwise. Note that, for generic distributions, we have $\delta^* = \frac{1 - \Phi(\delta^*)}{\Phi'(\delta^*)} + B > B$. Thus, we have the following corollary.

Corollary 3 *The optimal one-buyer mechanism assigns the restricted object too often (and the unrestricted object not often enough) relative to the first best.*

To give a simple example, if F is uniform on Θ, then $\Phi(\delta) = 2\delta - \delta^2$, which implies that $\delta^* = \frac{2}{3}B + \frac{1}{3}$. For example, if $B = \frac{1}{2}$, then $\delta^* = \frac{2}{3}$, so the unrestricted object is allocated if $h - l > \frac{2}{3}$ and otherwise the restricted object is allocated. In contrast, in the first-best allocation, the buyer would receive the unrestricted object whenever $h - l > \frac{1}{2}$.

The buyer's surplus from participating in the mechanism just characterized is
$$u_i(h, l) = \begin{cases} h - \delta^*, & \text{if } \delta^* < \delta \\ l, & \text{if } \delta^* > \delta. \end{cases}$$

As is well known, the "privileged buyer" auction (with no reserve price) yields the expected value of the second order statistic among $u_1, ..., u_n$. We conjecture that

a second-price auction for the right to participate in the optimal one-buyer mechanism described above constitutes an optimal mechanism, at least within the class of dominant strategy mechanisms that never retain the object.

If this conjecture is correct, then the optimal exclusive buyer mechanism dominates the contingent re-auction as far as expected seller surplus is concerned, and possibly expected total surplus as well. We have established that the efficient exclusive buyer mechanism dominates the contingent re-auction as far as expected total surplus is concerned, and possibly expected seller surplus as well. Comparing the optimal and efficient exclusive buyer mechanisms, of course neither dominates in terms of both expected seller surplus and efficiency.

8   Comparisons

This section remains a work in progress; however, we are able to provide some initial calculations. The numbers reported here are subject to refinement.

Consider an environment with two symmetric bidders drawing their values (l, h) from the uniform distribution on $[0, 1] \times [3, 4]$. Assume benefit to the seller of $B = 3.3$ if the object is sold in restricted form. In this case, one can use the numerically calculated function g shown in Figure 1 to show that for the contingent re-auction the seller's optimal reserve price from the set $R \equiv \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$ is $r = 0.5$. This reserve maximizes the seller's expected surplus (revenue plus the benefit from selling the restricted object) over the reserve prices in the set R. Increases in B result in lower optimal reserve prices so that the object is more likely to be sold in restricted form, and decreases in B result in higher optimal reserve prices so that the object is less likely to be sold in restricted form. The contingent re-auction that maximizes expected total surplus (again restricting attention to $r \in R$) has $r = 0.3$, so that the probability that the restricted object is sold is closer to the first-best level.

Table 2 shows the results of a Monte Carlo simulation based on 10,000 simulated auctions. We will refine these results in the future, but they provide rough estimates. Error introduced because of the coarseness of our estimate of g, coarseness in the reserve prices allowed, and approximation in the parameters of the optimal exclusive buyer mechanism.

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Table 2: Comparisons of mechanisms for two symmetric buyers
with (l, h) uniform on $[0, 1] \times [3, 4]$ and $B = 3.3$

	probability restricted object is sold	probability unrestricted object is sold	probability of no sale	expected seller revenue	expected seller surplus	expected bidder surplus	expected total surplus
optimal contingent re-auction	**0.708**	**0.292**	**0**	**1.368**	**3.704**	**0.266**	**3.969**
(r = 0.5)							
optimal contingent re-auction with truthful bidding	**0.747**	**0.253**	**0**				
(r = 0.5)							
change relative to optimal contingent re-auction	6%	-13%					
most efficient contingent re-auction	**0.883**	**0.117**	**0**	**0.764**	**3.678**	**0.304**	**3.982**
(r = 0.3)							
change relative to optimal contingent re-auction	24.75%	-59.96%		-44.14%	-0.69%	14.35%	0.32%
optimal mechanism s.t. seller can't retain object	**0.710**	**0.290**	**0**	**1.373**	**3.715**	**0.277**	**3.992**
(exclusive buyer with δ* = 3.18) change relative to optimal contingent re-auction	0.27%	-0.65%		0.36%	0.30%	4.27%	0.57%
efficient mechanism	**0.812**	**0.188**	**0**	**1.028**	**3.709**	**0.289**	**3.998**
(exclusive buyer with δ* = 3.3)							
change relative to optimal contingent re-auction	15%	-36%		-24.85%	0.14%	8.92%	0.73%

As shown in Table 2, the optimal contingent re-auction allocates the object in unrestricted form too often relative to the efficient outcome. The imposition of the reserve price and the resulting strategic bidding both decrease the probability that the object is allocated with restrictions. For the example used, the efficient mechanism allocates the restricted object with probability 0.812, but the optimal contingent re-auction only allocates the restricted object with probability 0.708. To see how much of this effect is due to strategic bidding, note that with truthful bidding in the optimal contingent re-auction, the restricted object is allocated with probability 0.747. So the effect of the reserve price itself and strategic bidding appear roughly equal. The most efficient contingent re-auction allocates the object in restricted form more often than the first-best, but this 'overshooting' may be because we only allow the reserve price to vary along a fairly coarse grid. Improvements on this are work in progress.

The optimal exclusive buyer mechanism conditional on the seller not retaining the object consists of first stage with a second-price auction with no reserve price and a second stage with price zero for the restricted object and price 3.18 for the unrestricted object. Comparing the optimal exclusive buyer mechanism with the optimal contingent re-auction, we see only a small increase in expected seller surplus

(0.30%), but a larger (4.27%) increase in expected bidder surplus.

Comparing the efficient exclusive buyer mechanism with the contingent re-auction, we see a large decrease in expected seller revenue (-24.85%), a small increase in expected seller surplus (0.14%), a larger increase in expected bidder surplus (8.92%), and a small increase in total expected surplus (0.73%). Thus, in this example, the efficient exclusive buyer mechanism dominates the contingent re-auction for both the seller and the buyers. The increase in total surplus from moving to an efficient mechanism is shared by both the seller and the buyers.

9   Conclusion

The conclusions presented here remain preliminary.

Based on our results, we can offer some comments regarding the contingent re-auction that may improve its implementation should sellers choose to use that mechanism. Recall that the contingent re-auction requires that the seller specify a reserve price for the initial auction for the restricted object, and potentially also a reserve price for the unrestricted object, although as a simplification in our analysis we assume a zero reserve price for the unrestricted object.

- Comments on the contingent re-auction

    - Strategic underbidding is most likely to be a problem when there are a small number of bidders, bidders believe they have some chance of winning the unrestricted object, and bidders have values for the restricted object that are less than or not substantially greater than the reserve price.

    - If the reserve price at the first auction is set so that $r + B$ is greater than or equal to the expected revenue at the second auction, then the contingent re-auction is at least as good (in expectation) as just holding a single auction for the unrestricted object with no reserve.

    - If B is greater than the expected revenue at the second auction, then a single auction for the restricted object with no reserve is better than a contingent re-auction.

Moving away from the contingent re-auction, for some environments, we can characterize the optimal mechanism. For example, if buyers are symmetric and their

values for the restricted object are simply a fixed fraction α of their values for the unrestricted object, then the optimal mechanism can implemented through an auction in which each bidder submits one bid, the high bidders wins, and receives the restricted object if its bid is above the reserve price but below a threshold set by the seller, and receives the unrestricted object if its bid is above the threshold. If the high bidder wins the restricted object it pays α times the second-highest bid (or the reserve price if there are no other bids above the reserve price), and if the high bidder wins the unrestricted object, it pays an amount calculated based on reserve price, the threshold, and the second-highest bid.

In a more general environment with two-dimensional types, we conjecture that an exclusive buyer mechanism is optimal, at least among dominant strategy mechanisms that never retain the object. Our numerical results suggest that the exclusive buyer mechanism offers some improvement over the contingent re-auction, although the improvement for the seller does not appear large. Interestingly, in our example, although the parameters of the exclusive buyer mechanism are chosen to maximize the seller's expected surplus, the mechanism is slightly more efficient that the contingent re-auction and so the buyers are also able to benefit from the change. Additional increases in seller revenue may be possible if we consider mechanisms in which the seller sometimes retains the object.

Finally, we identify an easily implemented efficient mechanism for our general environment. It is an exclusive buyer mechanism defined by a second-price auction with no reserve price and then prices of zero and B for the restricted and unrestricted objects, respectively. Thus, a seller interested in maximizing expected total surplus, can do so with no knowledge of the number of bidders or the underlying type distributions. The mechanism can be adapted to accommodate multiple possible restrictions. Furthermore, in our numerical example, although the seller forfeits substantial revenue in the efficient exclusive buyer mechanism relative to the contingent re-auction, the seller's expected surplus actually increases slightly. Thus, at least in our example, the seller can achieve an increase in efficiency and an increase it its own expected surplus simultaneously by switching from a contingent re-auction to the efficient exclusive buyer mechanism. In the example, such a switch also provides benefits to buyers, whose expected surplus increases.

In future work, we hope to expand and improve upon the results presented here. A particularly interesting extension would allow allocative externalities, where a buyer

that does not receive the object might have preferences over which buyer does receive it and whether the object is allocated in restricted or unrestricted form. Also, in our model some of the results for one-dimensional environments do not extend to the general multi-dimensional environment, so we hope to explore more generally when is it and is it not appropriate to focus on one-dimensional environments in generally multi-dimensional problems.

To conclude, our primary recommendation is that sellers in environments with seller-benefitting restrictions consider using an exclusive buyer mechanism, either one tailored to maximize efficiency or seller surplus, depending upon the seller's objectives. It appears that a type of exclusive buyer mechanism is currently being used by Japan's Nippon Professional Baseball. Japanese players are offered to Major League Baseball in a system described by Wikipedia as follows:

> "The MLB Office of the Commissioner then holds a four-day-long silent auction among its teams. The highest resulting bid on the player is sent to the Japanese team, which may or may not choose to accept it. If the bid is accepted, the bid amount is publicly revealed and the winning Major League team is granted the exclusive rights to negotiate with the player."[24]

Recently, the Red Sox paid $51.11 million for the opportunity to negotiate with Daisuke Matsuzaka, who they then signed to a six-year, $52 million contract.[25] This system has much of the flavor of the exclusive buyer mechanism, where buyers bid to be the privileged buyer to have an opportunity to buy the object for sale.

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[24]http://en.wikipedia.org/wiki/Posting_system

[25] "Matsuzaka, Red Sox reach agreement on six-year deal," ESPN.com news services, February 23, 2007, http://sports.espn.go.com/mlb/news/story?id=2696321.

A   Appendix – Characterization proofs

Proof of Proposition 2. It is obvious that, for each bidder $i = 1, ..., n$, the only bid in the second auction that is consistent with an undominated strategy is h_i. Consider the first auction. Let $b_j (l_j, h_j)$ denote the bidding functions used in the first auction by all bidders $j \neq i$ and consider the optimization problem for bidder i. The key observation is that by bidding $b_i \geq r$, bidder i rules out the possibility of a second auction. Therefore, conditional on bidding on or above the reserve price, bidder i's incentives are the same as in a standard second-price auction, where the only undominated bid is l_i.

The previous observation implies that, for any type (l_i, h_i) of bidder i, only two bids can be optimal in the first auction: 0 and l_i. Clearly, the optimal bid is 0 if $l_i < r$. For all other types, the expected payoff from bidding 0 is $V_i^2 (h_i) = E\left[s_i^2 \left(h_i, \widetilde{h}_{-i}, \widetilde{l}_{-i}; b_{-i} \right) \right]$, where

$$s_i^2 (h_i, h_{-i}, l_{-i}; b_{-i}) \equiv \begin{cases} h_i - \max_{j \neq i} h_j, & \text{if } \max_{j \neq i} h_j < h_i \text{ and } \max_{j \neq i} b_j (l_j, h_j) < r, \\ 0, & \text{otherwise}, \end{cases}$$

because in this case the second auction takes place only if all other bidders also bid below r, i.e. $\max_{j \neq i} b_j (l_j, h_j) < r$, and bidder i wins the second auction only if he has the highest value for the unrestricted object. Bidding $l_i \geq r$ instead yields expected payoff $V_i^1 (l_i) = E\left[s_i^1 \left(l_i, \widetilde{h}_{-i}, \widetilde{l}_{-i}; b_{-i} \right) \right]$, where

$$s_i^1 (l_i, h_{-i}, l_{-i}; b_{-i}) \equiv \begin{cases} l_i - \max \{r, \max_{j \neq i} l_j\} & \text{if } \max_{j \neq i} b_j (l_j, h_j) < l_i \\ 0, & \text{otherwise}. \end{cases}$$

Note that $V_i^2 (h_i)$ does not depend on l_i, and $V_i^1 (l_i)$ does not depend on h_i. Furthermore, since $\Pr[\max_{j \neq i} l_j < r]$ is strictly positive, $V_i^1 (l_i)$ is strictly increasing in l_i over the interval $[r, \bar{l}]$ and $V_i^2 (h_i)$ is strictly increasing in h_i over the interval $[\underline{h}, \bar{h}]$. This implies that if there is a type (l_i, h_i) such that $V_i^2 (l_i) \geq V_i^1 (h_i)$, then for all types (l_i', h_i) with $l_i' > l_i$ we have $V_i^2 (l_i') > V_i^1 (h_i)$. This in turn implies that for h_i there is a threshold value $g_i (h_i) \in [r, \bar{l}]$ such that for each $l_i > g_i (h_i)$, the bidder prefers to participate in the first auction rather than delay, and for each $l_i < g_i (h_i)$, the bidders prefer to offer zero. The function $g_i (h_i)$ is non-decreasing because $V_i^2 (h_i)$ is increasing in h_i. $\square$

Proof of Proposition 3. Suppose not. By Proposition 2 the equilibrium bidding functions in the first auction must be such that $g_i(h_i) = r$ for each i and $h_i \in [\underline{h}, \overline{h}]$. Suppose that all bidders other than i use this strategy and consider the best response of type $(r + \varepsilon, h_i)$ of bidder i, with $h_i > \underline{h}$. The expected payoff from bidding zero in the first auction is

$$V_i^2(h_i) = \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^r \cdots \int_{\underline{l}}^r \left(h_i - \max_{j \neq i} h_j \right) 1_{\max_{j \neq i} h_j < h_i} \Pi_{j \neq i} f(l_j, h_j)\, dl_{-i} dh_{-i},$$

and the expected payoff from bidding $l_i \geq r$ in the first auction is

$$V_i^1(r + \varepsilon) \leq \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^{r+\varepsilon} \cdots \int_{\underline{l}}^{r+\varepsilon} (r + \varepsilon - r) \Pi_{j \neq i} f(l_j, h_j)\, dl_{-i} dh_{-i} < \varepsilon.$$

(See the proof of Proposition 2 for expressions for V_i^2 and V_i^1.) For $h_i > \underline{h}$, $V_i^2(h_i) > 0$, which implies that there is some value $\overline{\varepsilon}(h_i) > 0$ such that $V_i^1(r + \varepsilon) < V_i^2(h_i)$ for each $\varepsilon < \overline{\varepsilon}(h_i)$. This implies that it is not a best response for types (l_i, h_i) with $l_i \in (r, r + \overline{\varepsilon}(h_i))$ to bid their value in the first auction. $\square$

B   Appendix − Existence proof

Proof of Proposition 4. Proposition 2 shows that every equilibrium can be characterized by a collection of 'threshold functions' $\{g_1(h_1), \ldots, g_n(h_n)\}$. We want to show that it is possible to find a function g such that $g_i = g$ for each bidder is an equilibrium. Consider the function

$$g(h) = \begin{cases} k, & \text{if } h \leq k \\ k + \tau(h - k), & \text{if } h > k, \end{cases} \tag{B.1}$$

where $k > r$, and $\tau : [k, \overline{h}] \to \mathbf{R}$ satisfies $\tau(0) = 0$ and $\tau'(x) \leq 1$ for $x \leq \overline{h}$.

Let $z(h)$ denote the marginal densities of h and let $Z(h)$ denote the cumulative distribution. The expected surplus for type (l, h) when bidding 0 in the first auction, if all other bidders bid according to g, can be written as

$$V^2(h; g) = \int_{\underline{h}}^h \cdots \int_{\underline{h}}^h \int_{\underline{l}}^{g(h_1)} \cdots \int_{\underline{l}}^{g(h_n)} \left(h - \max_{j \neq i} h_j \right) \Pi_{j \neq i} f(l_j, h_j)\, dl_{-i} dh_{-i}. \tag{B.2}$$

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Defining

$$s^* \left(l, l_{-i}, h_{-i}; g\right) \equiv \max \left\{ \left(l - r \left(\Pi_{j \neq i} 1_{l_j < g(h_j)} \right) - \left(\max_{j \neq i} l_j \right) \left(1 - \left(\Pi_{j \neq i} 1_{l_j < g(h_j)} \right) \right) \right), 0 \right\},$$

bidding $l_i \geq r$ yields an expected value equal to

$$V^1 (l; g) = \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^{\overline{l}} \cdots \int_{\underline{l}}^{\overline{l}} s^* (l, l_{-i}, h_{-i}; g) \, \Pi_{j \neq i} f (l_j, h_j) \, dl_{-i} dh_{-i}. \qquad \text{(B.3)}$$

Consider the case $k = \overline{l}$. In this case, V^2 and V^1 are independent of g and we have

$$\begin{aligned} V^2 \left(h; \overline{l}\right) &= \int_{\underline{h}}^{h} \cdots \int_{\underline{h}}^{h} \int_{\underline{l}}^{\overline{l}} \cdots \int_{\underline{l}}^{\overline{l}} \left(h - \max_{j \neq i} h_j \right) \Pi_{j \neq i} f (l_j, h_j) \, dl_{-i} dh_{-i}. \\ &= \left(h - E \left[\max_{j \neq i} h_j \mid h_j < h \text{ each } j \neq i \right] \right) (Z(h))^{n-1} \end{aligned}$$

and $V^1 \left(l; \overline{l}\right) = (l - r)$. Since $\frac{dV^2 \left(h; \overline{l}\right)}{dh} = (Z(h))^{n-1}$, the function $V^2 \left(h; \overline{l}\right)$ is strictly increasing and it has slope less than 1. The highest value l compatible with a given h is h and the function $V^2 \left(h; \overline{l}\right) = (h - r)$ increases with slope 1 and takes value zero at $h = r$. Since $V^1 \left(r; \overline{l}\right) > 0 = V^2 \left(r; \overline{l}\right)$, we conclude that if

$$\begin{aligned} V^1 \left(\overline{l}; \overline{l}\right) &= \left(\overline{l} - E \left[\max_{j \neq i} h_j \mid h_j < \overline{l} \text{ each } j \neq i \right] \right) \left(Z \left(\overline{l}\right) \right)^{n-1} \\ &\geq \overline{l} - r \\ &= V^2 \left(\overline{l}; \overline{l}\right), \end{aligned} \qquad \text{(B.4)}$$

then there exists an equilibrium in which all types bid zero in the first auction and compete in the second auction, i.e., an equilibrium with total postponement. We record this result in the following lemma.

Lemma B.1 *If* (B.4) *holds, then there exists an equilibrium in which bidders do not bid in the first auction, and bid their values in the second auction, i.e., delay occurs with probability one.*

In general the condition is easier to satisfy if r is large and it is in fact trivially satisfied if $r = \overline{l}$. Furthermore, as n increases $E \left[\max_{j \neq i} h_j \mid h_j < \overline{l} \text{ each } j \neq i \right]$ increases and $\left(Z \left(\overline{l}\right) \right)^{n-1}$ decreases, thus making the condition more difficult to satisfy. A simpler form of the condition is available for the case $\overline{l} = \overline{h}$, so that $Z \left(\overline{l}\right) = 1$. In

this case the condition can be written as $r \geq E\left[\max_{j \neq i} h_j\right]$, i.e., the reserve price in the first auction must be greater than the expected payment in the second auction for the highest type.

Suppose now that condition (B.4) is violated, that is

$$V^2\left(\bar{l}; \bar{l}\right) < V^1\left(\bar{l}; \bar{l}\right), \tag{B.5}$$

so that Lemma (B.1) does not apply. We want to show that there is an equilibrium with $k < \bar{l}$.

Since g can be represented by the real number k and the function τ we now write $V^i\left(\cdot; k, \tau\right)$ rather than $V^i\left(\cdot; g\right)$. In such an equilibrium, each type $(g(h), h)$ with $h \geq k$ has to be indifferent between bidding its value in the first auction and bidding zero. In particular, type (k, k) has to be indifferent between bidding in the first auction and waiting. Thus $V^2\left(k; k, \tau\right) = V^1\left(k; k, t\right)$. Since for $h_j \leq k$ we have $g(h_j) = k$, it follows that

$$V^2\left(k; k, t\right) = \int_{\underline{h}}^k \cdots \int_{\underline{h}}^k \int_{\underline{l}}^k \cdots \int_{\underline{l}}^k \left(k - \max_{j \neq i} h_j\right) \Pi_{j \neq i} f\left(l_j, h_j\right) dl_{-i} dh_{-i}.$$

Furthermore, since $g(h_j) \geq k$ for $l_i = k$, we have

$$V^1\left(k; k, t\right) = \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^{g(h_1)} \cdots \int_{\underline{l}}^{g(h_n)} \left(k - r\right) \Pi_{j \neq i} f\left(l_j, h_j\right) dl_{-i} dh_{-i}.$$

For a given t, both functions are continuous in k over $\left[k, \bar{l}\right]$. At $k = r$ we have $V_i^1\left(r; r, t\right) = 0 < V^2\left(r; r, t\right)$, and condition (B.5) implies that at $k = \bar{l}$ we have $V^1\left(\bar{l}; \bar{l}, t\right) > V^2\left(\bar{l}; \bar{l}, t\right)$. We therefore conclude that for each function τ, there is a $k^* \in \left(r, \bar{l}\right)$ such that $V^2\left(k^*; k^*, \tau\right) = V^1\left(k^*; k^*, \tau\right)$.

An additional condition for $g(h)$ to be an equilibrium is that for each $h > k$, the type $(g(h), h)$ be indifferent between bidding in the first auction and postponing. Thus, $V^2\left(h; k, \tau\right) = V^1\left(g(h); k, \tau\right)$, and given the assumption of differentiability

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$\frac{dV^2(h;k,\tau)}{dh} = \frac{dV^1(g(h);k,\tau)}{dl} g'(h)$, which implies $g'(h) = \frac{\frac{dV^2(h;k,\tau)}{dh}}{\frac{dV^1(g(h);k,\tau)}{dl}}$, where

$$\begin{aligned}
\frac{dV^2(h;k,\tau)}{dh} &= \int_{\underline{h}}^{h} \cdots \int_{\underline{h}}^{h} \int_{\underline{l}}^{g(h_1)} \cdots \int_{\underline{l}}^{g(h_n)} \Pi_{j\neq i} f(l_j, h_j) \, dl_{-i} dh_{-i} \\
&= \left(\int_{\underline{h}}^{h} F(g(x) \mid x) z(x) \, dx \right)^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{dV^1(l;k,\tau)}{dl} &= \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^{\max\{l,g(h_1)\}} \cdots \int_{\underline{l}}^{\max\{l,g(h_1)\}} \Pi_{j\neq i} f(l_j, h_j) \, dl_{-i} dh_{-i} \\
&= \left(\int_{\underline{h}}^{\overline{h}} F(\max\{l, g(x)\} \mid x) z(x) \, dx \right)^{n}.
\end{aligned}$$

Then the integral equation to be solved is

$$g'(h) = \left(\frac{\int_{\underline{h}}^{h} F(g(x) \mid x) z(x) \, dx}{\int_{\underline{h}}^{\overline{h}} F(\max\{g(h), g(x)\} \mid x) z(x) \, dx} \right)^{n-1}.$$

We are now ready to establish the main result of this section.

Lemma B.2 *For each n there is a symmetric perfect Bayesian equilibrium described by a function*

$$g_n(h) = \begin{cases} k_n, & \text{if } h \leq k_n \\ k_n + \tau_n(h - k), & \text{if } h > k_n, \end{cases}$$

where $k_n > r$, and τ_n is strictly increasing, differentiable and such that $\tau_n(0) = 0$ and $\tau_n'(x) \leq 1$ for $x \leq \overline{h}$. As $n \to \infty$ all symmetric equilibria converge to the constant function $g_\infty(h) = r$.

Proof of Lemma B.2. To prove existence, consider the problem with a fixed n. If inequality (B.4) is satisfied then there is an equilibrium with $g(h) = \overline{l}$. Thus, suppose that it is not satisfied. For each given $k \in [r, \overline{l}]$ consider the integral equation

$$g'(h) = \left(\frac{\int_{\underline{h}}^{h} F(g(x) \mid x) z(x) \, dx}{\int_{\underline{h}}^{\overline{h}} F(\max\{g(h), g(x)\} \mid x) z(x) \, dx} \right)^{n-1} \tag{B.6}$$

defined over the interval $[k, \overline{h}]$ with initial condition $g(k) = k$. Under regularity assumptions on the probability distribution there is a unique solution for each k, call it g_k^*. Clearly the solution has $\frac{dg_k^*(h)}{dh} \in (0,1)$. In fact, define $y(h) \equiv \frac{1}{1+\int_h^{\overline{h}} F(r|x)z(x)dx}$, and observe that for each $h < \overline{h}$ we have $y(h) < 1$. Observe that for every increasing function g we have

$$
\frac{\int_{\underline{h}}^h F(g(x)|x)z(x)\,dx}{\int_{\underline{h}}^{\overline{h}} F(\max\{g(h), g(x)\}|x)z(x)\,dx} = \frac{\int_{\underline{h}}^h F(g(x)|x)z(x)\,dx}{\int_{\underline{h}}^h F(g(h)|x)z(x)\,dx + \int_h^{\overline{h}} F(g(x)|x)z(x)\,dx}
$$

$$
< \frac{\int_{\underline{h}}^h F(g(h)|x)z(x)\,dx}{\int_{\underline{h}}^h F(g(h)|x)z(x)\,dx + \int_h^{\overline{h}} F(r|x)z(x)\,dx}
$$

$$
< y(h).
$$

It follows that $g'(h)$ is bounded above by $(y(h))^{n-1}$.

The solution is an equilibrium if we can find $k^* \in [r, \overline{l}]$ such that

$$
\Psi(k) \equiv V^2(k; g_k^*) - V^1(k; g_k^*) = 0. \tag{B.7}
$$

The function Ψ is continuous in k. Clearly $\Psi(r) < 0$ and, since inequality (B.4) is not satisfied, $\Psi(\overline{l}) > 0$. Thus, a solution exists.

Consider now the behavior of the solution as n goes to infinity. Inequality (B.4) can be written as

$$
\left(\overline{l} - E\left[\max_{j \neq i} h_j \mid h_j < \overline{l} \text{ each } j \neq i\right]\right) \left(\Pr\left(h_j < \overline{l}\right)\right)^{n-1} > \overline{l} - r.
$$

Since the LHS goes to zero as $n \to \infty$ while the RHS is constant and strictly positive, the inequality is not satisfied for n large enough. Thus, for n large the solution must be obtained as a solution of the integral equation (B.6) and the equation (B.7).

Since for each h we have $g_n'(h) \leq (y(h))^{n-1}$ and $y(h) < 1$ for each $h < \overline{h}$ it follows that $\lim_{n\to\infty} g_n'(h) = 0$, i.e., the solution must tend to a constant function. Notice further that for each n the solution must satisfy $\frac{V_n^2\left(k_n^*; g_{k_n^*}^*\right)}{V_n^1\left(k_n^*; g_{k_n^*}^*\right)} = 1$, which in turn implies

$$
\lim_{n\to\infty} \frac{V_n^2\left(k_n^*; g_{k_n^*}^*\right)}{V_{in}^1\left(k_n^*; g_{k_n^*}^*\right)} = 1. \tag{B.8}
$$

Let $k_\infty^* = \lim_{n\to\infty} k_n^*$. We want to show that $k_\infty^* = r$. Since we have established that $\lim_{n\to\infty} g_n^*(h) = k_\infty^*$ for each h, we have

$$\lim_{n\to\infty} \frac{V_n^2\left(k_n^*; g_{k_n^*}^*\right)}{V_n^1\left(k_n^*; g_{k_n^*}^*\right)}$$

$$= \lim_{n\to\infty} \frac{\left(k_\infty^* - E^n\left[\max_{j\neq i} h_j \mid h_j < k_\infty^*, l_j < k_\infty^* \text{ each } j \neq i\right]\right)}{(k_n^* - r)} \lim_{n\to\infty} \left(\frac{\Pr\left(h_j < k_\infty^*, l_j < k_\infty^*\right)}{\Pr\left(l_j < k_\infty^*\right)}\right)^{n-1},$$

where E^n denotes the expectation taken when there are n bidders. Since the second limit is zero, the only way in which (B.8) can be satisfied is if

$$\lim_{n\to\infty} \frac{\left(k_n^* - E^n\left[\max_{j\neq i} h_j \mid h_j < k_n^*, l_j < k_n^* \text{ each } j \neq i\right]\right)}{(k_n^* - r)} = \infty,$$

but for this to be the case the denominator must converge to zero, thus establishing $\lim_{n\to\infty} k_n^* = r$. $\square$

Using Lemma B.2, Proposition 4 follows. $\square$

C   Appendix – Optimal mechanism proofs

Proof of Lemma 2. The seller's expected benefit generated *ex ante* by buyer i is

$$\int_{\underline{h}}^{\bar{h}} \left(M_i(h_i) + B \, Q_i^L(h_i) \right) dF_i(h_i)$$

$$= \int_{\underline{h}}^{\bar{h}} \left(h_i \, Q_i^H(h_i) + \lambda_i(h_i) \, Q_i^L(h_i) - U_i(h_i) + B \, Q_i^L(h_i) \right) dF_i(h_i)$$

$$= \int_{\underline{h}}^{\bar{h}} \left(h_i \, Q_i^H(h_i) + (\lambda_i(h_i) + B) \, Q_i^L(h_i) \right) dF_i(h_i) - \int_{\underline{h}}^{\bar{h}} U_i(h_i) \, dF_i(h_i)$$

$$= \int_{\underline{h}}^{\bar{h}} \left(h_i \, Q_i^H(h_i) + (\lambda_i(h_i) + B) \, Q_i^L(h_i) \right) dF_i(h_i)$$

$$\quad - U_i(\underline{h}) - \int_{\underline{h}}^{\bar{h}} (1 - F_i(h_i)) \left(Q_i^H(h_i) + \lambda_i'(h_i) \, Q_i^L(h_i) \right) dh_i$$

$$= \int_{\underline{h}}^{\bar{h}} \left(\begin{array}{c} \left(h_i - \frac{1 - F_i(h_i)}{f_i(h_i)} \right) Q_i^H(h_i) + \\ \left(\lambda_i(h_i) + B - \frac{1 - F_i(h_i)}{f_i(h_i)} \lambda_i'(h_i) \right) Q_i^L(h_i) \end{array} \right) dF_i(h_i) - U_i(\underline{h})$$

$$= \int_{\underline{h}}^{\bar{h}} \left(v_i^H(h_i) \, Q_i^H(h_i) + v_i^L(h_i) \, Q_i^L(h_i) \right) dF_i(h_i) - U_i(\underline{h}),$$

where the first equality substitutes for $M_i(h_i)$ using (4), the second equality rearranges, and the third equality uses (8), the fourth equality rearranges, and the last equality uses the definitions of v_i^H and v_i^L. $\square$

Proof of Proposition 7. Conditions (13) and (14) guarantee that v_i^L and v_i^H are nondecreasing. To see this, note that (13) and (**??**) imply that $\frac{dv_i^H(h_i)}{dh_i} = 1 - \rho_i'(h_i) \geq 0$ and $\frac{dv_i^L(h_i)}{dh_i} = (1 - \rho_i'(h_i)) \lambda_i'(h_i) - \rho_i(h_i) \lambda_i''(h_i) > 0$. In addition, since

$$v_i^H(h_i) - v_i^L(h_i) = h_i - \rho_i(h_i) - (B + \lambda_i(h_i) - \rho_i(h_i) \lambda_i'(h_i)),$$

condition (14) guarantees that

$$\frac{d \left(v_i^H(h_i) - v_i^L(h_i) \right)}{dh_i} = (1 - \rho_i'(h_i)) (1 - \lambda_i'(h_i)) + \rho_i(h_i) \lambda_i''(h_i) > 0.$$

Thus, v_i^L crosses the x-axis at most once from below and v_i^H at most once from above. This implies the existence of the two threshold functions and the monotonicity of

A_i. It is routine to verify that the payment functions defined in (15) satisfy ex-post incentive compatibility and ex-post individually rationality. Finally, (χ, m^*) is optimal among all interim incentive compatible and interim individually rational mechanisms because it satisfies interim incentive compatibility and interim individual rationality and the assignment functions χ solve the 'relaxed' problem (subject to only 'local' interim incentive compatibility). Q.E.D.

D   Appendix – Exclusive buyer proofs

Proof of Lemma 3. As it is well known (Rochet, 1987; or Milgrom and Segal, 2002), the inequalities in (18) imply that the buyer's rent function $u(h, l)$ satisfies for almost all $(h, l) \in \Theta$,

$$\nabla u(h, l) = \begin{bmatrix} \frac{\partial u(h,l)}{\partial h} \\ \frac{\partial u(h,l)}{\partial l} \end{bmatrix} = \begin{bmatrix} q^H(h, l) \\ q^L(h, l) \end{bmatrix} \tag{D.1}$$

and for all $(h, l), (h', l') \in \Theta$, $u(h, l) - u(h', l') = \int_P \mathbf{q}(\boldsymbol{\tau}) \cdot d\boldsymbol{\tau}$, where the integral is a line integral of the gradient vector field $\mathbf{q} = [q^H, q^L]'$ along any path P from (h', l') to (h, l). Since $\mathbf{q} \geq \mathbf{0}$, by (D.1) we immediately have that $u(h, l) \geq u(\underline{h}, \underline{l})$ for all $(h, l) \in T$; hence all constraints in (19) are satisfied iff $u(\underline{h}, \underline{l}) \geq 0$. □

Proof of Lemma 4. Using the equality in (17) yields

$$
\begin{aligned}
h\, q^H(h', l') + l\, q^L(h', l') - m(h', l') &= h\, q^H(h', l') + l\, \left(1 - q^H(h', l')\right) - m(h', l') \\
&= (h - l)\, q^H(h', l') + l - m(h', l') \\
&= l + \delta\, q^H(h', l') - m(h', l').
\end{aligned}
$$

Changing variables $(h, l) \to (h, \delta)$ (with slight abuse of notation we keep similar notation), we rewrite (18) as, $\forall (\delta, l) \in \Delta$, $w(\delta, l) - w(\delta', l') \geq (l - l') + (\delta - \delta')\, q^H(\delta', l')$, where $w(\delta, l) \equiv l + \delta\, q_H(\delta, l) - m(\delta, l)$ for $(\delta, l) \in \Delta \equiv \left\{(x, y) \in \mathbf{R}_+^2 \mid x + y \leq 1\right\}$. By the envelope theorem we have for almost all $(\delta, l) \in \Delta$,

$$\nabla w(\delta, l) = \begin{bmatrix} \frac{\partial w(\delta,l)}{\partial \delta} \\ \frac{\partial w(\delta,l)}{\partial l} \end{bmatrix} = \begin{bmatrix} q^H(\delta, l) \\ 1 \end{bmatrix}.$$

Now conservativeness implies $\frac{\partial q^H(\delta,l)}{\partial l} = 0$. Recall that we assume that w is twice continuously differentiable. The set of all twice continuously differentiable convex functions on Δ is dense in the set of all convex functions on Δ. Thus, q^H must be independent of l, and (18) implies

$$\left\{ \begin{array}{l} l + \delta\, q^H\left(\delta\right) - m\left(\delta,l\right) \geq l + \delta\, q^H\left(\delta\right) - m\left(\delta,l'\right) \\ l' + \delta\, q^H\left(\delta\right) - m\left(\delta,l'\right) \geq l' + \delta\, q^H\left(\delta\right) - m\left(\delta,l\right) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} -m\left(\delta,l\right) \geq -m\left(\delta,l'\right) \\ -m\left(\delta,l'\right) \geq -m\left(\delta,l\right) \end{array} \right\},$$

which implies $m\left(\delta,l'\right) = m\left(\delta,l\right)$. $\square$

Proof of Proposition 12. The standard characterization of incentive compatibility now yields that q^H must be nondecreasing, and $\forall \delta \in [0,1]$, $\forall l \in [0, 1 - \delta]$, $w\left(\delta,l\right) = l + \int_0^\delta q^H\left(x\right) dx$ (see the proof of Lemma 4 for the definition of w). Taking expectations with respect to l and δ and rearranging yields

$$E\left[w\left(\delta,l\right)\right] = E\left[l\right] + \int_0^1 \left(1 - \Phi\left(\delta\right)\right) q^H\left(\delta\right) d\delta. \tag{D.2}$$

Now using (**??**) and (D.2) we can write the seller's expected surplus as

$$
\begin{aligned}
E\left[m\left(\delta\right) + B\left(1 - q^H\left(\delta\right)\right)\right] &= E\left[hq^H\left(\delta\right) + \left(l + B\right)\left(1 - q^H\left(\delta\right)\right) - w\left(\delta,l\right)\right] \\
&= E\left[\left(h - l - B\right) q^H\left(\delta\right)\right] + B - \int_0^1 \left(1 - \Phi\left(\delta\right)\right) q^H\left(\delta\right) d\delta \\
&= B + \int_0^1 \left(\delta - B - \frac{1 - \Phi\left(\delta\right)}{\Phi'\left(\delta\right)}\right) q^H\left(\delta\right) d\Phi\left(\delta\right).
\end{aligned}
$$

The result then follows. $\square$

E   Appendix – Calculation of g

Consider the case of two symmetric bidders, where each bidder i draws its type $\theta_i = (h_i, l_i)$ according to the same c.d.f. with rectangular support $\Theta_1 = [h_L, h_H] \times [l_L, l_H]$. Any *symmetric* equilibrium of the contingent re-auction is characterized by an indifference function $g : [h_L, h_H] \to [l_L, l_H]$, where (recall that F is the conditional

c.d.f. of l)

$$g'(h) = \frac{\int_{h_L}^{h} F_{Y|X}(g(x) \mid x) \; dF_X(x)}{\int_{h_L}^{h_H} F_{Y|X}(\max\{g(h), g(x)\} \mid x) \; dF_X(x)}. \tag{E.1}$$

The equation in (E.1) can be rewritten as a second order differential equation, where $g(h_L) = r$, which provides one initial condition. It is immediate to see that $g'(h_L) = 0$ for any $r \geq 0$; and for $r = 0$, $g(h) \equiv 0$ is a solution of (E.1), i.e., no reserve implies no delay. We can also show that for any $r > 0$, $\forall h \in (h_L, h_H)$, $0 < g'(h) < 1$.

With F uniform, we have

$$g'(h) = \frac{\int_{h_L}^{h} g(x) \, dx}{hg(h) + \int_{h}^{h_H} g(x) \, dx} = \frac{G(h)}{hg(h) + G(h_H) - G(h)}, \tag{E.2}$$

where $G(h) \equiv \int_{h_L}^{h} g(x) \, dx$.

One can then find a numerical solution $x(\cdot \mid \Gamma)$ of the second-order ODE $x''(h) = \dfrac{x(h)}{hx'(h) + \Gamma - x(h)}$, $h \in [h_L, h_H]$, with initial conditions $x(h_L) = 0$ and $g(h_L) = r$, for each $\Gamma \in \left[r, r + \frac{1}{2}(h_H - h_L)\right]$. Since $g' \in [0, 1]$ implies $G(h_H) \in \left[r, r + \frac{1}{2}(h_H - h_L)\right]$, we only look at solutions G where Γ is in (a grid within) that range. Then find the value of Γ_* such that $\Gamma_* = x(h_H \mid \Gamma_*)$. The function $g(h) = x'(h \mid \Gamma_*)$ is a solution to (E.1).

To solve the differential equation we first convert the above second-order ODE into a pair of first-order ODEs. This is a standard operation (Euler's method). We can then solve this system numerically, by discretizing the interval $[t_L, t_H]$, and solving the resulting difference equations recursively.

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