Grid Distributions to study single object auctions*

(Preliminary Version.)

Luciano I. de Castro[†]

March 2008

Abstract

This paper proposes a new approach for dealing with general dependence in auctions. The method allows both theoretical and numerical characterization of monotone pure strategy equilibria in first price auctions. Given the close connection between the theoretical and numerical parts, one can draw conjectures from the observation of computer experiments (simulations), which can suggest directions to the development of the theory. This is illustrated by the proof that the set of first price auctions with dependent private values which have monotonic pure strategy equilibrium has zero measure.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: dependence of types, asymmetric auctions, pure strategy equilibrium existence, revenue ranking.

1 Introduction

As Maskin (2004) points out, "by any standard measure, auction theory has been an enormous success." It has been influential not only in economic theory but also in the design of real world auctions, frequently involving millions of dollars. This success may suggest that auctions are well understood, but this is far from being true, even in the special (but important) case of single object private values auctions. In this case, there are at least two main issues that remain to be better explored: asymmetries and dependence of the private information of bidders.

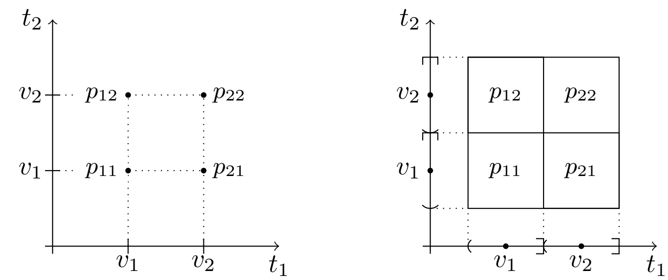
With asymmetric bidders, a complete characterization of the equilibrium strategies is not possible, even with independent types (see Lebrun, 2006). With dependence, the problem is not exactly characterization (which is possible in the symmetric case),

*This paper supersedes the second part of "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions". The first part of "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions" is now de Castro (2007). I am grateful to Vijay Krishna, Harry Paarsch, Steven Williams and Charles Zheng for helpful conversations.

†Department of Economics, University of Illinois at Urbana-Champaign. 1206 South Sixth St., Champaign, IL, 61822. E-mail: decastro.luciano@gmail.com.

but equilibrium existence itself. While monotone pure strategy equilibria (MPSE) existence is established for a special kind of positive dependence (affiliation),[1] there is no MPSE existence results with generic dependence. We know that mixed strategy equilibria exists under very mild assumptions (see Jackson and Swinkels, 2005).[2] However, it is not clear whether pure strategy equilibria are common but difficult to prove or, instead, they are rare and because of that it is impossible to give a general existence result. Since MPSE are more frequently used in auction theory, it is important to better understand the conditions that lead to MPSE existence. Now, if we want to address both asymmetry and general dependence, the conclusions seem to be beyond the reach of purely theoretical results. Thus, new methods are necessary to approach these problems.

This paper offers a new method, which is based on the simple idea of restricting the set of considered distributions to a special but sufficiently rich set. See Figure 1.



(a) $p_{ij} = \Pr(t_1 = v_i, t_2 = v_j)$ (b) $p_{ij} = f(v_i, v_j)$

Figure 1: Discrete values, such as in (a), capture the relevant economic possibilities in a private value model, but preclude the use of calculus. We use continuous variables, but consider only simple density functions (constant in squares), such as in (b).

Let us expand the above explanation. Consider a single object auction with risk neutral bidders and general dependence of types (private values). We depart from the observation that the bidder's values of the (indivisible) object can be specified only up to cents and are obviously bounded. In other words, there are finitely many different possible values and the economic problem is well described with discrete distributions. Nevertheless, to work with discrete values precludes us from using the convenient tools of differential calculus, which allow, for instance, a characterization of equilibrium strategies. Maintaining the advantage of continuum variables, but without requiring unnecessary richness in the set of distributions, we focus on the set of densities which are constant in small squares. This includes all the economically relevant cases. Moreover, it is dense in the set of all distributions because we can approximate any probability density function (pdf), including non-continuous ones, by taking arbitrarily small squares.

[1]Affiliation was introduced in auction theory by Milgrom and Weber (1982). See a discussion about affiliation in de Castro (2007).

[2]Jackson and Swinkels (2005)'s setup is that of private values, as this paper, but allows for multiple units, to the contrary of this paper, which focus only on single object auctions.

For this set of simple pdf's, we are able to provide an algorithm that completely characterizes whether or not pure strategy equilibrium exists. Computer experiments (simulations) allow an exploration of facts about the equilibrium. These experiments can illuminate directions of research. We illustrate how this can be done, by proving that pure strategy equilibrium existence in first price auction is a restrictive condition. In a sense, it holds in a set of zero measure in the set of all distributions.

This (preliminary) version of the paper mainly treats symmetric private values auctions, with two players, but the method can be extended to asymmetric private values with n players. The paper is organized as follows. Section 2 gives a brief exposition of the standard auction model. The class of distributions that we propose to use is described in section 3 and the theorem that states that the set of distributions with symmetric monotonic pure strategy equilibrium (SMPSE) has zero measure is stated in section 4. Section 5 presents the equilibrium existence algorithm for symmetric first price auctions, and illustrates the results with computer experiments (simulations). In this version, we just make some comments about asymmetric auctions and auctions with risk aversion in section 6. Section 7 deals with the problem of revenue ranking in auctions. Section 8 is a brief conclusion. Most proofs are collected in a separate supplement to this paper. The more important and short proofs are given in an appendix, while lengthy constructions and proofs are presented in a separate supplement to this paper.

2 Basic model and definitions

Our model and notations are standard. There are n bidders, $i = 1, ..., n$. Bidder i receives private information $t_i \in \left[\underline{t}, \bar{t}\right]$ which is the value of the object for himself. The usual notation $t = (t_i, t_{-i}) = (t_1, ..., t_n) \in \left[\underline{t}, \bar{t}\right]^n$ is adopted. The values are distributed according to a pdf $f : \left[\underline{t}, \bar{t}\right]^n \to \mathbb{R}_+$. For most of this version of the paper, we will assume symmetry. Thus, unless otherwise pointed, we will assume that f which is symmetric, that is, if $\pi : \{1, ..., n\} \to \{1, ..., n\}$ is a permutation, $f(t_1, ..., t_n) = f\left(t_{\pi(1)}, ..., t_{\pi(n)}\right)$.[3] Let $\overline{f}(x) = \int f(x, t_{-i})\, dt_{-i}$ be a marginal of f. Our main interest is the case where f *is not* the product of its marginals, that is, the case where the types are dependent. We denote by $f(t_{-i} \mid t_i)$ the conditional density $f(t_i, t_{-i}) / \overline{f}(t_i)$. After knowing his value, bidder i places a bid $b_i \in \mathbb{R}_+$. He receives the object if $b_i > \max_{j \neq i} b_j$. We consider both first and second price auctions. As Milgrom and Weber (1982a) argue, second price and English auctions are equivalent in the case of private values, as we assume here. In a first price auction, if $b_i > \max_{j \neq i} b_j$, bidder i's utility is $u(t_i - b_i)$ and is $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. In a second price auction, bidder i's utility is $u(t_i - \max_{j \neq i} b_j)$ if $b_i > \max_{j \neq i} b_j$ and $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. For both auctions, ties are randomly broken.

By reparametrization, we may assume, without loss of generality, $\left[\underline{t}, \bar{t}\right] = [0, 1]$. It is also useful to assume $n = 2$, but this is not needed for most of the results. For

[3]For the reader familiar with Mertens and Zamir (1986)'s construction of universal type spaces: we make the usual assumption in auction theory that the model is "closed" at the first level, that is, all higher level beliefs are consistently given by (and collapse to) f.

most of the paper, we assume risk neutrality, that is, $u(x) = x$. Thus, unless otherwise stated, the results will be presented under the following setup:

BASIC SETUP: *There are $n = 2$ risk neutrals bidders, that is, $u(x) = x$, with private values distributed according to a symmetric density function $f : [0,1]^2 \to \mathbb{R}_+$.*

A pure strategy is a function $b : [0,1] \to \mathbb{R}_+$, which specifies the bid $b(t_i)$ for each type t_i. The interim payoff of bidder i, who bids β when his opponent $j \neq i$, follows $b : [0,1] \to \mathbb{R}_+$ is given by

$$\Pi_i (t_i, \beta, b(\cdot)) = u(t_i - \beta) F\left(b^{-1}(\beta) \mid t_i\right) = u(t_i - \beta) \int_{\underline{t}}^{b^{-1}(\beta)} f(t_j \mid t_i)\, dt_j,$$

if it is a first price auction and

$$\Pi_i (t_i, \beta, b(\cdot)) = \int_{\underline{t}}^{b^{-1}(\beta)} u(t_i - b(t_j)) f(t_j \mid t_i)\, dt_j,$$

if it is a second price auction.

We focus attention on symmetric monotonic pure strategy equilibrium (SMPSE), which is defined as $b(\cdot)$ such that $\Pi_i(t_i, b(t_i), b(\cdot)) \geq \Pi_i(t_i, \beta, b(\cdot))$ for all β and t_i. The usual definition requires this inequality to be true only for almost all t_i. This stronger definition creates no problem and makes some statements simpler, as those about the differentiability and continuity of the equilibrium bidding function (otherwise, such properties should be always qualified by the expression "almost everywhere"). Under our assumptions, the second price auction always has a SMPSE in a weakly dominant strategy, which is $b(t_i) = t_i$. Finally, we say that a $b(\cdot)$ is an ε-equilibrium if $\Pi_i(t_i, b(t_i), b(\cdot)) + \varepsilon \geq \Pi_i(t_i, \beta, b(\cdot))$ for all β and t_i.

3 The class of distributions

Modeling types as continuous real variables is a widespread practice in auction theory. The reason for that is clear: continuous variables allow the use of the convenient tools of calculus, such as derivatives and integrals, to obtain precise characterizations and uniqueness results. This is a very important advantage that should not be underestimated. (See Remark 8 below for a consequence of this). From this, we are naturally led to work with continuous density functions to represent the distribution of private values of bidders. However, the set of continuous density functions is inconvenient in the sense that it is an excessively big set. Since this set includes so many possibilities, it is hard to study pure strategy equilibrium existence (MPSE) in such a set. Thus, auction theorists have to restrict their attention to smaller sets, that is, sets of densities that satisfy extra assumptions. An example is the restriction to affiliated densities, when one is interested in dependence (see Milgrom and Weber, 1982). Another example is the set of assumptions introduced by Maskin and Riley (2000) to study asymmetric auctions.

The restriction to special set of distributions seems unavoidable, but a basic problem it brings is that the set considered may not be truly representative of the general situation. If some result is true for a particular set of distributions but not in general, auction theorists may base their conclusions in misleading results. There are at least two ways to avoid this problem. One is to use sound economic reasons for the restriction. The other one is to work with sufficiently rich set of distributions that are dense in the general set, but there are still tractable. If some continuity properties exist (as they do), the results may be regarded as representative of the general situation. In this section we describe a set of distributions that have both advantages: it has a sound economic reason for its adoption and it is dense in the general set of distribution, thus providing a reliable picture of the problem through the continuities properties of equilibria.

Let us first discuss the economic motivation for adopting a set of distributions, and then describe our proposed set. Observe that the value of the single object in the auction is expressed up to cents and is obviously bounded. Thus, the number of actual possible values is finite. From this, why we do not use only a discrete distribution of values? Because this would prevent us from using the convenient tools of calculus, which are very helpful in the analysis of auctions, as we discussed above. However, instead of insisting in the (actual) case of discrete values, we may consider continuous values but but impose, on the other hand, that the density functions are simple (see Figure 1 in the Introduction). This seems even closer to the reality than the case of general continuous densities. Thus, we restrict attention to the set of simple functions $\mathcal{D}^k$, as defined below.

For simplicity, consider an auction between two players. Let $\mathcal{D}$ be the set of distributions in $[0,1]^2$. For $k \in \mathbb{N}$, define the transformation $T^k : \mathcal{D} \to \mathcal{D}$ by

$$T^k(f)(x,y) = k^2 \int_{\frac{p-1}{k}}^{\frac{p}{k}} \int_{\frac{m-1}{k}}^{\frac{m}{k}} f(\alpha,\beta)\, d\alpha d\beta,$$

whenever $(x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m, p \in \{1, 2, ..., k\}$. Observe that $T^k(f)$ is constant over each square $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Let $\mathcal{D}^k$ be the image of $\mathcal{D}$ by T^k, that is, $\mathcal{D}^k \equiv T^k(\mathcal{D})$. Thus, T^k is a projection from the infinite dimensional space $\mathcal{D}$ over the finite dimensional space $\mathcal{D}^k$. Indeed, $\mathcal{D}^k$ is finite dimensional set because any density function $f \in \mathcal{D}^k$ can be described by a matrix $A = (a_{ij})_{k \times k}$, as the figure below illustrates.

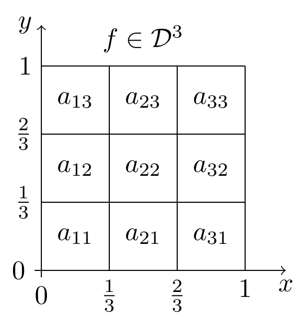


Figure 2: A density $f \in \mathcal{D}^k$ can be represented by a matrix $A = (a_{ij})$.

For general k, the description of $f \in \mathcal{D}^k$ can be made as follows:[4]

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right], \tag{1}$$

for $m, p \in \{1, 2, ..., k\}$. The definition of f at the zero measure set of points $\{(x,y) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

The restriction to the set $\mathcal{D}^\infty \equiv \cup_{k\in\mathbb{N}} \mathcal{D}^k$ of distributions implies no economic loss of generality in the problem we are studying. Note also that the closure $\overline{\mathcal{D}^\infty}$ is the set of all densities $\mathcal{D}$. Now we describe how the equilibrium existence problem can be completely solved in the set $\mathcal{D}^\infty$.

First, recall the standard result of auction theory on SMPSE in private value auctions: if there is a differentiable symmetric increasing equilibrium, it satisfies the differential equation (see Krishna 2002 or Menezes and Monteiro 2005):

$$b'(t) = \frac{t - b(t)}{F(t|t)} f(t|t).$$

If f is Lipschitz continuous, one can use Picard's theorem to show that this equation has a unique solution. Under some assumptions (basically, Property VI' of the previous subsection), it is possible to ensure that this solution is, in fact, equilibrium. Now, for $f \in \mathcal{D}^\infty$, the right hand side of the above equation is not continuous, and one cannot directly apply Picard's theorem. We proceed as follows.

First, we show that if there is a symmetric increasing equilibrium b, under mild conditions (satisfied by $f \in \mathcal{D}^\infty$), b is continuous. We also prove that b is differentiable at the points where f is continuous. Thus, for $f \in \mathcal{D}^\infty$, b is continuous everywhere and differentiable everywhere but, possibly, at the points of the form $\frac{m}{k}$. See figure 3.

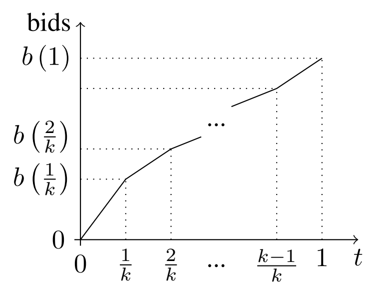


Figure 3: Bidding function for $f \in \mathcal{D}^k$.

With the initial condition $b(0) = 0$ and the above differential equation being valid for the first interval $\left(0, \frac{1}{k}\right)$, we have uniqueness of the solution on this interval and, thus, a unique value of $b\left(\frac{1}{k}\right)$. Since b is continuous, this value is the initial condition for the interval $\left(\frac{1}{k}, \frac{2}{k}\right)$, where we again obtain a unique solution and the uniqueness of the value $b\left(\frac{2}{k}\right)$. Proceeding in this way, we find that there is a unique b which can be

[4]For $n > 2$ players, the density function can be described by an array $[f] \in \mathbb{R}^{k^n}$. The reader can find details of this in the supplement of the paper.

a symmetric increasing equilibrium for an auction with $f \in \mathcal{D}^{\infty}$. In the supplement to this paper we prove the following:

Theorem 1 *Assume that u is twice continuously differentiable, $u' > 0$, $f \in \mathcal{D}^k$, f is symmetric and positive ($f > 0$).*[5] *If $b : [0,1] \to \mathbb{R}$ is a symmetric increasing equilibrium, then b is continuous in $(0,1)$ and is differentiable almost everywhere in $(0,1)$.*[6] *Moreover, b is the unique symmetric increasing equilibrium. If $u(x) = x^{1-c}$, for $c \in [0,1)$, b is given by*

$$b(x) = x - \int_0^x \exp\left[-\frac{1}{1-c}\int_\alpha^x \frac{f(s \mid s)}{F(s \mid s)} ds\right] d\alpha. \qquad (2)$$

Proof. *See the supplement to this paper.* ■

Having established the uniqueness of the candidate for equilibrium, our task is reduced to verifying whether this candidate is, indeed, an equilibrium. We complete this task in the section 5.

Remark 2 *The benefits of chosing $f \in \mathcal{D}^{\infty}$ are deeply related to our focus in pure strategy equilibrium. This kind of equilibrium is the most common in auction theory. It has an important advantage over mixed strategy equilibria: the latter hardly can be characterized, even for bimatrix games, while the former is explictly and uniquely determined in general. Since we already know that mixed strategy equilibrium exist, and little is said in general beyond its existence, it seems very natural to follow the standard practice in auction theory and restrict attention to pure strategy equilibrium, as we do.*

Even if the reader insists on considering the more general set of pdf's $\mathcal{D}$—being aware that this is a matter of mathematical generality, but not of economic generality—our set $\mathcal{D}^{\infty}$ is still dense in $\mathcal{D}$ and, thus, may arbitrarily approximate any conceivable pdf in $\mathcal{D}$. In fact, the following result shows that equilibrium existence in the set $\mathcal{D}^{\infty}$ is sufficient for equilibrium existence in $\mathcal{D}$. This provides an additional justification of the method.

Theorem 3 *Let $f \in \mathcal{D}$ be continuous and symmetric. If $T^k(f)$ has a differentiable symmetric pure strategy equilibrium for all $k \geq k_0$, then so does f, and it is the limit of the equilibria of $T^k(f)$ as k goes to infinity.*
Proof. *See the supplement to this paper.* ■

Although a complete converse of the above theorem is not possible, because the equilibrium inequality can be approximated by above (see below), the following theorem provides a partial converse.

[5] In fact, it is not necessary that f has full support. See the supplement of the paper for details.
[6] b may be non-differentiable only in the points $\frac{m}{k}$, for $m = 1, ..., k$

Theorem 4 *Let $f \in \mathcal{D}$ be continuous and symmetric and assume that f has a SMPSE. Then for each $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that b^k given by*[7]

$$b(x) = x - \int_0^x \exp\left[-\frac{1}{1-c}\int_\alpha^x \frac{f^k(s \mid s)}{F^k(s \mid s)} ds\right] d\alpha. \tag{3}$$

is an ε-equilibrium for T^k, for all $k \geq k_\varepsilon$.
Proof. *See the supplement to this paper.* ■

The above result is satisfactory for numerical applications, because numerical calculation will involve errors anyway. Thus, in this case the above restriction is innocuous. It is useful to emphasize, however, that the set $\mathcal{D}^\infty = \cup_{k \in \mathbb{N}} \mathcal{D}^k$ is meaningful for analysis of equilibrium in auctions even without numerical simulations. This is illustrated in the next section.

4 SMPSE are rare

As an illustration of the theoretical advantages of using the set of disbributions $\mathcal{D}^k$ described in the last section, we prove now that when there are two players, the the measure of the set of densities $f \in \mathcal{D}^k$ which has SMPSE goes to zero as $k \to \infty$. In order to formalize this, let λ^k be the Lebesgue measure on $\mathcal{D}^k$ and let $\mathcal{P}^k$ be the set of densities $f \in \mathcal{D}^k$ which have a SMPSE. We have the following:

Theorem 5 *Assume that there are two risk neutral players. Then $\lambda^k\left(\mathcal{P}^k\right) \to 0$ as $k \to \infty$.*
Proof. *See the supplement to this paper.* ■

The proof of this theorem follows a simple idea: the equilibrium existence depends on a series of inequalities, the number of which increases with k. Although some care is needed for rigorously establishing the result, this simple observation is the heart of the argument. This gives us the intuition that the equilibrium constraints defining equilibrium increase faster than the degrees of freedom of the problem, when k increases.

This theorem does not depend on numerical simulations, although it was inspired by them. Although this concerns SMPSE existence in $\mathcal{D}^k$, an analogous theorem holds for the set of all densities $\mathcal{D}$: the set of distributions with pure strategy equilibrium has zero measure. A precise statement of this fact requires some care, because $\mathcal{D}$ is an infinite dimensional set. Let us introduce some notation. Let μ be a measure over $\mathcal{D}$ and let μ^k be the corresponding measure induced on $\mathcal{D}^k$ by the projection T^k , that is, if $E \subset \mathcal{D}^k$ is a measurable subset, define $\mu^k(E)$ as $\mu\left(\left(T^k\right)^{-1}(E)\right)$. Let λ^k represent the Lebesgue measure of the finite dimensional set $\mathcal{D}^k$. Of course a measure μ over $\mathcal{D}$ can put all its weight in distributions with SMPSE. In fact, for general μ, the support can be even just one point. However, if μ is sufficiently "spread", that is, if μ^k is

[7]We denote by $f^k(s|s)$ and $F^k(s|s)$ the conditional pdf and cdf of T^k, respectively.

absolutely continuous with respect to λ^k, then the following theorem shows that μ puts zero measure on the set of distributions with SMPSE.[8,9]

Theorem 6 *Let μ and μ^k be as described above. Assume that there are two risk neutral players and there exists $M > 0$ such that $\mu^k \leq M\lambda^k$. Then, $\mu(\mathcal{P}) = 0$.*
Proof. *We just sketch the proof here. The proof of Theorem 5 can be adapted to show that the set of ε-equilibria $\mathcal{P}^{k,\varepsilon}$ also satisfy $\lambda^k\left(\mathcal{P}^{k,\varepsilon}\right) \to 0$ as $k \to \infty$ and $\varepsilon \to 0$. From Theorem 4, we have $\mathcal{P} \subset \left(T^k\right)^{-1}\left(\mathcal{P}^{k,\varepsilon}\right)$ for sufficiently large k and small ε. Thus,*

$$\mu(\mathcal{P}) \leq \mu\left(T^k\right)^{-1}\left(\mathcal{P}^{k,\varepsilon}\right) = \mu^k\left(\mathcal{P}^{k,\varepsilon}\right) \leq M\lambda^k\left(\mathcal{P}^{k,\varepsilon}\right).$$

Since $\lambda^k\left(\mathcal{P}^{k,\varepsilon}\right) \to 0$, then $\mu(\mathcal{P}) = 0$. ■

The result presented in Theorem 6 may be considered a negative result, because auction theory usually relies on pure strategy equilibria. This result suggests that the focus on symmetric monotonic equilibrium may be too narrow. Nevertheless, this is not yet sufficient to conclude that most of the equilibria are in mixed strategies. In fact, while we know that mixed strategy equilibria always exist (Jackson and Swinkels 2005), there is the possibility—not considered in our results—that there are equilibria in asymmetric or non-monotonic pure strategies.

5 SMPSE existence in $\mathcal{D}^k$

In this section we present a fast algorithm to test SMPSE existence for symmetric densities in $\mathcal{D}^k$. We begin with the two bidders case and then, generalize it to the n bidders case. Asymmetric auctions are considered in a separated section. In the supplement to this paper, we also discuss the algorithm with risk averse bidders, that is, when bidders' utility are $u(x) = x^{1-c}$ and $c > 0$.

5.1 Two risk neutral players case

Theorem 1 establishes the uniqueness of the candidate for symmetric increasing equilibrium for $f \in \mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$. We have now only to check if the unique candidate is indeed equilibrium, that is, whether it is an optimal reply. In economics, this is usually done by checking the second order condition. In auction theory, it is more common to appeal to monotonicity arguments based on a single crossing condition (see, among others, Milgrom and Weber, 1982 and especially Athey, 2001). These methods give sufficient conditions for equilibrium, but these conditions are, in general, not necessary. Sufficient and necessary conditions would not only provide grounds to understand what really entails equilibrium existence, but also to work with the more general possible setup. Thus, we take here another approach, which gives necessary

[8]The assumption of the theorem is, in fact, slightly stronger than just absolutely continuity.

[9] There is an alternative notion of "zero measure" for infinite dimensional sets, introduced by Christensen (1974). See also Hunt, Sauer and Yorke (1992) and Anderson and Zame (2001). Stinchcombe (2000) discuss some drawbacks of this approach.

and sufficient conditions for equilibrium existence. This is done by checking directly the equilibrium conditions, as we explain next.

Let $b(\cdot)$, given by (3) with $c = 0$, denote the candidate for equilibrium. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, z)$ represent the expected interim payoff of a player of type x who bids as a type z, that is, $\Delta(x, z) \equiv \Pi(x, b(z)) - \Pi(x, b(x))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $z \in [0, 1]^2$. In other words, the equilibrium condition require to check an inequality for an infinite pair of points. Of course, it is not possible to check an inequality at an infinite number of points.

If $\Delta(x, z)$ is continuous, then an approximation algorithm could check the inequality only at some points and, with some confidence, ensure equilibrium existence. Of course, this method would not be exact in the sense that approximation errors are inerent to the algorithm. However, for $n = 2$ players, the next theorem shows that when $f \in \mathcal{D}^k$ there is an exact algorithm, that does not introduce errors, and it is fast because it requires only a small number of comparisons.

Theorem 7 *Consider Symmetric Risk Neutral Private Value Auction with two players with $f \in \mathcal{D}^\infty = \cup_{k \geq 1} \mathcal{D}^k$. There exist an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. For $f \in \mathcal{D}^k$, the algorithm requires less than $3\left(k^2 + k\right)$ comparisons. The algorithm is exact, in the sense that errors can occur only in elementary operations.*[10]
Proof. *See the supplement to this paper.* ■

Remark 8 *It is important to compare this result with the best algorithms for solving simpler games as bimatrix games (see Savani and von Stengel 2006). While best known algorithms for bimatrix games requires operations that grow* exponentially *with the size of the matrix, our operations increases with k^2. We do not state that the algorithm runs in polynomial time because our problem is in continuous variables, not in discrete ones. "Polynomial time" would be slightly vague here, since errors of approximations are possible. Nevertheless, as stated, the possible errors are elementary and require a small number of operations. This allows one to realize the important benefits of working with continuous variables but density functions in $\mathcal{D}^k$, as we propose. The characterization of the strategies obtained through differential equations allows one to drastically reduce the computational effort, by reducing the equilibrium candidates to one. The fact that we work on $\mathcal{D}^k$ allows to precisely characterize a small number of points to be tested for the equilibrium condition. The speed of the method allows auction theorists to run simulations for a big number of trials and get a good figure of what happens in general. From this, conjectures for theoretical results can also be derived.*

[10]By elementary operations we mean sums, multiplications, divisions, comparisons and square and third degree roots.

The proof of this theorem is long, because $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and the determination of its maxima for each of these sets. It turns out that we need to check a different number of points (between 1 and 5) for some of these squares.

5.2 Equilibrium results for n players

The results of the previous subsection are restricted to the case of (symmetric, private values, risk neutral) 2 players. However, one can use grid distributions (those in $\mathcal{D}^{\infty}$) to study auctions in a much more general setup. In this subsection, we discuss the extension to the n bidders symmetric case.

The mathematical expression for this case (see the supplement) are obviously more complex, but the ideas are essentially the same of the $n = 2$ case. As before, the equilibrium candidate is unique and we have an expression for it. Thus, SMPSE will be established if and only if $\Delta(x, z) = \Pi(x, b(z)) - \Pi(x, b(x))$ is non-positive. We can test the signal of $\Delta(x, z)$ for $(x, z) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for m, $p \in \{1, ..., k\}$. This is simplified to check non-positiveness of a polynomial over $[0, 1]^2$. The only difference from the $n = 2$ case is that in this last case the polynomial is of degree 3 and we can analytically solve it. For $n > 2$, the polynomial (in the two variables, x and z) has a degree at least $n + 1$ and we have to rely in numerical methods for finding minimal points. The following establishes the existence of an algorithm to solve for this, that may make errors by approximation:

Theorem 9 *Consider symmetric risk neutral private value auction with n players with $f \in \mathcal{D}^{\infty}$. There exist an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. Errors are commited in finding roots of polynomials and in elementary operations.*
Proof. *See the supplement to this paper.* ■

Note that we did not make statements about the speed of the method. This is just because this speed depends on the numerical method used to find roots of polynomials. We were unable to find good characterizations of the running time of solutions to this problem.

6 Grid distributions in the general setup

The method can also be applied to the general setup, that is, asymmetric interdependent values auctions with n risk averse players. For this general case, there is an important difference with respect to the previous case. With symmetry and risk neutrality, it is possible to know exactly what is the unique candidate for equilibrium. In the asymmetric case or when there is risk aversion, one has just (a system of) differential equations for which there is no explicit solution, except in special cases (see Plum, 1992). Thus, an algorithm for testing for equilibrium has to include the additional step of finding the solution of the system of differential equations. After finding the candidate for equilibrium, the same idea of testing the equilibrium conditions are applied.

It is worth emphasizing that finding numerical solutions to differential equations has been done before. See Marshall et. al. (1994), for instance. There is absolutely no novelty in this. However, when one tries to consider all possible continuous functions, it becomes clear an important limitation of numerical studies: the fact that the set of different distributions is simply too big to be well explored. Since we cannot explore a good part of all possibilities, there is the suspicious that the knowledge derived from these simulations are not representable of the whole picture. This may be a reason for the low use of numerical methods in auction theory.

However, when one focus attention to the sets $\mathcal{D}^k$ for $k \in \mathbb{N}$, it is possible to explore a sufficiently large number of cases, if we have algorithms that are fast enough. We are not able to theoretically characterize the speed of our algorithm in the general case, but our simulations show that it is very fast.

In fact, the method can also be potentially applied to setups including interdependencies of values, that is, when the value functions $v_i(\mathbf{t})$ depend on the vector of all types $\mathbf{t} = (t_1, ..., t_n)$. The idea of using grid functions $f \in \mathcal{D}^\infty$ has to be extended now to grid functions $v_i(\mathbf{t})$. The procedures shall be similar, but it is beyond the scope of this paper to describe the algorithms for this case. The important point is that the method is useful even in this more general auction setup.

7 The Revenue Ranking of Auctions

Now, we illustrate how the method described in the previous section can be used to address the problem of revenue ranking of the first price and second price auctions. Let us denote by R_2^f the expected revenue (with respect to $f \in \mathcal{D}^k$) of the second price auction.[11] Similarly, R_1^f denotes the expected revenue (with respect to $f \in \mathcal{D}^k$) of the first price auction. When there is no need to emphasize the pdf $f \in \mathcal{D}^k$, we write R_1 and R_2 instead of R_1^f and R_2^f. Below, μ refers to the natural measure defined over $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$, as further explained in the supplement to this paper.

The following theorem gives the expression of the expected revenue difference $\Delta_R^f \equiv R_2^f - R_1^f$ between the second and the first price auctions and it is *not* restricted to densities in $f \in \mathcal{D}^\infty$.

Theorem 10 *Assume that f has a SMPSE in the first price auction. The revenue difference between the second and the first price auction is given by*

$$\int_0^1 \int_0^x b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \tag{4}$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$.

[11] Remember that, since we are working with private values, second price auctions are equivalent to English auctions.

Proof. *See the appendix.* ■

In order to make a relative comparison, we define $r \equiv \frac{R_2^f - R_1^f}{R_2^f}$, for each f. Generating a uniform sample of $f \in \mathcal{D}^k$, we can obtain the probabilistic distribution of Δ_R^f or of r. The procedure to generate $f \in \mathcal{D}^k$ uniformly is described in the supplement to this paper. The results are shown in subsection 7.1 below.

Moreover, we can also obtain theoretical results about what happens for $\mathcal{D}^k$ for a large k and even for $\mathcal{D}^\infty = \cup_{k=1}^{\infty} \mathcal{D}^k$. Nevertheless, for the last case, one has to be careful with the meaning of the "uniform" distribution. In the supplement to this paper we show that a natural measure can be defined for $\mathcal{D}^\infty$, which is analogous to Lebesgue measure, although it cannot have all the properties of the finite dimensional Lebesgue measure.

In this fashion, we are able to obtain previsions based on simulations and also theoretical results. One possible objection to this approach is that it considers too equally the pdf's in the sets $\mathcal{D}^k$. But this is just because we are not assuming any specific information about the context where the auction runs—in some sense, this is a "context-free"approach. If one has information on the environment where the auction runs, so that one can restrict the set of suitable pdf's, then the uniform measure should be substituted by the empirical measure obtained from this environment. Obviously, the method can be easily adapted to this, once one has such "empirical measure" of the possible distributions.

Now, we present the results that one can obtain using this approach.

7.1 Numerical results

[TO BE INCLUDED]

8 Related literature and conclusion

In this paper we advocated for the use a special class of distributions with dependence and symmetry in auction theory. This class of distribution is more general than normally considered ones, but can be treated both with theoretical and numerical methods. We illustrated the potential applications with theoretical results and computer experiments (simulations). It is shown that a fast algorithm exists for determining symmetric pure strategy equilibrium existence in auctions with n players.

Appendix

Proof of Theorem 10.

The dominant strategy for each bidder in the second price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$
\begin{aligned}
P^2 &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} y f(y|x)\, dy \cdot f(x)\, dx = \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} [y - b(y)]\, f(y|x)\, dy \cdot f(x)\, dx + \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\, dy \cdot f(x)\, dx,
\end{aligned}
$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx$, from the first order condition: $b'(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$
\begin{aligned}
& \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\, dy \cdot f(x)\, dx \\
= & \int_{[\underline{t},\bar{t}]} \left[b(x)\, F(x|x) - \int_{[\underline{t},x]} b'(y)\, F(y|x)\, dy \right] \cdot f(x)\, dx \\
= & \int_{[\underline{t},\bar{t}]} b(x)\, F(x|x) \cdot f(x)\, dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y)\, F(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

In the last line, the first integral is just the expected payment for the first price auction, P^1. Thus, we have

$$
\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y)\, F(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$
\begin{aligned}
b^{'}(y) &= 1-L(y|y)-\int_{[\underline{t},y]}\partial_y L(\alpha|y)\,d\alpha \\
&= \frac{f(y|y)}{F(y|y)}\int_{[\underline{t},y]}L(\alpha|y)\,d\alpha.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]}\frac{f(y|y)}{F(y|y)}\int_{[\underline{t},y]}L(\alpha|y)\,d\alpha\left[\frac{F(y|y)}{f(y|y)}-\frac{F(y|x)}{f(y|x)}\right]f(y|x)\,dy\cdot f(x)\,dx \\
&= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]}\left[\int_{[\underline{t},y]}L(\alpha|y)\,d\alpha\right]\cdot\left[1-\frac{F(y|x)}{f(y|x)}\cdot\frac{f(y|y)}{F(y|y)}\right]\cdot f(y|x)\,dy\cdot f(x)\,dx
\end{aligned}
$$

This is the desired expression. ■

References

[1] ALIPRANTIS, C. D. AND K. C. BORDER (1999): *Infinite Dimensional Analysis: a Hitchhiker's guide*. Springer-Verlag, Berlin, Heidelberg.

[2] ANDERSON, R. M. AND W. R. ZAME (2001): "Genericity with Infinitely Many Parameters," Advances in Theoretical Economics, 1, 1, article 1.

[3] ATHEY, S. (2001): "Single Crossing Properties and The Existence of Pure Strategy Equilibria in Games of Incomplete Information," Econometrica, 69, 861-899.

[4] ATHEY, S. AND P. H. HAILE. (2002): "Identification of Standard Auction Models," Econometrica, 70, 2107-2140.

[5] BARLOW, R. E. AND F. PROSCHAN (1965): "Mathematical Theory of Reliability," John Wiley & Sons, Inc., New York.

[6] CHRISTENSEN, J. P. R. (1974): *Topology and Borel Structure*, vol. 10 in North Holland Mathematics Studies. Amsterdam: North-Holland Publishing Company.

[7] DIACONIS, P. AND D. FREEDMAN (1980): "Finite Exchangeable Sequences," The Annals of Probability, 8, 745-764.

[8] ENGELBRECHT-WIGGANS, R., P. MILGROM AND R. J. WEBER (1983): "Competitive Bidding and Proprietary Information," Journal of Mathematical Economics, 11, 161-169.

[9] ESARY, J. B., PROSCHAN, F. AND WALKUP, D. W. (1967): "Association of Random Variables with Applications," Annals of Mathematical Statistics, 38, 1466-1474.

[10] FRIEDMAN, M. (1953): “The Methodology of Positive Economics,” in Essays in Positive Economics, University of Chicago Press, Chicago, 1953, p. 1-43.

[11] HEWITT, E. AND SAVAGE, L. J. (1955): “Symmetric measures on Cartesian products,” Transactions of the American Mathematical Society, 80, 470-501.

[12] HUNT, B. R., SAUER, T. AND YORKE, J. A. (1992): “Prevalence: A Translation-Invariant ‘Almost Every’ on Infinite-Dimensional Spaces,” Bulletin (New Series) of the American Mathematical Society, 27, 217-238.

[13] JACKSON, M. O. AND J. M. SWINKELS (2005): “Existence of Equilibrium in Single and Double Private Value Auctions,” Econometrica, 73, 93-140.

[14] JOGDEO, K. (1977): “Association and Probability Inequalities,” Annals of Statistics, 5, 495-504.

[15] KARLIN, S. (1968): “Total Positivity,” vol. 1, Stanford Univ. Press, Stanford, California.

[16] KLEMPERER, P., (2003): “Using and Abusing Economic Theory,” Journal of the European Economic Association, 1, 272-300.

[17] KLEMPERER, P., (2004): “Auctions: Theory and Practice,” Princeton University Press, Princeton, New Jersey.

[18] KOOPMANS, T. C. (1957): “Three Essays on the State of Economic Science” New York: McGraw-Hill.

[19] KOTZ, S., Q. WANG AND K. HUNG (1988): “Interrelations Among Various Definitions of Bivariate Positive Dependence,” in: Block, et. al., IMS Lecture Notes in Statistics, 16, p. 333-349.

[20] KRISHNA, V. (2002): “Auction Theory,” Elsevier Science, San Diego, USA.

[21] LEBRUN, B. (2002): “Continuity of the First Price Auction Nash Equilibrium Correspondence,” Economic Theory, 20, 435-453.

[22] LEBRUN, B. (2006): “Uniqueness of the equilibrium in first-price auctions,” Games and Economic Behavior, 55, 131-151.

[23] LEHMANN, E. L. (1966): “Some Concepts of Dependence,” Annals of Mathematical Statistics, 37, 1137–1153.

[24] LEHMANN, E. L. (1959): “Testing Statistical Hypothesis,” New York: Wiley.

[25] MASKIN, E. (2004): “The Unity of Auction Theory: Milgrom’s Masterclass,” Journal of Economic Literature, 42, 1102-1115.

[26] MASKIN, E. AND J. RILEY (1984): “Optimal Auctions With Risk Averse Buyers,” Econometrica 52, 1473–1518.

[27] MASKIN, E. AND J. RILEY (2000): "Asymmetric Auctions," Review of Economic Studies, 67, 413-438.

[28] MCMILLAN, J. (1994): "Selling Spectrum Rights," The Journal of Economic Perspectives, 8, 145-162.

[29] MENEZES, F. AND MONTEIRO, P. K. (2005): "An Introduction to Auction Theory," Oxford University Press, New York, USA.

[30] MILGROM, P. R. (1981a): "Good News and Bad News: Representation Theorems and Applications," Bell Journal of Economics, 12, 380-391.

[31] MILGROM, P. R. (1981b): "Rational Expectations, Information Acquisition and Competitive Bidding," Econometrica, 49, 921-943.

[32] MILGROM, P. R (1989): "Auctions and Bidding: A Primer," Journal of Economic Perspectives, 3, 3-22.

[33] MILGROM, P. R AND SHANNON, C. (1994): "Monotone Comparative Statics," Econometrica, 62, 157-180.

[34] MILGROM, P. R AND WEBER, R. J. (1982a): "A Theory of Auctions and Competitive Bidding," Econometrica, 50, 1089-1122.

[35] MILGROM, P. R AND WEBER, R. J. (1982b): "The Value of Information in a Sealed-Bid Auction," Journal of Mathematical Economics, 10, 105-114.

[36] MONTEIRO, P. K. AND H. MOREIRA (2006): "First Price Auctions without Affiliation," Economic Letters, forthcoming.

[37] MYERSON, R. (1981): "Optimal Auction Design," Mathematics of Operations Research, 6, 58-73.

[38] LAFFONT, J.-J. (1997): "Game Theory and Empirical Economics: The Case of Auction Data," European Economic Review, 41, 1-35.

[39] LI, T., H. J. PAARSCH, AND T. P. HUBBARD (2007): "Semiparametric Estimation in Models of First-Price, Sealed-Bid Auctions with Affiliation," working paper.

[40] PERRY, M. AND RENY, P. J. (1999): "On the failure of the linkage principle in multi-unit auctions," Econometrica, 67, 895-900.

[41] PINKSE, J. AND TAN, G. (2005): "The Affiliation Effect in First-Price Auctions," Econometrica, 73, 263-277.

[42] RENY, P. J. AND ZAMIR, S. (2004): "On Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions", Econometrica, 72, 1105-1125.

[43] ROTH, A. (2002): "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics", Econometrica, 70, 1341-1378.

[44] SAVANI, R. AND B. VON STENGEL (2006): "Hard-to-Solve Bimatrix Games", Econometrica, 74, 397-429.

[45] STINCHCOMBE, M. B. (2000): "The Gap Between Probability and Prevalence: Loneliness in Vector Spaces," Proceedings of American Mathematical Society, 129, 2, 451-457.

[46] TONG, Y. L. (1980): "Probability Inequalities for Multivariate Distributions," New York: Academic Press.

[47] TOPKIS, D. M. (1978): "Minimizing a Submodular Function on a Lattice," Mathematics of Operations Research, 26, 8-37.

[48] VICKREY, W. (1961): "Counterspeculation, Auctions and Competitive Sealed Tenders," Journal of Finance, 16, 8-37.

[49] WILSON, R. (1969): "Competitive Bidding with Disparate Information," Management Science, 15, 446-448.

[50] WILSON, R. (1977): "A Bidding Model of Perfect Competition," Review of Economic Studies, 4, 511-518.

[51] Wilson, R. (1987): "Game-Theoretic Approaches to Trading Processes," in Advances in Economic Theory: Fifth World Congress, ed. by T. Bewley, chap. 2, pp. 33 - 77. Cambridge University Press, Cambridge.

[52] YAMASAKI, Y. (1985): "Measures on Infinite Dimensional Spaces". Series in Pure Mathematics, vol. 5, World Scientific, Singapore, Philadelphia.

[53] YANAGIMOTO, T. (1972): "Families of Positively Dependent Random Variables," Annals of the Institute of Statistical Mathematics, 24, 559-573.