On the Analysis of Asymmetric First Price Auctions*

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Abstract

Asymmetric first price auctions are notoriously intractable. We provide a new set of tools for thinking about such auctions, One key step is to connect the behavior of equilibria to the pho-concavity of the underlying distributions. Another is to show how one can use surplus expressions related to symmetric auctions to bound behavior in asymmetric auctions. We apply these tools to studying procurement auctions in which one seller is preferred, based on better reliability or quality, or may have lower costs. We study the performance of simple first and second price mechanisms in such settings, and how they relate to the optimal mechanism. Under a variety of conditions on cost distributions, a second price auction with bonuses outperforms, on an outcome by outcome basis, any of a class of first price auctions that includes on one extreme a request for proposals and on the other a standard first price auction.

Abstract

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1 Introduction

This paper contributes to our understanding of asymmetric first price auctions. We study the case of an auction with two sellers, 1 and 2, having costs c_1 and c_2 distributed independently but according to distributions F_1 and F_2 that need not be symmetric.[1] We show a strong connection between the equilibria of such auctions and the local concavity of F_1 and F_2 and various associated objects. Along the way, we provide a useful extension to a result by Prekopa and Borel on the relationship between the concavity of a function and the concavity of its integral.

The key object in understanding asymmetric auctions is a function ϕ that pairs each cost type c_2 with the c_1 that is tied with c_2 in equilibrium. Perhaps the most difficult problem in auctions like this is that behavior of the equilibrium at any cost pair c_1, c_2, $c_1 = \phi(c_2)$ is determined by the surplus each player has at that point. But, this is determined in turn by the behavior of ϕ at all higher costs. One key part of our analysis is to relate the behavior of surplus in these auctions to surplus in various counterpart symmetric auctions. These allow new results on the slopes and shape of equilibrium bid functions, and the allocations ϕ they generate.

We think these tools will have broader general applicability. But, for much of the paper, we focus on a specific application: we study *offset auctions*, in which the distribution of costs for one player is a horizontal shift of that for the other. We think such auctions are of immense practical interest. To see why, consider a buyer with potential suppliers 1 and 2. Each supplier has privately known costs c_i, drawn independently from F. The buyer places known value v on procuring from 2, but value $v + \Delta$, $\Delta > 0$, on procuring from 1.[2] The difference Δ could reflect higher quality from 1, more reliable delivery, or a premium that the buyer places on encouraging 1 to remain in the industry.

[1] We work in terms of a procurement auction, because our lead application is one of procurement. But, as is the norm in this literature, all of the results have their obvious counterparts in a standard auction setting.

[2] So, this is different than the setting of Manelli and Vincent (1995) in which low costs suggest low quality.

A common mechanism for procurement in such settings is the Request for Proposals (RFP). In the simplest RFP, each supplier submits a proposal as to what will be provided, and at what price. The supplier chooses his favorite proposal at the stated price. In our setting, this reduces to a first price auction, but one in which 2 wins only if his bid is at least Δ below that of 1.[3] But, as we will show, this is isomorphic to an auction in which the buyer is indifferent between the sellers, but 1 has a cost distribution shifted by Δ to the left of 2.

An alternative is to run a standard sealed bid first price auction: despite his preference for 1, the buyer commits to taking the lowest bid. The auction is thus a standard symmetric one. The RFP process allows the buyer to choose his favorite proposal, and so might be more efficient than a standard auction. But, the standard auction makes the market more competitive, since it converts the market from the point of view of the sellers into one with homogeneous products. While simple, this is a potential benefit of auctions that is generally overlooked.[4]

A natural question is whether some intermediate form might be preferred. If the incremental value of buying from 1 is $\Delta = \$100,000$, might it be optimal to commit to act as if Δ were some intermediate number? In practice, auctions very close to this are also used: firms like Boeing use a first price mechanism, but choose the winning bidder based on price augmented by scores on dimensions such as reliability and technological capability. These scores are private to Boeing, but their total possible impact on the bid is known. So, if Boeing prefers one supplier over the other and the bids are close, they can chose their favorite. But, when bids are sufficiently far apart relative to the announced scoring rule, they are stuck with the low bidder, regardless of which bid is ex-post more attractive.

To explore this, let the *first price handicap auction* (FPHA) with handi-

[3] In some settings, an open RFP is used, with multiple rounds of offers. An example is the 2003 competition between Boeing and Airbus to sell plane to Iberia Airlines. If Δ is common knowledge, this is equivalent to a second price auction with bonus Δ paid to 1, since he needs only match the costs of 2 plus Δ to be picked.

[4] It has been observed that public agencies such as the Tennessee Valley Authority paid more on average for electrical turbines than did private utilities. This is attributed to the power of information in sustaining tacit collusion in a repeated game, since the public utilities ran public auctions while pricing to the private utilities remained secret. But, the result of lower prices to the private utilities is more surprising than commonly understood: The auction format forced the public agencies to act as if the product was homogeneous. In contrast, pricing to the private utilities was done in a setting where the utilities paid attention to their preferences between manufacturers. There was thus a powerful force in the direction of *higher* pricing to the private utilities.

cap A be such that 1 wins when $b_1 < b_2 + A$, and 2 otherwise.[5] The winning bidder receives his bid. The case $A = 0$ is the standard first price auction. The case $A = \Delta$ is, in our setting, equivalent to a RFP, since the buyer will then always choose the bid which is ex-post most attractive. As might be expected, however, it will turn out that the optimal A is intermediate between 0 and Δ, trading off efficiency and the amount of competition generated.

These auctions have an odd feature. Imagine that costs have support $[0, 1]$. Then, $\beta_1(0) = \beta_2(0) + A$, where β_i is the bid function of i.[6] So, when costs are low, all of the allocative effects of the handicap format are undone in equilibrium. On the other hand, we will also show that when 2 has costs above $1 - A$, 1 wins for sure. So, in equilibrium, the FPHA creates lots of distortion away from the symmetric case when costs are high, but very little when costs are low.

Now consider a simple second price mechanism. A bonus B is specified. Each supplier submits a bid, and the low bidder wins. If 2 wins, he receives b_1. If 1 wins, he receives $b_2 + A$. The use of bonuses of this form is actually fairly prevalent: one can write, as part of the rules for a a second price (or open) auction, who is responsible for various costs associated with the work, such as specialized tooling. As such, one can have the extra costs involved with a new supplier competing against an old be born by the buyer, the seller, or some combination.

Our main result for offset auctions is this. For a large class of cost distributions, the second price mechanism dominates the first. For any first price mechanism, with handicap A_{FP}, there is a second price mechanism, with A_{SP}, such that, for each (c_1, c_2), the outcome achieved by the second price mechanism is never worse, and sometimes better, than that of the first price mechanism. The key to this result is that the second price mechanism creates a more even distortion away from a symmetric mechanism, which is typically more efficient than having low distortion at low costs, but high distortion at high costs.[7]

We show the result to be true under a variety of different conditions on the distribution over costs. First, any distribution with non-decreasing density works. So does any distribution attained from such a distribution by

[5] We show an isomorphism between auctions with offset cost distributions and a handicap. So, as the designer varies A, he is changing the asymmetry of costs in a particular structured way.

[6] A bid by 1 below $\beta_2(0) + A$ or by 2 below $\beta_1(0) - A$ can be raised a little and still win for sure, and so is not optimal.

[7] Another advantage of the second price mechanism is that it remains trivial to bid in even with bonuses.

taking the reverse cumulative and raising it to a power greater than 1. For integer powers, this corresponds to taking minima from multiple draws from the original distribution. We also show two conditions on distributions that seem commonly satisfied for distributions which are decreasing or hump-shaped, but equal to 0 at 1. Each is related to the relative concavity of f and $\bar{F}$, the reverse cumulative of F. Our results also continue to hold if there are multiple players of each of the two types, as long as the number of sellers from whom the value of buying is $v + \Delta$ is at least as large as the number of sellers from whom the value of buying is v. Finally, we show how our results on the dominance of first price mechanisms by second price mechanisms are robust to certain asymmetries in costs.

There are indications in our analysis that even in settings where our key condition is not satisfied, the second price mechanism will continue to outperform the first price. This is based on numerical analysis of a (fairly large) class of examples. We hope to understand this better in future work.

Section 2 discusses the literature. Section 3 presents the model. Section 13 derives the optimal mechanism. Section 14 examines second price mechanisms, and Section 5 discusses first price mechanisms. Section 12 looks at the choice between mechanisms. Section **??** discusses the robustness of our central results. Section 15 concludes. As we view much of the contribution of this paper as technical, many proofs are kept in main text. The appendix contains less central proofs.

2 Related Literature

Myerson (1981) begins a long discussion of implementation with asymmetric cost distributions. McAffee and McMillan (1988) look at optimal mechanisms with asymmetric cost distributions and argue that one doesn't want to always buy from the low-cost bidder.

In Che (1993), suppliers have different costs and can provide goods of different qualities. Suppliers submit a bid (p, q) which is evaluated via a quasi-linear scoring rule $S(p, q)$. Under the *first score rule* the high scorer executes the submitted bid. Under the *second score rule* the high scorer executes a contract equivalent in score and cost to the next highest bid. This transforms a multidimensional problem into a standard asymmetric cost environment. The optimal scoring rule distorts quality downward, and can be implemented by either the first or second score rules. Branco (1997) adds common value aspects and correlation to costs. Asker and Cantillion (2006) expand the results to multi-dimensional quality.

Shachat and Swarthout (2003) consider a model somewhat similar to ours, with a uniform distributions over costs and over qualities. Their main result (in line with Che (1993)) is that the optimal English auction (theoretically equivalent to a second price mechanism) in which the auctioneer observes qualities and sets a bonus should favor bidder 1 by more than 0 but less that Δ. We show that this feature generalizes beyond their example. They provide experimental evidence that the optimal English auction with a bonus outperforms a standard sealed bid RFP setting (in a setting where bidders are aware of their own quality but not that of their opponent).

Our results complement Maskin and Riley (2000), who also study "structured asymmetry." We discuss this in Section 14.2.

3 Model

A buyer faces sellers 1 and 2 with costs c_1 and c_2. Costs are independent, from cumulatives F_1 and F_2, where F_i has density f_i which is log-concave and continuously differentiable on $[\underline{c}_i, \bar{c}_i]$. The reverse cumulative is $\bar{F}_i = 1 - F_i$. We assume that $\frac{f_i}{\bar{F}_i}$ is strictly increasing. This is very mild.[8]

The buyer's utility from purchasing from i is

$$U_B(p, i) = v_i - p,$$

where p is the transaction price. Let $\Delta = v_1 - v_2$ be the amount by which 1 is preferred to 2. We assume Δ is common knowledge and, without loss of generality, $\Delta \geq 0$.[9]

For simplicity, we focus our analysis on how to allocate the contract between 1 and 2, setting aside when it is better to not buy at all. This is optimal if the buyer's outside option, v_0, is sufficiently negative. In what follows, Δ, rather than the absolute levels of v_1 and v_2, will be central.

4 Local Concavity

One contribution of this paper is to connect aspects of auctions and mechanism design to the local concavity of various functions. To recall, a positive

[8] By log-concavity, $\frac{f}{\bar{F}}$ is weakly increasing (see Lemma 46). If $\frac{f}{\bar{F}} = \gamma \geq 0$ on $[a, b]$, then $\bar{F}(c) = \bar{F}(a)e^{-\gamma c}$ for $c \in [a, b]$. Since $\bar{F}(a) > 0$, $\bar{F}(a)e^{-\gamma c}$ does not go to 0 for any finite c. Hence, what we are ruling out is that the distribution has a segment of an exponential distribution "patched" into it.

[9] Except in drawing the close connection between the RFP and various bonus auctions, it is irrelevant whether the sellers know Δ.

C^2 function h is *ρ-concave* if

$$\left(\frac{h^\rho}{\rho}\right)'' \leq 0$$

for all x. Standard concavity is thus equivalent to 1-concavity, while log-concavity is equivalent to 0 concavity. See Prekopa (1971,1973) and Borell (1975)

Noting that

$$\begin{aligned}\left(\frac{h^\rho}{\rho}\right)'' &= \left(h^{\rho-1}h'\right)' \\ &= (\rho-1)\,h^{\rho-2}\left(h'\right)^2 + h^{\rho-1}h'',\end{aligned}$$

we have that $\frac{h^\rho}{\rho}$ is concave at x iff

$$\rho \leq 1 - W_h\,(x)\,.$$

where

$$W_h\,(x) \equiv \frac{hh''}{\left(h'\right)^2}\,(x)$$

We can thus define $\rho_h\,(x) \equiv 1 - W_h\,(x)$ as the local (rho) concavity of h at x. Note that by definition, a function h is ρ concave if $\rho_h\,(x) \geq \rho$ for all x in the range of h. We will assume that any h we deal with is sufficiently well behaved that $W_h\,(1) \equiv \lim_{s\to 1} W_h\,(s)$ is well defined and finite when $h\,(1) = 0$, while $W_{\bar{H}}\,(1) \equiv \lim_{s\to 1} W_{\bar{H}}\,(s)$ are well defined and finite. Finiteness is *extremely* mild. See the first appendix for a discussion and primitives. For intuition, note that when $h'\,(1) \neq 0$, $W_h\,(1) = 0$.

Perhaps the most important property of $\rho-$concavity is the following result due to Prekopa and Borel:

Claim 1 *If h is ρ concave, then $\bar{H}$ given by $\bar{H}\,(c) = \int_c^1 h\,(s)\,ds$ is $\frac{\rho}{1+\rho}$ concave. That is ,*

$$\rho_{\bar{H}}\,(c) \geq \frac{\rho}{1+\rho}.$$

It turns out that we can strengthen this result in a very useful way. Let $\bar{\rho}_h\,(c)$ be $\max_{s\in[c,1]} \rho_h\,(s)\,,$ and $\underline{\rho}_h = \min_{s\in[c,1]} \rho_h\,(s)\,.$ Let $\underline{W}_h\,(c) = \min_{s\in[c,1]} W_h\,(s)$ and $\bar{W}_h\,(c) = \max_{s\in[c,1]} W_h\,(s)\,.$ Note that $\underline{W}_h\,(c)$ is weakly increasing, and $\bar{W}_h\,(c)$ is weakly decreasing. Then,

Proposition 2 *Let h log-concave satisfy $h(1) = 0$ and that h is decreasing on $[\hat{c}, 1]$. Then,*

$$\frac{\bar{\rho}_h(c)}{1 + \bar{\rho}_h(c)} \geq \rho_{\bar{H}}(c) \geq \frac{\underline{\rho}_h(c)}{1 + \underline{\rho}_h(c)} \tag{1}$$

for all $c \in [\hat{c}, 1]$.

Thus, the result is stronger than the standard one in the (not very surprising) dimension that only the properties of h on one side of c matter, but more importantly in the ability to provide an analogous inequality bounding the concavity of $\bar{H}$ from above.

Since $\rho(c) = 1 - W(c)$, we can rewrite (1) as

$$\frac{1 - \underline{W}_h(c)}{2 - \underline{W}_h(c)} \geq 1 - W_{\bar{H}}(c) \geq \frac{1 - \bar{W}_h(c)}{2 - \bar{W}_h(c)}.$$

Straightforward manipulation then yields

Remark 3 *Two equivalent expressions to* (1) *are*

$$\frac{1}{2 - \underline{W}_h(c)} \leq W_{\bar{H}}(c) \leq \frac{1}{2 - \bar{W}_h(c)}, \tag{2}$$

and

$$2 - \bar{W}_h(c) \leq \frac{1}{W_{\bar{H}}(c)} \leq 2 - \underline{W}_h(c).\text{[10]} \tag{3}$$

To prove this result, we begin with the following lemma.

Lemma 4 *If $h(1) = 0$ then*

$$2 - \bar{W}_h(1) = \frac{1}{W_{\bar{H}}(1)} = 2 - \underline{W}_h(1). \tag{4}$$

The proof follows from l'Hopital's rule. This in hand, we turn to the proof of Proposition 4.

Proof of Proposition 4: We will work with (3). Let

$$\begin{aligned}
\bar{J}(c) &\equiv \frac{-1}{W_{\bar{H}}(c)} + 2 - \underline{W}_h(c), \\
J(c) &\equiv \frac{-1}{W_{\bar{H}}(c)} + 2 - W_h(c), \\
&\text{and} \\
\underline{J}(c) &\equiv -\frac{1}{W_{\bar{H}}(c)} + 2 - \bar{W}_h(c),
\end{aligned}$$

[10] Log concavity of h gaurantees that $\underline{W}_h(c) \leq \bar{W}_h(c) \leq 1$, so that the end terms of (2) are positive, while $W_{\bar{H}} > 0$ when $h' < 0$, and so the cross multiplication is valid.

so that $\underline{J}(c) \leq J(c) \leq \bar{J}(c)$. Note that

$$\begin{aligned} W'_{\bar{H}}(c) &= W_{\bar{H}}(c)\left(\frac{-h(c)}{\bar{H}(c)} - 2\frac{h'(c)}{h(c)} + \frac{h''(c)}{h'(c)}\right) \\ &= W_{\bar{H}}(c)\left(\frac{-h'(c)}{h(c)}\right)\left(\frac{-1}{W_{\bar{H}}(c)} + 2 - W_h(c)\right) \end{aligned}$$

and that

$$W_{\bar{H}}(c)\left(\frac{-h'(c)}{h(c)}\right) = \frac{\bar{H}(c)}{h(c)}\left(\frac{-h'(c)}{h(c)}\right)^2 > 0,$$

and so

$$W'_{\bar{H}}(c) =_s J(c). \tag{5}$$

Assume that at some c, $\frac{1}{W_{\bar{H}}(c)} > 2 - \underline{W}_h(c)$, or, equivalently, $\bar{J}(c) < 0$. Then, $J(c) \leq \bar{J}(c) < 0$, and so

$$W'_{\bar{H}}(c) < 0.$$

Since $\underline{W}_h(c)$ is weakly increasing,

$$\bar{J}'(c) = \frac{W'_{\bar{H}}(c)}{(W_{\bar{H}}(c))^2} - \underline{W}'_h(c) < 0$$

as well. Thus, if $\bar{J}(c) < 0$, $\bar{J}'(c) < 0$, and it follows that $\bar{J}'(s) < 0$ for all $s \in [c, 1]$. But then, $\bar{J}(1) < 0$, contradicting Lemma 4.

Similarly, assume $2 - \bar{W}_h(1) > \frac{1}{W_{\bar{H}}(1)}$, or equivalently $\underline{J}(c) > 0$. Then, $W'_{\bar{H}}(c) > 0$ and so since $\bar{W}'_h(c) \leq 0$,

$$\underline{J}'(c) = \frac{W'_{\bar{H}}(c)}{(W_{\bar{H}}(c))^2} - \bar{W}'_h(c) > 0.$$

Thus, $\underline{J}(c) > 0$, again contradicting Lemma 4.

Note that what has been proved here is $\bar{J}(c) \geq 0$, and $\underline{J}(c) \leq 0$. Together with 5, we thus have the following immediate corollary.

Corollary 5 *Assume $h(1) = 0$. If on some interval $[\tilde{c}, 1]$ h is decreasing while W_h is increasing, then, $W_{\bar{H}}$ is increasing on $[\tilde{c}, 1]$. If h is decreasing, while W_h is decreasing on $[\tilde{c}, 1]$, then $W_{\bar{H}}$ is decreasing on $[\tilde{c}, 1]$.*

This is immediate, since if W_h is increasing then $\underline{W}_h(c) = W_h(c)$, and so $J(c) = \bar{J}(c) \geq 0$, and similarly if W_h is decreasing

As a useful adjunct to this, we have

Claim 6 *If h is log-concave and increasing at c then $W_{\bar{H}}$ is increasing at c.*

To see this, note that if h at c is increasing and log-concave then $\frac{h'}{h}$ is positive and decreasing, as is $\frac{\bar{H}}{h}$. Thus, $-W_{\bar{H}} = \frac{h'\bar{H}}{h^2}$ is decreasing.

One common use of Proposition 2 is when $h(c) = \bar{F}_i(c)$, and so $\bar{H}(c) = \int_c^1 \bar{F}_i(s)\, ds$.

Corollary 7 *If h is increasing on $[c, 1]$, then $W_{\bar{H}} = 1 - \rho_{\bar{H}} \leq \frac{1}{2}$. If f is decreasing on $[c, 1]$ then $W_{\bar{H}} = 1 - \rho_{\bar{H}} \geq \frac{1}{2}$.*

To see this, note that where f_i is increasing, $\bar{F}_i$ is concave, and so $\rho_{\bar{F}_i}(c) \geq 1$. Similarly, where f is decreasing, $\bar{F}_i$ is convex, and so $\rho_{\bar{F}_i}(c) \leq 1$. The claim is then immediate from Proposition 2.

5 First Price Mechanisms

We consider three first price auction settings.

Auction A^I: A *First Price Handicap Auction* (FPHA) with handicap A. Bidders 1 and 2 draw costs independently from F_i, and submit bids b_1, b_2. Bidder 1 wins iff $b_1 < b_2 + A$. The winner receives their bid. Player i is restricted to bid at most $\bar{c}_i$.[11]

Auction A^{II}: A *First Price Bonus Auction* (FPBA) with bonus A. Bidders 1 and 2 draw costs independently from F_i and submit bids b_1, b_2. Bidder 1 wins iff $b_1 < b_2$. If 2 wins, he receives b_2. If 1 wins,he receives $b_1 + A$. Player 1 is restricted to bid at most $1 - A$.

Auction A^{III}: A *First Price Shifted Cost Auction* with shift A. Bidders 1 and 2 draw costs independently and submit bids b_1, b_2. Bidder 2 draws his cost from F_2. Bidder 1 draws his cost from $F_{1,A}$ defined on $[\underline{c}_1 - A, \bar{c}_1 - A]$ by

$$\bar{F}_{1,A}(c_1) = \bar{F}_1(c_1 + A). \tag{6}$$

Bidder 1 wins iff $b_1 < b_2$. The winner receives their bid. Player 1 is restricted to bid at most $1 - A$.

So, for example, if $F_1 = F_2$, then in A^I, the bidders and payment rules are symmetric but the allocation rule is not. In A^{II} bidders and the allocation rule are symmetric, but the payment rule is not. In A^{III}, the allocation

[11] Depending on the relation between $\bar{c}_1$ and $\bar{c}_2$, one of these will be irrelevant. XXX we should think about more general reserves

and payment rules are symmetric, but the bidders have asymmetrically distributed costs.

A commonly used procurement mechanism is the sealed bid *request for proposal* (RFP) in which the buyer requests sealed bids on a project and then chooses his ex-post favorite. In a setting where Δ is common knowledge, this corresponds to a FPHA with $A = \Delta$.[12]

6 Equivalence, Existence and Basic Properties

We consider Bayesian equilibrium in which for A^I and A^{III}, $b_i \geq c_i$, while for A^{II}, $b_1 \geq c_1 - A$ and $b_2 \geq c_2$.[13]

Lemma 8 *Any equilibrium of A^t, $t \in \{I, II, III\}$, is in pure, continuous, and strictly increasing strategies, with the exception that one type may have an interval of high types over which he bids the maximum possible. If $\beta_1(\cdot), \beta_2(\cdot)$ is an equilibrium of A^I, then $\beta_1(\underline{c}_1) = \beta_2(\underline{c}_2) + A$. If $\beta_1(\cdot), \beta_2(\cdot)$ is an equilibrium of A^{II} or A^{III}, then $\beta_1(\underline{c}_1) = \beta_2(\underline{c}_2)$.*

XXX needs work

We sketch the proof and omit a formal version.[14] Consider A^I (the other cases are similar). Assume $\bar{c}_1 \leq \bar{c}_2 + A$ (other case again similar) Any $c_1 < \bar{c}_1$ and $c_2 < \bar{c}_1 - A$ earns strictly positive profits (since, e.g., 2 can submit $b_2 \in (c_2, \bar{c}_1 - A)$ and win when $c_1 > b + A$). Thus, equilibrium bids win with positive probability and earn a positive amount when they win. By single crossing best responses are thus weakly increasing in the strong set order, and so unique almost everywhere.[15] Hence strategies can be taken to be pure and weakly increasing.

It cannot be that for some $\hat{b}$, $b_1 = \hat{b}$ and $b_2 = \hat{b} - A$ both occur with positive probability, else an arbitrarily small drop in bid would win strictly more often, and earn essentially the same (positive amount) when it wins, for a profitable deviation. Nor can it be that say $b_1 = \hat{b}$ with positive probability. If it is, then a bid b_2 in $(\hat{b} - A, \hat{b} - A + \varepsilon]$ is inferior to b_2 just below $\hat{b} - A$ which wins discretely more often at essentially the same positive profit when it wins. But then, given that ties are zero probability, $b_1 = \hat{b}$

[12] Analysis of the RFP when Δ is unknown, or equivalently, of a $FPHA$ with stochastic allocation rule, is challenging.

[13] Ruling out such weakly dominated strategies rules out uninteresting pathologies.

[14] See Jackson and Swinkels (2005) for details of a similar argument.

[15] Since $b_2 \geq c_2$ and $b_1 \leq \bar{c}_1$, b_2 is irrelevant when $c_2 > \bar{c}_1 - A$. We set $\beta_2(c_2) = c_2$ wlog.

can be raised a bit at no loss in probability of winning (the exception to this argument is when $b_1 = \bar{c}_1$).

Finally, given that there are no atoms, there cannot be jumps in bids: Assume that as c_1 approaches $\hat{c}_1$ from below, $\beta_1(c_1)$ approaches b_L, while as c_1 approaches $\hat{c}_1$ from above, $\beta_1(c_1)$ approaches $b_H > b_L$. Since b_1 is never in (b_L, b_H), b_2 is never in $(b_L - A, b_H - A)$, since such bids earn less than $b_H - A$. But then, bids b_1 near b_L are less profitable than $b_1 = b_H$, contradicting that bids b_1 near b_L are optimal.

Our next theorem says that these three settings are isomorphic.

Theorem 9 *Let pure, continuous, and strictly increasing strategy profiles* (β_1^I, β_2^I), $(\beta_1^{II}, \beta_2^{II})$ *and* $(\beta_1^{III}, \beta_2^{III})$ *(again with the possible exception that one player bids the maximum possible over some range) be related by*

$$\beta_2^I(c_2) = \beta_2^{II}(c_2) = \beta_2^{III}(c_2), \tag{7}$$

$$\beta_1^{II}(c_1) = \beta_1^I(c_1) - A, \tag{8}$$

and

$$\beta_1^{III}(c_1 - A) = \beta_1^{II}(c_1) \tag{9}$$

for $(c_1, c_2) \in [\underline{c}_1, \bar{c}_1] \times [\underline{c}_2, \bar{c}_2]$. *Then, either each strategy profile is an equilibrium of its respective setting or none is.*

So, it makes no difference whether one runs an auction with a handicap A or a bonus A, and each is tightly related to a standard auction in which the cost distribution for player 1 is modified by shifting it A to the left.

To see the idea of the proof of Theorem 9, we begin with a definition is central to much of the rest of the paper.

Definition 10 *Let functions* ϕ^I, ϕ^{II}, *and* ϕ^{III} *be defined implicitly by*

$$\beta_1^I(\phi^I(c_2)) = \beta_2^I(c_2) + A \tag{10}$$

$$\beta_1^{II}(\phi^{II}(c_2)) = \beta_2^{II}(c_2) \tag{11}$$

and

$$\beta_1^{III}(\phi^{III}(c_2)) = \beta_2^{III}(c_2), \tag{12}$$

and let ψ^I, ψ^{II}, *and* ψ^{III} *be their respective inverses.*

Since bid functions are continuous and strictly increasing, these are well-defined, continuous and increasing. Each ϕ^t, $t \in \{I, II, III\}$ connects c_2 to the c_1 that "ties" it: when $c_1 < \phi^t(c_2)$, 1 wins, and when $c_1 > \phi^t(c_2)$ 2 wins.

The key to the proof is that because (β_1^I, β_2^I), $(\beta_1^{II}, \beta_2^{II})$ and $(\beta_1^{III}, \beta_2^{III})$ are related as in the statement of Theorem 9,

$$\phi^I(c_2) = \phi^{II}(c_2) = \phi^{III}(c_2) + A.$$

From this, we show that each player has an incentive to mimic another type in A^I if and only if he has an incentive to mimic in A^{II} and A^{III}. Hence (β_1^I, β_2^I), $(\beta_1^{II}, \beta_2^{II})$ and $(\beta_1^{III}, \beta_2^{III})$ are either all equilibria or none.

XXX some work here to deal with potential flats at top with general distributions.

While Theorem 9 draws a tight formal connection across the three settings, the interpretation of the auctions differs. Consider a *value advantage* case where the buyer places incremental value Δ on buying from 1 but costs are symmetric, and a *cost advantage* case where the buyer is indifferent, but 2 draws from F, while 1 draws costs from F_Δ obtained by shifting F Δ to the left.

In the value advantage case, treating the two sellers the "same" corresponds to A^I or A^{II} with $A = 0$, and so the bidders face a completely symmetric setting. In the cost advantage case, this corresponds to the (fairly unnatural) design in which 1 is penalized by Δ, his whole cost advantage.

In the cost advantage case, the natural auction that treats the two sellers the "same" is A^{III} with $A = \Delta$, so that 1 retains all of his cost advantage (this is effectively the case studied by Maskin and Riley (2000)). In A^I this corresponds to the RFP, while in A^{II}, 1 receives a bonus A equal to his whole value advantage Δ.

In many settings, the boundary between these cases is blurry: lower transportation costs for 1 are a cost advantage, a value advantage, or a blend, depending on how the auction rules specify transportation costs are shared. Because of this, even in settings where there may be constraints against explicitly favoring a given bidder (e.g., legal, or in terms of the perception of fairness among the sellers), there may be considerable latitude to choose A indirectly. Similarly, with an incumbent and new supplier one can specify who pays the transition costs, and when dealing with a foreign and domestic bidder, a government might have the use of differential tax treatments.

Theorem 9 has the following useful corollary.

Corollary 11 *For each of A^I, A^{II} and A^{III} an equilibrium exists, is unique, and is in pure and strictly increasing strategies.*

This is direct from Maskin and Riley (2000) applied to A^{III}.[16] Given Corollary 11, ϕ^t is uniquely determined by F and A regardless of which first price auction format is used. We thus write simply ϕ_{FP} (FP mnemonic for first price), and ψ_{FP} for $(\phi_{FP})^{-1}$, dropping the subscript when possible.

In what follows, we analyze A^I. Since $\beta_1(\underline{c}_1) = \beta_2(\underline{c}_2) + A$, $\phi(\underline{c}_1) = \underline{c}_2$, and since the equilibrium is strictly increasing, $\phi(\underline{c}_2) = \underline{c}_1$.

7 Basic Characterization

(There is considerable work to be done here and in what follows to deal with asymmetries. To a first approximation, every 1 at the top of an integral should be $\bar{c}_1$, and every $1 - A$ should be $\bar{c}_1 - A$. If $\bar{c}_1 - \bar{c}_2 > A$, then all sorts of details need to be thought thought. There are a few results where the distinction matters. I've tried to flag to ones that count).

We now turn to a more detailed examination of the equilibrium bid and allocation functions. We begin with a set of basic results.

Theorem 12 *In the FPHA with handicap A, Player 1's surplus at type c is*

$$S_1(c) = \int_c^1 \bar{F}_2(\psi(s))ds. \tag{13}$$

Player 2's surplus at type c is

$$S_2(c) = \int_c^{1-A} \bar{F}_1(\phi(s))ds. \tag{14}$$

Bid functions are

$$\beta_1(c) = c + \frac{\int_c^1 \bar{F}_2(\psi(s))ds}{\bar{F}_2(\psi(c))} \tag{15}$$

and

$$\beta_2(c) = c + \frac{\int_c^{1-A} \bar{F}_1(\phi(s))ds}{\bar{F}_1(\phi(c))}. \tag{16}$$

[16] Their key assumptions are that the type of one player conditionally stochastically dominates the other, and (to ensure that there is no atom in bid functions at the top, that the supports of values overlap. Two good entry points to the considerable literature on existence for asymmetric first price auctions are Lebrun (1996,1999), who assumes costs have identical support, and Reny and Zamir (2004). Reny and Zamir and Jackson and Swinkels (2005) show existence in increasing pure strategies for the setting of this paper. A (fairly straightforward) separate proof can be constructed that there is no atom at the top.XXXanything new with the new general supports?

If f_1 and f_2 are C^k, then β_1, β_2, and ϕ are C^{k+1}. On their domains

$$\beta_1'(c) = \frac{1}{\phi'(\psi(c))} S_1(c) \frac{f_2(\psi(c))}{\bar{F}_2^2(\psi(c))} > 0, \tag{17}$$

$$\beta_2'(c) = \phi'(c) S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))} > 0, \tag{18}$$

and

$$\phi'(c) = \frac{S_1(\phi(c))}{S_2(c)} \frac{\frac{f_2(c)}{\bar{F}_2^2(c)}}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} > 0. \tag{19}$$

Let us sketch the proof. Equations (13) and (14) follow from a simple envelope theorem argument, noting that $\bar{F}_2(\psi(s))$ is the probability that 1 wins with value s and $\bar{F}_1(\phi(s))$ is the probability that 2 wins with value s. Equation (15) follows from (13), noting that

$$S_1(c) = \bar{F}_2(\psi(c))(\beta_1(c) - c)$$

and rearranging, and similarly for equation (16).

Because β_1,β_2, ϕ, and ψ are increasing, they are differentiable almost everywhere. Imagine that $\beta_2'(c_2) = 0$ at some $c_2 < 1 - A$. Then near $c_1 = \phi(c_2)$, the probability of winning changes arbitrarily fast in b_1. But, since for $c_1 < 1$, $\beta_1(c_1) > c_1$, a small decrease in bid is optimal. Thus, $\beta_2'(c_2) > 0$ where it exists, and similarly for β_1'.

Differentiating $\beta_1(\phi(c)) = \beta_2(c) + A$, gives that where β_1' and β_2' exist,

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0. \tag{20}$$

Equations (17) and (18) are a matter of calculation. Substituting (17) and (18) into (20) and rearranging yields (19).

Substituting for S_1 and S_2 in (20) gives

$$\phi'(c) = \frac{\int_{\phi(c)}^{1} \bar{F}_2(\psi(s))ds}{\int_{c}^{1-A} \bar{F}_1(\phi(s))ds} \frac{\frac{f_2(c)}{\bar{F}_2^2(c)}}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}}. \tag{21}$$

For $1 \leq \hat{k} \leq k$, assume that $\phi \in C^{\hat{k}}[0, 1-A)$. Then, each building block of the RHS of (21) belongs to $C^{\hat{k}}[0, 1-A)$ and so the RHS as a whole belongs to $C^{\hat{k}}[0, 1-A)$. But then $\phi' \in C^{\hat{k}}[0, 1-A)$, and so $\phi = \int \phi' \in C^{\hat{k}+1}[0, 1-A)$. By induction, $\phi \in C^{k+1}[0, 1-A)$, and the analogous claims follow for β_1 and β_2.

This in hand, let's first note two potentially useful ways of looking at the previous expressions. First, note that since $S_1(c) = \int_c^1 \bar{F}_2(\psi(s))ds$, $S_1'(c) = -\bar{F}_2(\psi(c))$, and $S_1''(c) = f_2(\psi(c))$. Thus,

$$W_{S_1}(c) = \frac{S_1(c)\, f_2(\psi(c))}{\bar{F}_2^2(\psi(c))} \frac{1}{\phi'(\psi(c))}.$$

Comparing to (17),

$$\beta_1'(c) = W_{S_1}(c),$$

another suggestion that the connection between $\rho-$*concavity* and first price auctions is worth some attention.

Similarly, since $S_2(c) = \int_c^{1-A} \bar{F}_1(\phi(s))ds$, $S_2'(c) = -\bar{F}_1(\phi(c))$ and $S_2''(c) = f_1(\phi(c))\,\phi'(c)$. Hence,

$$\beta_2'(c) = W_{S_2}(c) = \frac{S_2(c)\, f_1(\phi(c))}{\bar{F}_1^2(\phi(c))} \phi'(c),$$

and

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} = \frac{W_{S_2}(c)}{W_{S_1}(\phi(c))}.$$

As a small application, we have

Remark 13 *Consider the symmetric FPA where $A = 0$ and $F_1 = F_2 = F$, with support $[0, 1]$. If f is increasing, then $\beta'(c) \leq \frac{1}{2}$ for all c. If f is decreasing, then $\beta'(c) \geq \frac{1}{2}$ for all c.*

To see this, note that in this case $S_1(c) = S_2(c) = \int_c^1 \bar{F}(s)\,ds$, and so

$$\beta'(c) = W_S(c) = W_{\int \bar{F}}(c) = 1 - \rho_{\int \bar{F}}(c).$$

A final rewriting of the slopes of the bidding functions is of some use (this needs to be integrated better). Since $\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))}$, and since from (17)

$$\beta_1'(\phi(c)) = \frac{1}{\phi'(c)} S_1(\phi(c)) \frac{f_2(c)}{\bar{F}_2^2(c)},$$

it follows that

$$\beta_2'(c) = S_1(\phi(c)) \frac{f_2(c)}{\bar{F}_2^2(c)}.$$

Similarly, from (18),

$$\beta_2'(c) = \phi'(c)\, S_2(c)\, \frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))} > 0,$$

and so

$$\beta_1'(\phi(c)) = S_2(c)\, \frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}.$$

8 Some Further Preliminaries

(This whole section a mess). Recall that by Lemma 8, $\beta_1(\underline{c}_1) = \beta_2(\underline{c}_2) + A$. Thus, $\phi(\underline{c}_1) = \underline{c}_2$.

Lemma 14 *Assume that* $\frac{f_1(\underline{c}_1)}{F_1(\underline{c}_1)} \geq \frac{f_2(\underline{c}_2)}{F_2(\underline{c}_2)}$. *Then, for* $A > \underline{c}_1 - \underline{c}_2$, $\phi'(\underline{c}_2) > 1$.

Proof of Lemma 14: From (19),

$$\begin{aligned}
\phi'(\underline{c}_2) &= \frac{S_1(\underline{c}_1)\, \frac{f_2(\underline{c}_2)}{\bar{F}_2^2(\underline{c}_2)}}{S_2(\underline{c}_2)\, \frac{f_1(\underline{c}_1)}{\bar{F}_1^2(\underline{c}_1)}} \\
&\geq \frac{\frac{S_1(\underline{c}_1)}{\bar{F}_2(\underline{c}_2)}}{\frac{S_2(\underline{c}_2)}{\bar{F}_1(\underline{c}_1)}}. \\
&= \frac{\beta_1(\underline{c}_1) - \underline{c}_1}{\beta_2(\underline{c}_2) - \underline{c}_2} \\
&= \frac{\beta_2(\underline{c}_2) + A - \underline{c}_1}{\beta_2(\underline{c}_2) - \underline{c}_2} \\
&= \frac{\beta_2(\underline{c}_2) - \underline{c}_2 + A - (\underline{c}_1 - \underline{c}_2)}{\beta_2(\underline{c}_2) - \underline{c}_2} \\
&= 1 + \frac{A - (\underline{c}_1 - \underline{c}_2)}{\beta_2(\underline{c}_2) - \underline{c}_2}.
\end{aligned}$$

Lemma 15 *If* $F_1 = F_2 = F$, *with support* $[0,1]$, *then* $\phi(c) > c$ *for all* $c \in (0,1)$.

Proof of Lemma 15: By Lemma 8, $\phi(0) = 0$. Consider any point in $[0,1)$ at which $\phi(c) = c$. Then,

$$\begin{aligned} \phi'(c) &= \frac{\frac{S_1(\phi(c))}{\bar{F}_2(c)} \frac{f_2(c)}{\bar{F}_2(c)}}{\frac{S_2(c)}{\bar{F}_1(\phi(c))} \frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))}} \\ &= \frac{\beta_1(c) - c}{\beta_2(c) - c} \\ &= \frac{\beta_2(c) + A - c}{\beta_2(c) - c} > 1. \end{aligned}$$

More generally, (add correct version of asymmetric analogs to $\phi(c_2) > c_2$, and $\phi(c_2) < c_2 + A$ for asymmetric distributions).

Next, let us show that both bid functions tend to a slope of zero.

Lemma 16 *β_1 is differentiable at 1 with $\beta'(1) = 0$. β_2 is continuously differentiable at $1 - A$ with $\beta_2'(1 - A) = 0$.*

For β_1, we claim differentiability at 1, but not continuous differentiability.[17] For β_2, we assert continuous differentiability.

The proofs of the two cases are quite different: with $c_1 = 1$, 1 already has probability $\bar{F}(1 - A)$ of winning while with $c_2 = 1 - A$, 2 has no chance of winning and so S_1 and S_2 have different shapes at the top. Thus, even bidding 1 already earns 1 a great deal of surplus, and this bounds his bid from below sufficiently to guarantee $\beta_1'(1) = 0$. Showing that $\beta_2'(c_2) \to 0$ as $c_2 \to 1 - A$ involves an examination of 1's incentives.

An easy but useful implication of this is:

Lemma 17 *As $c \to 1 - A$,*

$$\frac{\beta_1(\phi(c)) - \phi(c)}{1 - \phi(c)} \to 1 \text{ and } \frac{\beta_2(c) - c}{1 - A - c} \to 1.$$

(needs updating)

It is central to our main result to understand the behavior of $\phi'(c)$ as c goes to $1 - A$.

Theorem 18 *In the FPHA,* $\limsup \phi'(c) = \infty$, *and* $\liminf \phi'(c) > 1$.

So, (modulo the annoying possibility of a discontinuity of the second type in ϕ' at $1 - A$), ϕ' becomes arbitrarily large as $c \to 1 - A$.

[17] If h is continuously differentiable on $[0,1)$, its derivative at 1 need not equal $\lim_{x \to 1} h'(x)$ (which need not exist). We establish continuous differentiability of β_1 later in a rather round-about way.

9 The Slope of the Allocation Function in First Price Handicap Auctions

Let us now turn to our central set of results. We will use the connection between equilibria of first price auctions and $\rho-$concavity to establish conditions under which ϕ, the equilibrium allocation function, has slope greater than 1. That is to say, the gap between the costs of tied types increases as one moves to higher costs. In a later section, we show how to use this result to rank first and second price handicap auctions from the point of view of the buyer.

The strategy will be a proof by contradiction. In particular, the last section shows (Theorem 18) that for all c above some $\hat{c} < \bar{c}_2$, $\phi'(c) > 1$, while $\phi'(\underline{c}_2) > 1$ by Corollary 14. So, if $\phi' \leq 1$ anywhere on $[0, \hat{c}]$, then since ϕ' is continuously differentiable on $[0, 1-A)$, ϕ' achieves an interior minimum at some $r \in (0, \hat{c}]$, so that $\phi''(r) = 0$. The trick will be to find conditions under which it cannot simultaneously be that $\phi'(r) \leq 1$, and $\phi''(r) = 0$.

The first essential step in doing this is the following lemma.

Lemma 19 *Assume that $\phi'(c) \leq 1$. Then,*

$$\phi''(c) \geq_s \left(\frac{1}{S_1(\phi(c)) \frac{f_2(c)}{\bar{F}_2^2(c)}} - 2 \right) \left(\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} - \frac{f_2(c)}{\bar{F}_2(c)} \right) + \left(\frac{f_2'(c)}{f_2(c)} - \frac{f_1'(\phi(c))}{f_1(\phi(c))} \right), \tag{22}$$

where $\geq_s$ denotes that the LHS is positive whenever the RHS is.

Proof of Lemma 19: Since

$$\phi'(c) = \frac{\int_{\phi(c)}^{1} \bar{F}_2(\psi(s))ds}{\int_{c}^{1-A} \bar{F}_1(\phi(s))ds} \frac{\frac{f_2(c)}{\bar{F}_2^2(c)}}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} = \frac{S_1(\phi(c))}{S_2(c)} \frac{\frac{f_2(c)}{\bar{F}_2^2(c)}}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}}$$

$$\begin{aligned} \frac{\phi''(c)}{\phi'(c)} &= \left(\log \phi'(c)\right)' = \frac{(S_1(\phi(c))'}{S_1(\phi(c))} - \frac{(S_2(c))'}{S_2(c)} + \frac{f_2'(c)}{f_2(c)} + 2\frac{f_2(c)}{\bar{F}_2(c)} \\ &\quad -\phi'(c) \left(\frac{f_1'(\phi(c))}{f_1(\phi(c))} + 2\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} \right) \end{aligned}$$

Since $S_1(\phi(c)) = \int_{\phi(c)}^{1} \bar{F}_2(\psi(s))ds$, $(S_1(\phi(c))' = -\phi'(c)\bar{F}_2(c)$, and similarly

$(S_2(c))' = -\bar{F}_1(\phi(c))$. So,

$$\begin{aligned}
\frac{(S_1(\phi(c))'}{S_1(\phi(c))} - \frac{(S_2(c))'}{S_2(c)} &= \phi'(c)\frac{-\bar{F}_2(c)}{S_1(\phi(c))} + \frac{\bar{F}_1(\phi(c))}{S_2(c)} \\
&= \phi'(c)\frac{-\bar{F}_2(c)\frac{f_2(c)}{\bar{F}_2^2(c)}}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}} + \frac{\bar{F}_1(\phi(c))\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}}{S_2(c)\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} \\
&= \phi'(c)\frac{-\bar{F}_2(c)\frac{f_2(c)}{\bar{F}_2^2(c)}}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}} + \frac{\bar{F}_1(\phi(c))\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}}{S_2(c)\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} \\
&= \phi'(c)\frac{-\frac{f_2(c)}{\bar{F}_2(c)}}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}} + \frac{\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))}}{S_2(c)\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} \\
&= \frac{\phi'(c)}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}}\left(\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} - \frac{f_2(c)}{\bar{F}_2(c)}\right).
\end{aligned}$$

An expression for $\frac{\phi''(c)}{\phi'(c)}$ is thus

$$\begin{aligned}
\frac{\phi''(c)}{\phi'(c)} &= \frac{\phi'(c)}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}}\left(\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} - \frac{f_2(c)}{\bar{F}_2(c)}\right) \\
&\quad + \left(\frac{f_2'(\phi(c))}{f_2(\phi(c))} + 2\frac{f_2(\phi(c))}{\bar{F}_2(\phi(c))}\right) - \phi'(c)\left(\frac{f_1'(\phi(c))}{f_1(\phi(c))} + 2\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))}\right).
\end{aligned} \tag{23}$$

Now, note that

$$\left(\frac{f_2'(\phi(c))}{f_2(\phi(c))} + 2\frac{f_2(\phi(c))}{\bar{F}_2(\phi(c))}\right) = \left(\log\frac{f_2}{\bar{F}_2^2}\right)'\bigg|_{x=\phi(c)} = \left(\log\frac{1}{\bar{F}_2} + \log\frac{f_2}{\bar{F}_2}\right)'\bigg|_{x=\phi(c)} > 0$$

by log concavity of $\bar{F}_2$. Thus, if $\phi'(c) \leq 1$,

$$\begin{aligned}
\frac{\phi''(c)}{\phi'(c)} &\geq \frac{\phi'(c)}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}}\left(\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} - \frac{f_2(c)}{\bar{F}_2(c)}\right) \\
&\quad + \phi'(c)\left(\frac{f_2'(\phi(c))}{f_2(\phi(c))} + 2\frac{f_2(\phi(c))}{\bar{F}_2(\phi(c))}\right) - \phi'(c)\left(\frac{f_1'(\phi(c))}{f_1(\phi(c))} + 2\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))}\right)
\end{aligned}$$

and so

$$\frac{\phi''(c)}{(\phi'(c))^2} \geq \frac{1}{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}}\left(\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} - \frac{f_2(c)}{\bar{F}_2(c)}\right)$$
$$+\left(\frac{f_2'(\phi(c))}{f_2(\phi(c))} + 2\frac{f_2(\phi(c))}{\bar{F}_2(\phi(c))}\right) - \left(\frac{f_1'(\phi(c))}{f_1(\phi(c))} + 2\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))}\right).$$

Collecting terms, and recalling $\phi'(c) > 0$, we are done. ■

The expression in Lemma 19 is compact, but involves the rather annoying object $S_1(\phi(c))$. This is a function of the entire equilibrium to the right of $\phi(c)$, and so inherently forbidding. The following lemma is very helpful in this regard.

Lemma 20 *Let $r \in \arg\min_{c\in[0,1-A)} \phi'(c)$. Then, for all c_2,*

$$S_1(\phi(r))\frac{f_2(c)}{\bar{F}_2^2(c)} < W_{\int \bar{F}}(\phi(c)).$$

Proof of Lemma 20: By a change of variables,

$$\begin{aligned} S_2(c) &= \int_c^{1-A} \bar{F}(\phi(s))ds \\ &= \int_{\phi(c)}^{\phi(1-A)} \bar{F}(s)\psi'(s)\,ds \\ &< \frac{1}{\phi'(r)}\int_{\phi(c)}^{1} \bar{F}(s)ds, \end{aligned}$$

where the strict inequality follows since ϕ' is continuous and $\limsup \phi' = \infty$. Multiplying both sides by $\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))} = \frac{f(\phi(c))}{\bar{F}^2(\phi(c))}$ gives

$$W_{S_2}(c) < \frac{1}{\phi'(r)}W_{\int \bar{F}}(\phi(c)). \tag{24}$$

But then,

$$S_1(\phi(r))\,g(r) = \phi'(r)\,S_2(r)\,g(\phi(r)) < W_{\int \bar{F}}(\phi(c)). \quad ■ \tag{25}$$

Putting these together yields.

Proposition 21 *Assume that $\phi'(r) \leq 1$, and that $\frac{f_1(\phi(r))}{\bar{F}_1(\phi(r))} - \frac{f_2(r)}{\bar{F}_2(r)} > 0$. Then,*

$$\phi''(r) >_s \left(\frac{1}{W_{\int \bar{F}}(r)} - 2\right)\left(\frac{f_1(\phi(r))}{\bar{F}_1(\phi(r))} - \frac{f_2(r)}{\bar{F}_2(r)}\right) + \frac{f_2'(r)}{f_2(r)} - \frac{f_1'(\phi(r))}{f_1(\phi(r))}. \tag{26}$$

10 Slope Results for Symmetric Cost Distributions

Let us begin with the case where $F_1 = F_2 = F$. Wlog, we take the support of F to be $[0, 1]$. There are then several paths to using Proposition 21 to establish a slope result. We begin with the easiest.

Theorem 22 *Assume that $F_1 = F_2 = F$, and that f is weakly increasing. Then, $\phi'(c) > 1$ for all $c \in [0, 1 - A)$.*

Proof of Theorem 22: Assume, that $\phi'(c) \leq 1$ for any c. Then, there is an interior minimum r, with $\phi'(r) \leq 1$. But, since $F_1 = F_2 = F$, Lemma 15 establishes that $\phi(r) > r$, and hence, using log-concavity, that $\frac{f_1(\phi(r))}{\bar{F}_1(\phi(r))} - \frac{f_2(r)}{\bar{F}_2(r)} > 0$ and $\frac{f_2'(r)}{f_2(r)} - \frac{f_1'(\phi(r))}{f_1(\phi(r))} \geq 0$. But, since f is increasing, by Corollary 7 $\frac{1}{W_{\int \bar{F}}(r)} - 2 \geq 0$, and so by Proposition 21, $\phi''(r) > 0$, a contradiction. ■

Now, let us turn to more general densities. Since f is log-concave, either f is monotone increasing, monotone decreasing, or is hump-shaped, but with $f'(c)$ strictly negative on some interval $(\hat{c}, 1)$.

We need two conditions. The first is that $W_{\int \bar{F}}$ is non-decreasing. Recall first that this is automatic for any f increasing, by Claim 6 and Corollary 5. On the other hand, note that

$$W_{\bar{F}} = \frac{\bar{F}(-f')}{f^2} = \frac{\bar{F}}{f}\frac{(-f')}{f}.$$

and so, if f is decreasing on $(\hat{c}, 1)$, then $W_{\bar{F}} > 0$ on $(\hat{c}, 1)$. If $f(1) > 0$, but $f'(1)$ is finite, then, since $\frac{\bar{F}}{f}(1) = 0$, $W_{\bar{F}}(1) = 0$. Thus (unless $W_{\bar{F}}'$ changes sign an infinite number of times), there will be an interval near 1 on which $W_{\bar{F}}$ is decreasing. But then, by Corollary 5, $W_{\int \bar{F}}$ is decreasing on this interval as well. Hence, this approach has no traction on such densities. Thus, no such density can have $W_{\bar{F}}$ increasing. For densities where $f(1) = 0$, the condition that $W_{\bar{F}}$ is increasing is neither vacuous nor difficult to satisfy, as is illustrated by the following example.

Example 23 *Choose h increasing and log-concave. Then, $W_{\bar{H}}$ is increasing (Lemma* **??***). Setting $f = \bar{H}$, we have that $W_f = W_{\bar{H}}$ is increasing, and thus so is $W_{\int \bar{F}}$ (Corollary 5 applied twice). On the other hand, if h is decreasing with $h(1) > 0$, then $W_{\bar{H}}(1) = \frac{1}{2}$, while, by Corollary 7, $W_{\bar{H}}(c) > \frac{1}{2}$ for $c < 1$. So, (again excepting infinitely many sign changes in $W_{\bar{H}}'$), $W_{\bar{H}}$ will be decreasing over some interval near 1. But then, on this interval, $W_{\int \int \bar{H}}$*

is also decreasing. So, $f = \bar{H}$*, while satisfying* $f(1) = 0$*, does not satisfy* $W_{\int \bar{F}}$ *increasing.*

Our second requirement is that $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ has a well defined limit, and reaches its minimum at $c = 1$. This is satisfied very broadly in examples we have checked. It also has intuitive content, saying simply that the concavity of $\log f$ versus that of $\log \bar{F}$ is at its smallest at 1. For some intuition, note that

$$\frac{\partial^2 \log f}{\partial^2 \log \bar{F}} = \frac{\left(\frac{f'}{f}\right)'}{\left(-\frac{f}{\bar{F}}\right)'} = \frac{\left(\frac{-f'}{f}\right)'}{\left(\frac{f}{\bar{F}}\right)'}.$$

Now, $\left(\frac{f}{\bar{F}}\right)'$ is strictly positive by assumption and continuous on $(0,1)$, while $\left(\frac{-f'}{f}\right)'$ is non-negative since f is log-concave. Hence, $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ is continuous and non-negative on $(0,1)$. Consider any f for which $\left(\frac{-f'}{f}\right)'(1) < \infty$. This includes any case where $f(1) > 0$, and $f'(1)$ is finite (and so in particular includes any increasing f). Then, since $\left(\frac{f}{\bar{F}}\right)'(1) = \infty$, $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1) = 0$, and $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ is minimized at 1. Summarizing,

Claim 24 $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ *is continuous and non-negative on* $(0,1)$*. If* $\left(\frac{-f'}{f}\right)'(1) < \infty$*, then* $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ *is minimized at* 1*.*

The following lemma ties this condition to the underlying rho-concavities of f and $\bar{F}$.

Lemma 25 *At any point in* $[0,1]$ *where* $\frac{f'}{f} \neq 0$*,*

$$\frac{\partial^2 \log f}{\partial^2 \log \bar{F}} = \frac{W_{\bar{F}}^2(1 - W_f)}{(1 - W_{\bar{F}})}.$$

If $f(1) = 0$*,* $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1)$ *is well defined, then* $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1) = \frac{W_{\bar{F}}^2(1-W_f)}{(1-W_{\bar{F}})}(1)$ *regardless of* $\frac{f'}{f}(1)$*.*

Proof of Lemma ??: Note that

$$\frac{\partial^2 \log f}{\partial^2 \log \bar{F}} = \frac{\left(\frac{-f'}{f}\right)'}{\left(\frac{f}{\bar{F}}\right)'} = \frac{\left(-\frac{f''}{f} + \left(\frac{f'}{f}\right)^2\right)}{\left(\frac{f'}{\bar{F}} + \frac{f}{\bar{F}}^2\right)}. \tag{27}$$

On $[0,1)$, $\frac{f}{\bar{F}} \in (0,\infty)$, and $\frac{f'}{f} \in (-\infty,\infty)$. So, where $\frac{f'}{f} \neq 0$, we have

$$\frac{\partial^2 \log f}{\partial^2 \log \bar{F}} = \frac{-\left(\frac{f'}{f}\right)^2 \left(\frac{f''f}{(f')^2} - 1\right)}{\left(\frac{f}{\bar{F}}\right)^2 \left(\frac{f'\bar{F}}{f^2} + 1\right)} = \frac{W_{\bar{F}}^2 \left(1 - W_f\right)}{\left(1 - W_{\bar{F}}\right)}. \blacksquare \tag{28}$$

Assume that $f(1) = 0$. Then, on some interval $(\tilde{c}, 1)$ $f' \neq 0$ (since, by log-concavity, f' crosses 0 on at most one point or interval). Hence, on this interval, $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}} = \frac{W_{\bar{F}}^2(1-W_f)}{(1-W_{\bar{F}})}$. Since $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1)$ is well-defined by assumption, and since both sides are continuous on $(\tilde{c}, 1)$, the claim follows.

Theorem 26 *Assume that* $W_{\int \bar{F}}$ *is increasing, and that* $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(c) \geq \frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1)$ *for all c. Then,* $\phi'(c) > 1$ *for all* $c \in [0, 1-A]$.

We begin with a small lemma:

Lemma 27 *If* $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1)$ *exists, then*

$$\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1) = W_{\bar{F}}(1).$$

Proof of Lemma 27: Assume first that $f(1) = 0$. Then, from (4) with $h = f$, we have, with a little rearrangement,

$$1 - W_f(1) = \frac{1}{W_{\bar{F}}(1)} - 1$$

or

$$\frac{W_{\bar{F}}(1)\left(1 - W_f(1)\right)}{1 - W_{\bar{F}}(1)} = 1$$

from which

$$\frac{W_{\bar{F}}^2(1)\left(1 - W_f(1)\right)}{1 - W_{\bar{F}}(1)} = W_{\bar{F}}(1).$$

If $f'(1) \neq 0$, we are thus done, by Lemma 25.

On the other hand, assume that $f(1) > 0$. Then, $W_{\bar{F}}(1) = 0$, and so $\frac{W_{\bar{F}}^2(1)\left(1-W_f(1)\right)}{1-W_{\bar{F}}(1)} = W_{\bar{F}}(1)$ holds again (recall that we assume $W_f(1)$ to be finite).

Proof of Theorem 26: Let $\delta \equiv \phi(r) - r$. From (26), we have

$$\begin{aligned} 0 &= \frac{\phi''(r)}{\phi'(r)} > \left(\frac{1}{W_{\int \bar{F}}(r+\delta)} - 2 \right) \left(\frac{f}{\bar{F}}(r+\delta) - \frac{f}{\bar{F}}(r) \right) + \frac{f'}{f}(r) - \frac{f'}{f}(r+\delta) \\ &=_s \frac{1}{W_{\int \bar{F}}(r+\delta)} - 2 + \frac{\frac{f'}{f}(r) - \frac{f'}{f}(r+\delta)}{\frac{f}{\bar{F}}(r+\delta) - \frac{f}{\bar{F}}(r)}. \end{aligned}$$

Using Cauchy's theorem, there exists $\xi \in (r, r+\delta)$ such that

$$\begin{aligned} \frac{\phi''(r)}{\phi'(r)} &>_s \frac{1}{W_{\int \bar{F}}(r+\delta)} - 2 + \frac{\left(-\frac{f'}{f}(\xi)\right)'}{\left(\frac{f}{\bar{F}}(\xi)\right)'} \\ &= \frac{1}{W_{\int \bar{F}}(r+\delta)} - 2 + \frac{\partial^2 \log f(\xi)}{\partial^2 \log \bar{F}(\xi)} \end{aligned}$$

Since $W_{\bar{F}}$ is increasing, $W_{\int \bar{F}}$ is too (Corollary 5 applied to $h = \bar{F}$). Since $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ is minimized at 1,we thus have

$$\frac{\phi''(r)}{\phi'(r)} >_s \frac{1}{W_{\int \bar{F}}(1)} - 2 + \frac{\partial^2 \log f(1)}{\partial^2 \log \bar{F}(1)},$$

or, using Lemma (27)

$$\begin{aligned} \frac{\phi''(r)}{\phi'(r)} &>_s \frac{1}{W_{\int \bar{F}}(1)} - 2 + W_{\bar{F}}(1) \\ &= 0, \end{aligned}$$

using (4) with $h = \bar{F}$. This is a contradiction. ■

One very nice application of these conditions is to the situation where $\bar{F}$ itself reflects multiple draws from a base distribution $\bar{H}$.

Theorem 28 *Let $\bar{H}$ be a reverse cumulative, and let $\bar{F} = \bar{H}^n$, for some $n \in (0, \infty)$ (not neccasarily an integer). Assume that $W_{\bar{H}}$ is increasing, and that $\frac{\partial^2 \log h}{\partial^2 \log \bar{H}}(c) \geq \frac{\partial^2 \log h}{\partial^2 \log \bar{H}}(1)$ for all c. Then, $W_{\bar{F}}$ is increasing, and $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(c) \geq \frac{\partial^2 \log f}{\partial^2 \log \bar{F}}(1)$ for all c. So $\phi'(c) > 1$ for all $c \in [0, 1-A]$.*

When n is greater than one and an integer, $\bar{F}$ represents the result of taking the most favorable from n draws from $\bar{H}$, while when n is less than one, and $\frac{1}{n}$ is an integer, $\bar{H}$ can be thought of as the most favorable from n draws from $\bar{F}$.

Proof of Theorem 28: Note that

$$f = -\bar{F}' = n\bar{H}^{n-1}h,$$

and so

$$\frac{f}{\bar{F}} = n\frac{h}{\bar{H}} \tag{29}$$

and

$$\frac{f'}{f} = -(n-1)\frac{h}{\bar{H}} + \frac{h'}{h}. \tag{30}$$

Thus,

$$\begin{aligned} W_{\bar{F}} &= \frac{\bar{F}}{f}\frac{-f'}{f} \\ &= \frac{1}{n}\frac{\bar{H}}{h}\left((n-1)\frac{h}{\bar{H}} - \frac{h'}{h}\right) \\ &= \frac{n-1}{n} - \frac{1}{n}\frac{\bar{H}}{h}\frac{h'}{h} \\ &= \frac{n-1}{n} + \frac{1}{n}W_{\bar{H}}. \end{aligned}$$

So, $W_{\bar{F}}$ has the same monotonicity as $W_{\bar{H}}$.

Using (29) and (30), note that

$$\begin{aligned} \frac{\partial^2 \log f}{\partial^2 \log \bar{F}} &= \frac{\left(\frac{-f'}{f}\right)'}{\left(\frac{f}{\bar{F}}\right)'} = \frac{(n-1)\left(\frac{h}{\bar{H}}\right)' - \left(\frac{h'}{h}\right)'}{n\left(\frac{h}{\bar{H}}\right)'} \\ &= \frac{(n-1)}{n} + \frac{1}{n}\frac{\left(\frac{-h'}{h}\right)'}{\left(\frac{h}{\bar{H}}\right)'} \\ &= \frac{(n-1)}{n} + \frac{1}{n}\frac{\partial^2 \log h}{\partial^2 \log \bar{H}}. \end{aligned}$$

So, if $\frac{\partial^2 \log h}{\partial^2 \log \bar{H}}$ reaches its minimum at c, so does $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$

Example 29 *Note that this construction is a ready source of both hump-shaped and decreasing examples. Recall that for any h increasing, the two conditions of Theorem **28** are automatic (Lemma* **??***). By* (30)

$$\begin{aligned}\frac{f'}{f} &= -(n-1)\frac{h}{\bar{H}} + \frac{h'}{h} \\ &= {}_s - (n-1) - W_{\bar{H}}.\end{aligned}$$

For h increasing, $W_{\bar{H}}(1) = 0$. So, for any $n > 1$, $f' < 0$ near 1. And, for n large enough, $f' < 0$ everywhere.

Let us now turn to a different way one might think about all of this. From (26) , we have

$$0 = \frac{\phi''(r)}{\phi'(r)} > \left(\frac{1}{W_{\int \bar{F}}(r+\delta)} - 2\right)\left(\frac{f}{\bar{F}}(r+\delta) - \frac{f}{\bar{F}}(r)\right) + \frac{f'}{f}(r) - \frac{f'}{f}(r+\delta)$$

If $W_{\int \bar{F}}(r+\delta)$ is nondecreasing, then $\left(\frac{1}{W_{\int \bar{F}}(r+\delta)} - 2\right) \geq \left(\frac{1}{W_{\int \bar{F}}(1)} - 2\right) \equiv \alpha$, and so

$$\frac{\phi''(r)}{\phi'(r)} > \alpha\left(\frac{f}{\bar{F}}(r+\delta) - \frac{f}{\bar{F}}(r)\right) + \frac{f'}{f}(r) - \frac{f'}{f}(r+\delta)$$

This can be re-written as

$$\begin{aligned}\frac{\phi''(r)}{\phi'(r)} &> \alpha\left(\frac{f}{\bar{F}}(r+\delta) - \frac{f}{\bar{F}}(r)\right) + \frac{f'}{f}(r) - \frac{f'}{f}(r+\delta) \\ &= \alpha\left(\frac{\partial}{\partial r}\log\frac{\bar{F}(r)}{\bar{F}(r+\delta)}\right) + \frac{\partial}{\partial r}\log\frac{f(r)}{f(r+\delta)} \\ &= \frac{\partial}{\partial r}\log\frac{\bar{F}(r)^{\alpha}f(r)}{\bar{F}(r+\delta)^{\alpha}f(r+\delta)} \\ &= \frac{\partial}{\partial r}\log\bar{F}(r)^{\alpha}f(r) - \frac{\partial}{\partial r}\log\bar{F}(r+\delta)^{\alpha}f(r+\delta).\end{aligned}$$

If this is positive, then we again have a contradiction. Thus, we have

Theorem 30 *Assume that $W_{\int \bar{F}}$ is increasing, and let $\alpha = \frac{1}{W_{\int \bar{F}}} - 2$. Then, if $\bar{F}(c)^{\alpha}f(c)$ is log-concave, then, $\phi'(c) > 1$ for all $c \in [0, 1-A)$.*

Indeed, a different way of seeing the proof of Theorem 22 is that when $f' \geq 0$, then for all c,

$$\frac{1}{W(c)} - 2 \geq 0$$

and so $\bar{F}(c)^{\alpha} f(c)$ inherits the log-concavity of $\bar{F}$ and f. However, $\bar{F}(c)^{\frac{1}{w}-2} f(c)$ is also log-concave for many distributions that do not have this property.

Example 31 *Consider the strictly unimodal distribution $f(c) = 6c(1-c)$. Then, it can be verified that $W(c) = \frac{3c(1+c)}{(1+2c)^2}$ which is increasing, and so $W(c) \leq \frac{2}{3}$. It is therefore enough to show that $\bar{F}(c)^{\frac{3}{2}-2} f(c) = \bar{F}(c)^{-\frac{1}{2}} f(c)$ is log-concave, which is satisfied. Effectively, f is sufficiently log-concave as to overcome the log-convexity of $\bar{F}(c)^{-\frac{1}{2}}$.*

Example 32 *For any $\alpha \geq 1$, consider the distribution $\bar{F}(c) = (1-c)^{\alpha}$. Then, $f(c) = \alpha(1-c)^{\alpha-1}$ and so $W(c) = w = \frac{\alpha}{1+\alpha}$ for all c. Thus,*

$$\begin{aligned} & \bar{F}(c)^{\frac{1}{w}-2} f(c) \\ = \; & \bar{F}(c)^{\frac{\alpha+1}{\alpha}-2} f(c) \\ = \; & (1-c)^{\alpha\left(\frac{1-\alpha}{\alpha}\right)} \alpha(1-c)^{\alpha-1} \\ = \; & \alpha \end{aligned}$$

which is trivially log-concave.

It is interesting to learn that the two sets of conditions are in fact equivalent!

Note that for any given α,

$$\left(\log f \bar{F}^{\alpha}\right)' = \frac{f'}{f} - \alpha \frac{f}{\bar{F}}$$

and so

$$\begin{aligned} \left(\log f \bar{F}^{\alpha}\right)'' &= \left(\frac{f'}{f}\right)' - \alpha \left(\frac{f}{\bar{F}}\right)' \\ &= \left(\frac{f'}{f}\right)\left(\frac{f''}{f'} - \frac{f'}{f}\right) - \alpha \left(\frac{f}{\bar{F}}\right)\left(\frac{f'}{f} + \frac{f}{\bar{F}}\right) \\ &= \left(\frac{f'}{f}\right)^2 (W_f - 1) - \alpha \left(\frac{f}{\bar{F}}\right)^2 (1 - W_{\bar{F}}) \\ &=_s \left(\frac{f'}{f}\right)^2 \left(\frac{\bar{F}}{f}\right)^2 (W_f - 1) - \alpha\,(1 - W_{\bar{F}}) \\ &= \frac{W_{\bar{F}}^2 (W_f - 1)}{(1 - W_{\bar{F}})} - \alpha \end{aligned}$$

So, we would need

$$\frac{W_{\bar{F}}^2\left(1-W_f\right)}{\left(1-W_{\bar{F}}\right)}(c) \geq 2-\frac{1}{W_{\int \bar{F}}(1)}.$$

But, recall that $2-\frac{1}{W_{\int \bar{F}}(1)} = W_{\bar{F}}(1) = \frac{W_{\bar{F}}^2\left(1-W_f\right)}{\left(1-W_{\bar{F}}\right)}(1)$, and we have precisely returned to the condition that $\frac{\partial^2 \log f}{\partial^2 \log \bar{F}}$ attains a minimum at 0.

11 Stuff we Missed Putting in this Version of the Paper

Sorry, simply ran out of time.

- First, we have a bunch of results about asymmetric auctions. One set of them shows how, for any given A, one can get away with certain amounts and types of asymmetry and still be sure that $\phi'' > 1$. Another (less developed) tries to draw a tighter connection to the general structure in Maskin-Riley.
- Second, there is a nice way of constructing decreasing densities meeting our conditions.
- Third, we have results for multiple favored and non-favored bidders.
- There is also a lot of related work on comparative statics for SPHA's, and on why the optimal allocation when f is log concave but $\frac{F}{f}$ is concave has slope very close to one.

12 Comparing the First Price, Second Price and Optimal Mechanism

13 Optimal Mechanisms

Let

$$\omega_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$$

be the virtual cost of i. Because F is log-concave, $\omega'(\cdot) \geq 1$. Note, by the way, that for many purposes in mechanism design, all that is required is that $\omega'(c_i) > 0$. A conjecture, corroborated by a conversation with one of the early users of log-concavity in this setting, is that the base assumption of logconcavity was used because the assumption $\omega'(c_i) > 0$ was difficult to interpret. However, through the lens of concavity, things become somewhat clearer:

Claim 33

$$\left(\frac{h}{h'}\right)' = 1 - \frac{hh''}{(h')^2} = \rho_h.$$

Hence,

$$\begin{aligned}\omega'(c_i) &= \left(c_i + \frac{F(c_i)}{f(c_i)}\right)' \\ &= 1 + \rho_F.\end{aligned}$$

So, the condition $\omega'(c_i) > 0$ can be interpreted as the (fairly weak) condition that $\rho_F \geq -1$. This means that $\frac{F^{-1}}{-1}$ is concave, or equivalently that $\frac{1}{F}$ is convex.

Let the *Myerson Difference* be

$$\begin{aligned}\eta(c_1, c_2) &= \Delta - (c_1 - c_2) - \left(\frac{F(c_1)}{f(c_1)} - \frac{F(c_2)}{f(c_2)}\right) \qquad (31) \\ &= \Delta - (\omega(c_1) - \omega(c_2)).\end{aligned}$$

Since $\Delta = v_1 - v_2$, η is the value difference between 1 and 2, minus the virtual cost difference between 1 and 2. The next Lemma collects some observations about η.

Lemma 34

$$\begin{array}{ll} \eta_{c_1}(c_1, c_2) = -\omega'(c_1) \leq -1 & \forall (c_1, c_2) \\ \eta_{c_2}(c_1, c_2) = \omega'(c_2) \geq 1 & \forall (c_1, c_2) \\ \eta(c, c) = \Delta \geq 0 & \forall c \\ \eta(c + \Delta, c) = \frac{F(c)}{f(c)} - \frac{F(c+\Delta)}{f(c+\Delta)} \leq 0 & \forall c \end{array}.$$

Since we assume that the buyer always buys, any mechanism can be represented by $\gamma(\cdot, \cdot)$, where $\gamma(c_1, c_2)$ is the probability that 1 gets the job for given (c_1, c_2). Following Myerson (1981), γ ties down the entire mechanism under the condition that the highest cost types of 1 and 2 receive zero surplus.

Lemma 35 *The buyer's surplus, $BS(\gamma)$ is*

$$BS(\gamma) = v_2 - 1 + \int \int \gamma(c_1, c_2)\, \eta(c_1, c_2)\, f(c_1)\, f(c_2)\, dc_1 dc_2. \tag{32}$$

This is intuitive. Always buying from 2 gives the buyer surplus $v_2 - 1$, since 2 must receive 1 if he is to sell for all c_2. The second term represents the change in buyer surplus from buying from 1 according to γ.

From Lemma 35, it follows directly that

Corollary 36 *In the optimal mechanism,* 1 *wins if*

$$\Delta > \omega(c_1) - \omega(c_2) \tag{33}$$

and 2 *wins otherwise.*[18]

Thus, 1 wins if his virtual cost is no more than Δ above 2's. This follows since by (33), $\gamma(c_1, c_2)\, \eta(c_1, c_2)$ is maximized point-wise. Since ω is increasing, the allocation rule is monotone and hence incentive compatible.

Define the Myerson Line, $\phi_M(\cdot)$, by

$$\Delta = \omega(\phi_M(c_2)) - \omega(c_2),^{19} \tag{34}$$

or, equivalently, by

$$\eta(\phi_M(c_2), c_2) = 0. \tag{35}$$

By Lemma 34, 1 wins for $c_1 < \phi_M(c_2)$ and 2 wins for $c_1 > \phi_M(c_2)$.

Example 37 *Fig. 1 shows $\phi_{M,1}$ (the lighter line) for $f_1(c) = 6c(1-c)$ and $\phi_{M,2}$ (the heavier line) for $f_2(c) = \frac{2(1+c)}{3}$.*

13.1 Properties of the Myerson Line

One geometric properties of the Myerson line is at the core of our basic results. First, we have

Lemma 38 *Fix F. If $\Delta = 0$, then $\phi_M(c_2) = c_2$ for all c_2. If $\Delta > 0$, then for all c_2, $c_2 < \phi_M(c_2) < c_2 + \Delta$ and $\phi'_M(c_2) > 0$.*

[18] We do not specify what happens in the zero probability event of a tie.If $v_0 < v_1 - \omega(1)$ then it is indeed optimal to always buy.

[19] This is well defined since ω is monotonic.

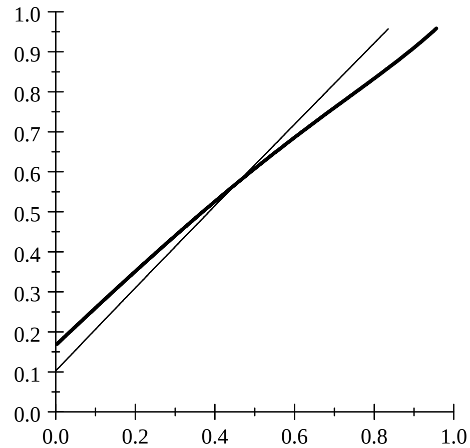

Figure 1: Myerson lines for $f_1(c) = 6c(1-c)$ (thin) and $f_2(c) = \frac{2(1+c)}{3}$ (thick) for $\Delta = .2$

So, the Myerson line lies above the diagonal but by less than Δ, and has positive slope. Each of these is trivial from (35) and Lemma 34. Key to our central result is to relate ϕ_M' to the concavity or convexity of $\frac{F}{f}$.

Lemma 39 *If $\frac{F}{f}$ is convex, then $\phi_M' \leq 1$. If $\frac{F}{f}$ is concave then $\phi_M' \geq 1$.*

Note that by Claim 33, $\left(\frac{F}{f}\right)' = \rho_F$. So, $\frac{F}{f}$ is convex if and only if ρ_h is increasing - convex virtual costs are the result of F becoming increasingly concave as x increases. Similarly, concave virtual costs are the result of decreasing local convexity.

The class of log-concave distributions contains many examples of both convex and concave virtual costs:

Example 40 *Both $f_1(c) = 6c(1-c)$ and $f_2(c) = \frac{2(1+c)}{3}$ are log-concave but $\frac{F_1}{f_1}$ is convex while $\frac{F_2}{f_2}$ is concave. As seen in Fig. 1, $\phi_{M,1}' < 1$ while $\phi_{M,2}' > 1$.*

Lemma 41 *If f is decreasing then $\frac{F}{f}$ is convex.*

This condition is by no means necessary.

Example 42 *Take $f(c) = ce^{-c}$. Then, f is increasing, but $\frac{F}{f}$ is convex.*

For one nice class of distributions, the Myerson line is particularly simple:

Example 43 *Consider the power distributions $F(c) = c^{\alpha}$. These are log-concave. Virtual costs are*

$$\omega(c) = c + \frac{F(c)}{f(c)} = c + \frac{c^{\alpha}}{\alpha c^{\alpha-1}} = c\left(1 + \frac{1}{\alpha}\right),$$

and thus linear. So,

$$\begin{aligned} \Delta &\geq \omega(c_1) - \omega(c_2) \\ &\text{iff} \\ c_1 - c_2 &\leq \frac{\Delta}{\left(1 + \frac{1}{\alpha}\right)}, \end{aligned}$$

and the Myerson line is parallel to, and $\frac{\Delta}{(1+\frac{1}{\alpha})}$ above, the diagonal. As α increases, $\frac{\Delta}{(1+\frac{1}{\alpha})}$ increases, and as $\alpha \to \infty$, F becomes highly convex and $\frac{\Delta}{(1+\frac{1}{\alpha})} \to \Delta$. As $\alpha \to 0$, F becomes very concave and $\frac{\Delta}{(1+\frac{1}{\alpha})} \to 0$. For the uniform distribution ($\alpha = 1$), $\frac{\Delta}{(1+\frac{1}{\alpha})} = \frac{\Delta}{2}$.

This idea - that as the cost distribution deteriorates by becoming more convex, the allocation moves towards 1 - is quite general. See...

14 Second Price Mechanisms

For general distributions, the optimal mechanism will specify bonuses that are complicated functions of revealed costs. We rarely see such mechanisms in practice. In this section, we focus on simple second price mechanisms that allow the auctioneer to reflect his preferences. In the next section, we look at first price implementations of this type of auction. Such mechanisms are simple insofar as the distortion is not dependent on the actual bids.

One reason why such simplicity may be favored in practice is that complicated rules can be susceptible to interim strategic manipulation by the buyer. Another reason is that it may be either illegal or unpalatable to the bidders to write rules that explicitly favor one bidder as a function of the others costs, but easier to write rules of the form "the buyer will pay transportation costs" implicitly favoring one bidder, but in a coarser manner.

An example of a simple mechanism is a scoring rule, in which the allocation depends on a weighted sum of the submitted bids and other characteristics of the deal.[20] Another example of auctions with simpler bonus rules

[20] See Wolfstetter and Lengwiler (2006) and Asker and Cantillion (2006).

is various versions of the request for proposal process, which, because they lack commitment power, are based only on the ex-post attractiveness of the competing bids.

14.1 Second Price Auction Formats

In a *second price bonus auction* (SPBA) the auctioneer announces a bonus A, and requests sealed bids from 1 and 2. The low bidder wins.[21] When 1 wins, he receives $\min(b_2 + A, 1)$, while if 2 wins, he receives $\min(b_1, 1)$. Putting a maximum of 1 on payments guarantees that the bidders do not receive an amount above their highest possible cost.

For 2, it is weakly dominant to set $\beta_2(c_2) = c_2$, while for 1, it is weakly dominant to set $\beta_1(c_1) = c_1 - A$. Thus, using SPBA mechanisms, we can implement allocations of the form

$$A \geq c_1 - c_2.$$

In general, this allocation need not be optimal, but the SPBA has the advantage of being very simple to both explain and to bid in. It also may respect important limitations on the sorts of distortion real world mechanism designers face...

Because $\beta_1(c_1) = c_1 - A$, 2 never wins when $c_2 > 1 - A$. Any $A > 1$ is thus equivalent to $A = 1$: 1 wins for sure, and does so at price 1, and so, without loss of generality, we restrict $A \leq 1$.

Add a very minimal amount from XXX.

The result that $\phi'_{FP} > 1$ has significant practical import. Say that mechanism 1 *ex-post dominates* mechanism 2 if it agrees with the optimal mechanism on strict superset of the set of (c_1, c_2) for which mechanism 2 agrees with the optimal mechanism.[22] So, mechanism 1 always gets it right when mechanism 2 does, and gets it right more often.

Theorem 44 *Assume that $\frac{F}{f}$ is convex. Consider the first price auction with handicap A, for any given A. If $\phi'_{FP} > 1$, then for suitably chosen $\hat{A}$, the first price auction is ex-post dominated by the second price auction with handicap $\hat{A}$.*

[21] Ties are zero probability in equilibrium and the tie breaking rule is inessential (see Jackson and Swinkels (2005)).

[22] Recall that for the various mechanisms the allocation is deterministic on a measure one set of types.

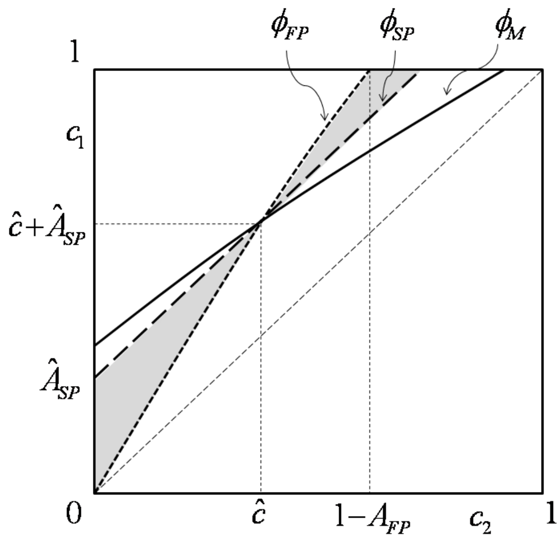

Figure 2: Illustration of Theorem 44.

So, for any first price auction (including, of course the optimal one), there is a second price mechanism that strictly improves upon it, and does so outcome by outcome.

The proof is illustrated by Fig. 2. When $\frac{F}{f}$ is convex, $\phi'_M\ (c_2) \leq 1$. So, pick any given A_{FP}. By assumption, $\phi'_{FP} > 1$. If ϕ_M and ϕ_{FP} never cross, then trivially one can find $\hat{A}$ such that the associated ϕ_{SP} lies between ϕ_M and $\phi_{FP.}$ If ϕ_M and ϕ_{FP} cross, then they do so once at some $\hat{c}$. Set $\hat{A} = \phi_{FP}(\hat{c}) - \hat{c}$, and consider ϕ_{SP} for bonus $\hat{A}$. Then, for all $c < \hat{c}$, $\phi_M\,(c) > \phi_{SP}\,(c) > \phi_{FP}(c)$, and for all $c > \hat{c}$, $\phi_M\,(c) < \phi_{SP}\,(c) < \phi_{FP}(c)$. So, for any (c_1, c_2) where the second price auction allocates differently than the first price auction (in the shaded areas of Fig. 2), the second price auction allocates optimally while the first price auction allocates incorrectly.

14.2 On the Relationship to Maskin and Riley

Maskin and Riley (2000) analyze auctions with a seller and two buyers with asymmetrically distributed values. In one specification a "weak" buyer draws his value from F with support $[0, 1]$ and a "strong" buyer draws his value from F_s with support $[s, s+1]$ given by

$$F_s(x) = F(x+s).$$

When s is large, in the standard first price auction the strong bidder bids 1 in equilibrium and always win. In the second price auction, the strong bidder will still always win but pays the value of his opponent. So, the first price auction raises more revenue than the second price auction. Maskin and Riley extend this revenue ranking to settings where s is smaller, and so there is a range of bids over which the allocation is competitive.[23]

To translate into our setting, begin with A^{III}, in which 1 has a Δ cost advantage but the rules are symmetric. The Maskin and Riley result says that in A^{III}, a symmetric first price mechanism is better than a symmetric second price mechanism. Translated into A^{I}, the setting with symmetrically distributed costs but a Δ value advantage for 1, this says that running a second price mechanism with handicap $A = \Delta$ is worse than running a second price mechanism with handicap $A = \Delta$. So, if one is going to run an RFP process, then it is better to do so in a sealed bid than in an open manner.

This result, that a first price auction can be better than a second, does *not* contradict ours. Maskin and Riley compare the first and second prices auction for a fixed $A = \Delta$. While natural, this turns out to be a pretty bad choice for the auctioneer, especially in the second price case, given that Myerson line lies strictly below $c_2 + \Delta$ (Lemma 38), and so second price auction universally distorts too far in favor of 1. In the FPHA with $A = \Delta$, $\phi_{FP}(c) - c$ is strictly below Δ except at $c = 1 - \Delta$ (Lemma **??**), and so does not always distort too far, providing one intuition for their ranking.[24]

In contrast, what we show is that for *any* first price auction, there is a handicap such that the second price auction does better in the very strong sense of ex-post dominance.[25] So, while Maskin and Riley show that between a focal pair of asymmetric auctions, one prefers the first price mechanism, we show that if one can choose which handicap to offer, one will prefer a second price mechanism.

[23] To get this result, they assume that F is convex (f is weakly increasing).

[24] We limit bidder 1 to receive at most his highest possible cost. We are working to better understand the degree to which Maskin and Riley's ranking depends on the absense of a reserve price in their framework. In their motivating example of bidders with very different supports over values, it seems critical.

[25] Our requirement that $\bar{F}$ be concave is identical to theirs, but we also require convex virtual costs. Our numerical exercises suggest that neither of these is critical.

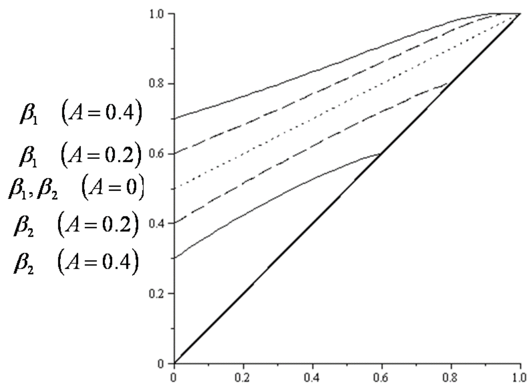

Figure 3: The equilibrium bid functions for $A = 0$ (dotted), $A = .2$ (dashed) and $A = .4$ (solid) for uniformly distributed costs.

14.3 Comparing the FPHA with the symmetric FPA

When $\phi' > 1$ holds everywhere, we have a simple characterization of how the FPHA with $A > 0$ compares to the symmetric first price auction:

Theorem 45 *If $\phi' \geq 1$ everywhere, then β_1 and β_2 lie on either side of the symmetric equilibrium strategy β_s of a standard first price auction*

$$\beta_1 \geq \beta_s \geq \beta_2.$$

This is intuitive: 1 bids less aggressively than if he were not favored, and 2 more. Overall, costs go up, but, for appropriately chosen A, the improvement in efficiency more than compensates the buyer. Fig. 3 shows how β_1 and β_2 vary in A for the uniform case. It is an interesting conjecture that β_1 and β_2 should move monotonically further apart as A grows for general f.

15 Conclusion

Needs to be updated...

In our setting a buyer puts premium Δ on procuring from 1 instead of 2. We derive the optimal mechanism, and provide a number of properties showing how the optimal, first, and second price mechanisms vary with

Δ and the underlying distribution F over costs. Our central result shows conditions under which a first price auction with a handicap will always be dominated by an appropriately chosen second price mechanism.

An intuition for the result is that the second price mechanism creates a constant distortion away from a symmetric allocation rule, while in the first price handicap auction, bidder optimization results in no distortion when 2's cost c_2 is low, but a great deal of distortion when c_2 is high. In particular, when f is non-decreasing, we show that the slope of the allocation generated by the first price auction, ϕ'_{FP}, is at least 1. For more general densities, we do not have theoretical results, but have failed to find a case in which $\phi'_{FP} < 1$.

The key to the dominance result is to compare these allocations with the optimal one. When virtual costs, $\frac{F}{f}$, are convex, the optimal mechanism specifies a distortion that decreases in c_2. So, while a well chosen second price mechanism need not be optimal in such a setting, it is better, on a case by case basis, than the first price mechanism, which gets things precisely backward. When $\frac{F}{f}$ is concave our numerical examples suggest that under fairly general conditions, the optimal mechanism will specify a distortion that is similar at different c_2. Thus, the second price mechanism is not only likely to be better than the first price mechanism, but also close to optimal!

Our analysis contains a number of novel points. The connection between the concavity or convexity of $\frac{F}{f}$ and the shape of the optimal allocation is new, and suggests that there may be other interesting properties of how the optimal allocation relates to the structure of F. We also make some headway on the question of how F feeds into the choice of an optimal second price mechanism.

Our derivation of the result that $\phi'_{FP} > 1$ uses techniques that we have not seen before, but that seem likely have more generally applicability. In particular, the degree to which one can generate bounds on the surplus that can be available to each player, and use that to partially characterize equilibrium bid functions seems intriguing.

Two obvious topics for further research are to get a better understanding of the examples suggesting that $\phi'_{FP} > 1$ holds much more widely than when f is non-decreasing and to get a better understanding of why the optimal mechanism seems so generally to have slope near 1 when $\frac{F}{f}$ is concave. It would also be worth exploring how much broader the result on the superiority of the second price mechanism is if one relaxes the criterion from a case by case to expected buyer surplus basis. A model with more than two bidders seems highly relevant. Other simple auction forms, such as percentage auctions, deserve more consideration. Finally, our techniques suggest that it may be useful to make a further study of the properties of asymmetric first price auctions more generally.

Our results should be interesting to an economic theorist, but also to firms that engage in procurement. While simple, our model seems a good match for many practical settings in which the most common practice is either an open or sealed bid request for proposal. The open RFP corresponds to a second price bonus auction with $A = \Delta$. As we show, this is non-optimal. There is useful insight to firms in saying simply "look, you are better off to commit yourself to act as if your preferences are weaker than they truly are. What you lose in not always getting your favorite guy you will more than make up for in lower costs."

The stronger message is that because they are of a first price nature, sealed bid style RFPs are a very bad way to go once one understands how bidder optimization undoes the desired impact of the handicap structure. By inducing a more even distortion, second price mechanisms are likely to perform better. On a practical level, the fact that their equilibria are so simple to calculate may also have significant advantages. Industry practice is routinely at a variance to the advice we give here. We think it probable that these firms could improve their practices.[26]

Firms should make a distinction between ways in $\Delta \neq 0$ can arise. In some cases, Δ reflects some sort of incompatibility or lock-in rather than an innate preference for one supplier or another. In such cases, designing a better auction mechanism is good, but better still would be to design a setting where Δ is smaller in the first place. In the process of designed the Dreamliner, Boeing made it a priority to have both the physical connection between the aircraft engine and the wing and the software interface between the engine and the cockpit standardized across aircraft engine manufacturers. Moving from one engine manufacturer to another is thus considerably easier, and Δ is lowered. When Δ reflects more fundamental issues, the goal of a good auction design should be to allow the allocation to depend on whose product is better without giving away the store in terms of muting competition.

It is also worthwhile to think about relevant features of many real world settings that are not captured in our model. In our model, Δ is fixed and exogenous. Assume Δ is determined by pre-auction effort. Since $\frac{\partial A}{\partial \Delta} < 1$, the optimal auction in our model provides muted quality incentives at the first stage, while a request for proposals might do better. Exploring this formally and thinking about good mechanisms in such a setting are topics for future research.

[26] We accept our expulsion from the Chicago school with quiet dignity.

16 Appendix

16.1 Proofs for Section 4

First, Karlin (1968) provides a useful inventory of properties related to log-concavity.

Lemma 46 *If f is a log-concave density then F and $\bar{F}$ are themselves log-concave. If f is a (strictly) log-concave density then $\frac{f}{F}$ is (strictly) decreasing and $\frac{f}{\bar{F}}$ is (strictly) increasing.*

Lemma 47 *Let g be log-concave with $g(1) = 0$. Then,*

$$W_{\bar{G}}(1) = \frac{1}{2 - W_g(1)}.$$

Proof of Lemma 47: By assumption $g(1) = 0$. Thus, l'Hopital's rule applies to give have

$$\begin{aligned}
W_{\bar{G}}(1) &= \lim_{s\to 1}\left(-\frac{g'(s)\bar{G}(s)}{g^2(s)}\right) = -\lim_{s\to 1}\frac{g''(s)\bar{G}(s) - g'(s)g(s)}{2g'(s)g(s)} \\
&= -\lim_{s\to 1}\frac{g''(s)\bar{G}(s)}{2g'(s)g(s)} + \frac{1}{2} = \frac{1}{2}\lim_{s\to 1}\left(-\frac{g''(s)g(s)g'(s)\bar{G}(s)}{(g')^2(s)g^2(s)}\right) + \frac{1}{2} \\
&= \frac{1}{2}\lim_{s\to 1}\left(W_{\bar{G}}(s)\, W_g(s)\right) + \frac{1}{2} \\
&= \frac{1}{2}\left(W_{\bar{G}}(1) W_g(1) + 1\right),
\end{aligned}$$

where the last step is valid noting that since $\bar{G}$ is log-concave $W_{\bar{G}}(1) \leq 1$, while since g is decreasing near zero, $W_{\bar{G}}(1) \geq 0$, and so $W_{\bar{G}}(1)$ is finite. Rearranging yields the result. ■

In main text, we contended that it is mild to assume that $W_h(1)$ is finite when $h(1) = 0$, and very mild to assume that $W_{\bar{H}}$ was finite. The following Lemma shows why.

Lemma 48 *Assume that g is $C^\infty[0,1]$ and has $g(1) = 0$. Then, $W_g(1)$ is finite.*

Proof of Lemma 48: Let n be such that $g^{(n)}(1) \neq 0$ while $g^{(k)}(1) = 0$ for all $k < n$. Note that n must be finite[27]. Since $g^{(n)}(1) \neq 0$ while $g^{(n-1)}(1) = 0$.

$$W_{g^{(n-1)}}(1) \equiv \frac{g^{(n-1)}(1) g^{(n+1)}(1)}{\left(g^{(n)}(1)\right)^2} = 0.^{28}$$

Assume by induction that $W_{g^{(n-k)}}(1) = \frac{k-1}{k}$ for some $k \in \{1, 2, \cdots, n\}$. Then, since $g^{n-k}(1) = 0$, Lemma 47 applies to $g = g^{(n-k)}$ to yield

$$\begin{aligned} W_{g^{(n-(k+1))}}(1) &= \frac{1}{2 - W_{g^{(n-1)}}(1)} \\ &= \frac{1}{2 - \frac{k-1}{k}} \\ &= \frac{k}{2k - (k-1)} \\ &= \frac{k}{k+1}. \ \blacksquare \end{aligned}$$

So, when $h(1) = 0$, or, noting that $\bar{H}(1) = 0$ by definition, all that is required is enough differentiability that the first non-zero derivative is not swamped by the next derivative up.

Proof of Lemma 4: Since W_h is continuous and $W_h(1)$ is finite,

$$\underline{W}_h(1) = W_h(1) = \bar{W}_h(1).$$

The result is then immediate from Lemma 47 applied to $g = h$.

16.2 Proofs for Section 5

Proof of Theorem 9**:** Let ϕ^I, ϕ^{II}, and ϕ^{III} be as in Definition 10. Then, we claim

$$\phi^I(c_2) = \phi^{II}(c_2) \tag{36}$$

and thus

$$\psi^I(c_1) = \psi^{II}(c_1). \tag{37}$$

[27] Otherwise by Taylor's formula $f \equiv 0$.

[28] The sole use of the assumption that $g \in C^\infty[0,1]$ is to assure that $\frac{g^{(n+1)}(1)}{\left(g^{(n)}(1)\right)^2}$ is finite. Clearly substantially weaker conditions than $h \in C^\infty[0,1]$ would suffice.

To see (36), note that

$$\begin{aligned}
\beta_1^{II}(\phi^I(c_2)) &= \beta_1^I(\phi^I(c_2)) - A && \text{(by (8))} \\
&= \beta_2^I(c_2) && \text{(by definition of } \phi^I\text{)} \\
&= \beta_2^{II}(c_2) && \text{(by (7))} \\
&= \beta_1^{II}(\phi^{II}(c_2)) && \text{(by definition of } \phi^{II}\text{).}
\end{aligned}$$

Thus, since β_1^{II} is increasing, $\phi^I(c_2) = \phi^{II}(c_2)$.

Similarly, we claim

$$\phi^{II}(c_2) = \phi^{III}(c_2) + A \tag{38}$$

and thus

$$\psi^{III}(c_1 - A) = \psi^{II}(c_1).\text{[29]} \tag{39}$$

To see (38), note that

$$\begin{aligned}
\beta_2^{III}(c_2) &= \beta_1^{III}(\phi^{III}(c_2)) && \text{(by definition of } \phi^{III}\text{)} \\
&= \beta_1^{II}(\phi^{III}(c_2) + A) && \text{(by (9)).}
\end{aligned}$$

Also

$$\begin{aligned}
\beta_2^{III}(c_2) &= \beta_2^{II}(c_2) && \text{(by (7))} \\
&= \beta_1^{II}(\phi^{II}(c_2)) && \text{(by definition of } \phi^{II}\text{).}
\end{aligned}$$

Combining, we have

$$\beta_1^{II}(\phi^{III}(c_2) + A) = \beta_1^{II}(\phi^{II}(c_2))$$

and so, since β_1^{II} is increasing,

$$\phi^{II}(c_2) = \phi^{III}(c_2) + A.$$

For $t \in \{I, II, III\}$, let $S_i^t(\widetilde{c}_i; c_i)$ be i's surplus in A^t when his true type

[29] To see that (38) implies (39) note that (38) holds at $\psi^{II}(c_1)$ for all c_1 in the range of ϕ^{II}. Thus,

$$\begin{aligned}
\phi^{II}(\psi^{II}(c_1)) &= \phi^{III}(\psi^{II}(c_1)) + A \\
c_1 - A &= \phi^{III}(\psi^{II}(c_1)) \\
\psi^{III}(c_1 - A) &= \psi^{II}(c_1).
\end{aligned}$$

is c_i but he submits the bid prescribed for $\tilde{c}_i$, given $\left(\beta_1^t, \beta_2^t\right)$.[30] Then,

$$\begin{aligned} S_1^{II}(\widetilde{c}_1; c_1) &= \bar{F}(\psi^{II}(\widetilde{c}_1)) \left(\beta_1^{II}(\widetilde{c}_1) + A - c_1\right) \\ &= \bar{F}(\psi^{I}(\widetilde{c}_1)) \left(\beta_1^{II}(\widetilde{c}_1) + A - c_1\right) && \text{(by (37)} \\ &= \bar{F}(\psi^{I}(\widetilde{c}_1)) \left(\beta_1^{I}(\widetilde{c}_1) - c_1\right) && \text{(by (8))} \\ &= S_1^{I}(\widetilde{c}_1; c_1), \end{aligned}$$

so that 1 has an incentive to deviate in A^I iff he has an incentive to deviate in A^{II}.

For 2, we similarly have

$$\begin{aligned} S_2^{II}(\widetilde{c}_2; c_2) &= \bar{F}(\phi^{II}(\widetilde{c}_2)) \left(\beta_2^{II}(\widetilde{c}_2) - c_2\right) \\ &= \bar{F}(\phi^{I}(\widetilde{c}_2)) \left(\beta_2^{II}(\widetilde{c}_2) - c_2\right) && \text{(by (36))} \\ &= \bar{F}(\phi^{I}(\widetilde{c}_2)) \left(\beta_2^{I}(\widetilde{c}_2) - c_2\right) && \text{(by (7))} \\ &= S_2^{I}(\widetilde{c}_2; c_2). \end{aligned}$$

Thus, $\left(\beta_1^I, \beta_2^I\right)$ is an equilibrium for A^I if and only $\left(\beta_1^{II}, \beta_2^{II}\right)$ is an equilibrium for A^{II}.

Similarly,

$$\begin{aligned} S_1^{III}(\tilde{c}_1; c_1) &= \bar{F}\left(\psi^{III}(\widetilde{c}_1)\right) \left(\beta_1^{III}(\widetilde{c}_1) - c_1\right) \\ &= \bar{F}\left(\psi^{II}(\widetilde{c}_1 + A)\right) \left(\beta_1^{III}(\widetilde{c}_1) - c_1\right) && \text{(by (39))} \\ &= \bar{F}_A\left(\psi^{II}(\widetilde{c}_1 + A)\right) \left(\beta_1^{II}(\widetilde{c}_1 + A) - c_1\right) && \text{(by (9))} \\ &= \bar{F}_A\left(\psi^{II}(\widetilde{c}_1 + A)\right) \left(\beta_1^{II}(\widetilde{c}_1 + A) + A - (c_1 + A)\right) \\ &= S_1^{II}(\tilde{c}_1 + A; c_1 + A) \end{aligned}$$

so that $c_1 \in [-A, 1-A]$ has an incentive to deviate in in A^{III} iff $c_1 + A \in [0, 1]$ has an incentive to deviate in A^{II}, and

$$\begin{aligned} S_2^{III}(\tilde{c}_2; c_2) &= \bar{F}_A\left(\phi^{III}(\widetilde{c}_2)\right) \left(\beta_2^{III}(\widetilde{c}_2) - c_2\right) \\ &= \bar{F}_A\left(\phi^{III}(\widetilde{c}_2)\right) \left(\beta_2^{II}(\widetilde{c}_2) - c_2\right) && \text{(by (7))} \\ &= \bar{F}_A\left(\phi^{II}(\widetilde{c}_2) - A\right) \left(\beta_2^{II}(\widetilde{c}_2) - c_2\right) && \text{(by (38))} \\ &= \bar{F}\left(\phi^{II}(\widetilde{c}_2)\right) \left(\beta_2^{II}(\widetilde{c}_2) - c_2\right) && \text{(by definition of } \bar{F}_A\text{)} \\ &= S_2^{III}(\tilde{c}_2; c_2) \end{aligned}$$

so that $\left(\beta_1^{III}, \beta_2^{III}\right)$ is an equilibrium for A^{III} if and only if $\left(\beta_1^{II}, \beta_2^{II}\right)$ is an equilibrium for A^{II}. ■

[30] By Lemma 8, the range of β_1^t is an interval for $t \in \{I, II, III\}$. Bids below $\beta_1^t(0)$ earn less than $\beta_1^t(0)$ since $\beta_1^t(0)$ already wins with probability 1. Bids above $\beta_1^t(1)$ never win. So, to be an equilibrium, it is necessary and sufficient that player 1 never wants to mimic another cost type, and analogously for player 2.

Proof of Theorem 12: We proceed in a sequence of steps.

Step 1: Derivation of (13) **and** (14)**.** If 1 with type c bids as if his type is $\tilde{c}$, his surplus is $\hat{S}_1(\tilde{c};c) = \bar{F}(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. By the envelope theorem,

$$\frac{\partial}{\partial c} S_1(c) = \frac{\partial}{\partial c} \hat{S}_1(\tilde{c};c)\bigg|_{\tilde{c}=c} = \bar{F}(\psi(c)).$$

Given $b_1 \leq 1$, $S_1(1) = 0$, yielding (13). Similarly, $\frac{\partial}{\partial c} S_1(c) = \bar{F}(\phi(c))$, and for $c_2 > 1 - A$ no $b_2 > c_2$ ever wins, and so $S_2(1 - A) = 0$, yielding (14). ■

Step 2: Derivation of (15) **and** (16)**.** From (13) we have

$$\bar{F}(\psi(c))(\beta_1(c) - c) = S_1(c) = \int_c^1 \bar{F}(\psi(s))ds,$$

and (15) follows by rearranging, and similarly for (16).

Step 3: Positive derivatives of β_1**,** β_2 **and** ϕ'**.** As increasing functions, $\beta_1(\cdot)$, $\beta_2(\cdot)$, $\phi(\cdot)$, and $\psi(\cdot)$ are differentiable almost everywhere. Note that a bid of $b_1 \in [\beta_2(0) + A, 1)$ by 1 wins with probability $\bar{F}\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)$. Pick $\hat{b}_1$ such that β_2 is differentiable at $\beta_2^{-1}\left(\hat{b}_1 - A\right)$, let $\hat{c}_2 = \beta_2^{-1}\left(\hat{b}_1 - A\right)$, and let $\hat{c}_1 = \beta_1^{-1}\left(\hat{b}_1\right)$. Then, since

$$\bar{F}\left(\beta_2^{-1}(b_1 - A)\right)(b_1 - \hat{c}_1)$$

is maximized at $\hat{b}_1$,

$$-f\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)\frac{1}{\beta_2'\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)}\left(\hat{b}_1 - \hat{c}_1\right) + \bar{F}\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right) = 0.^{31} \tag{40}$$

Since $(b_1 - \hat{c}_1) > 0$ (because 1 earns positive surplus with $\hat{c}_1 < 1$, and $\hat{c}_1 < 1$ since $\hat{b}_1 < 1$), and since the other terms in (40) are finite but positive,

$$\frac{1}{\beta_2'\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)}$$

[31] At $\hat{b}_1 = \beta_2(0) + A$, this is weakly negative.

is finite and positive as well, and so $\beta_2'\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right) > 0$. Hence, wherever β_2' exists, $\beta_2' > 0$. Similarly, for $c_2 < 1 - A$, if β_1' exists at $\phi(c_2)$, then $\beta_1' > 0$. As $\beta_1(\phi(c)) = \beta_2(c) + A$, where β_1' and β_2' exist,

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0. \tag{41}$$

Step 4: Derivation of (17) **and** (18)**.** Using (15)

$$\begin{aligned}
& \beta_1'(c) \\
= \; & 1 + \left(\frac{\int_c^1 \bar{F}(\psi(s))ds}{\bar{F}(\psi(c))}\right)' \\
= \; & 1 + \frac{\left(\bar{F}(\psi(c))\right)\left(\int_c^1 \bar{F}(\psi(s))ds\right)' - \left(\int_c^1 \bar{F}(\psi(s))ds\right)\left(\bar{F}(\psi(c))\right)'}{\left(\bar{F}(\psi(c))\right)^2} \\
= \; & 1 + \frac{\left(\bar{F}(\psi(c))\right)\left(-\bar{F}(\psi(c))\right) + \left(\int_c^1 \bar{F}(\psi(s))ds\right) f(\psi(c))\psi'(c)}{\left(\bar{F}(\psi(c))\right)^2} \\
= \; & \int_c^1 \bar{F}(\psi(s))ds \frac{f(\psi(c))}{\bar{F}^2(\psi(c))}\psi'(c) \\
= \; & S_1(c)\, g(\psi(c)) \frac{1}{\phi'(\psi(c))},
\end{aligned}$$

using that $\phi' > 0$ wherever β_1' and β_2' exist. Similarly,

$$\begin{aligned}
\beta_2'(c) &= \int_c^1 \bar{F}(\phi(s))ds \frac{f(\phi(c))}{\bar{F}^2(\phi(c))}\phi'(c) \\
&= S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}\phi'(c).
\end{aligned}$$

Step 5: Derivation of (19)**.** Substituting (17) and (18) into (41) gives

$$\phi'(c) = \frac{S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}\phi'(c)}{S_1(\phi(c))\, g(\psi(\phi(c))) \frac{1}{\phi'(\psi(\phi(c)))}} = \frac{S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}\phi'(c)}{S_1(\phi(c))\, g(c) \frac{1}{\phi'(c)}}.$$

Canceling $\phi'(c) > 0$ and rearranging yields (19).

Step 6: Continuous differentiability of ϕ. By (19),

$$\phi'(c) = \frac{S_1(\phi(c))}{S_2(c)} \frac{g(c)}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}}$$

almost everywhere. As a bounded, continuous function on a compact interval, ϕ is absolutely continuous (see, e.g., Wade (1995)) and so (see, e.g., Billingsley, Theorem 31.8).

$$\phi(c) = \phi(0) + \int_0^c \phi'(t)\, dt = \int_0^c \frac{S_1(\phi(t))}{S_2(t)} \frac{g(t)}{g(\phi(t))} dt.$$

Since $\phi(\cdot)$ is continuous, $\frac{S_1(\phi(\cdot))}{S_2(\cdot)} \frac{g(\cdot)}{g(\phi(\cdot))} \in C^1[0, 1-A)$ and so $\phi(c) \in C^1[0, 1-A)$ by the fundamental theorem of calculus.

Step 7: $\phi \in C^{k+1}[0, 1-A)$. We show the result on $[0, a]$, $a < 1 - A$. Since a is arbitrary, the result follows. Pick $a < 1 - A$. From Step 6, $\phi(c)$ is C^1 on $[0, 1-A)$, with

$$\phi'(c) = \frac{\int_{\phi(c)}^1 \bar{F}(\psi(s))ds}{\int_c^{1-A} \bar{F}(\phi(c))} \frac{\frac{f(c)}{\bar{F}^2(c)}}{\frac{f(\phi(c))}{\bar{F}^2(\phi(c))}}. \tag{42}$$

For $1 \leq \hat{k} \leq k$, assume that $\phi \in C^{\hat{k}}[0, a]$. Note that,

$$\left(\int_{\phi(c)}^1 \bar{F}(\psi(s))ds \right)' = -\phi'(c)\, \bar{F}(c).$$

Since $\phi' \in C^{\hat{k}-1}[0, a]$ and $\bar{F} \in C^k[0, 1]$, $-\phi'(c)\, \bar{F}(c) \in C^{\hat{k}-1}[0, a]$, and so $\int_{\phi(c)}^1 \bar{F}(\psi(s))ds \in C^{\hat{k}}[0, a]$. Similarly, $\left(\int_c^{1-A} \bar{F}(\phi(c))\right)' = -\bar{F}(\phi(c))$, and so $\left(\int_c^{1-A} \bar{F}(\phi(c))\right) \in C^{\hat{k}+1}[0, a]$. Since ϕ, f and $\bar{F}$ each belong to $C^k[0, a]$, and since all components of the RHS of (42) are everywhere positive on $[0, a]$, the RHS of (42) belongs to $C^{\hat{k}}[0, a]$ (see Shilov (1997). Thus, $\phi' \in C^{\hat{k}}[0, a]$, and so $\phi \in C^{\hat{k}+1}[0, a]$. By induction, $\phi \in C^{k+1}[0, a]$.

Step 8: $\beta_1 \in C^{k+1}[0, 1)$ and $\beta_2 \in C^{k+1}[0, 1-A)$: This follows immediately using the argument and conclusion of Step 7 applied to (17) and (18). ■

Proof of Lemma ??: Since $\phi(0) - 0 = 0$, and since ϕ is continuous, if there is $c < 1 - A$ where $\phi(c) - c \geq A$, then there is $c^* < 1 - A$ such that $\phi(c^*) - c^* = A$ and such that $\phi(c) - c$ is weakly increasing at c^*.

Player 1's first order condition at $\phi(c^*)$ can be expressed as

$$\frac{\beta_1'(\phi(c^*))}{\beta_1(\phi(c^*)) - \phi(c^*)} = \frac{f(c^*)}{\bar{F}(c^*)}\psi'(\phi(c^*)) = \frac{f(c^*)}{\bar{F}(c^*)}\frac{1}{\phi'(c^*)},$$

or

$$\frac{\bar{F}(c^*)}{f(c^*)}\frac{\phi'(c^*)\beta_1'(\phi(c^*))}{\beta_1(\phi(c^*)) - \phi(c^*)} = 1. \tag{43}$$

Since $\beta_1(\phi(c)) = \beta_2(c) + A$ for all c,

$$\phi'(c)\beta_1'(\phi(c)) = \beta_2'(c). \tag{44}$$

Since $\beta_1(\phi(c^*)) = \beta_2(c^*) + A$ and $\phi(c^*) - c^* = A$,

$$\beta_1(\phi(c^*)) = \beta_2(c^*) + \phi(c^*) - c^*$$

and so

$$\beta_1(\phi(c^*)) - \phi(c^*) = \beta_2(c^*) - c^*. \tag{45}$$

Substituting (44) and (45) into (43) gives

$$\frac{\bar{F}(c^*)}{f(c^*)}\frac{\beta_2'(c^*)}{\beta_2(c^*) - c^*} = 1.$$

But, 2's first order condition at c^* is

$$\frac{\beta_2'(c^*)}{\beta_2(c^*) - c^*} = \frac{f(\phi(c^*))}{\bar{F}(\phi(c^*))}\phi'(c^*).$$

Substituting,

$$\frac{\bar{F}(c^*)}{f(c^*)}\frac{f(\phi(c^*))}{\bar{F}(\phi(c^*))}\phi'(c^*) = 1,$$

and so

$$\phi'(c^*) = \frac{\frac{f(c^*)}{\bar{F}(c^*)}}{\frac{f(\phi(c^*))}{\bar{F}(\phi(c^*))}}.$$

But, $\frac{f}{\bar{F}}$ is strictly increasing and $\phi(c^*) > c^*$. Thus, $\phi'(c^*) < 1$, contradicting that $\phi(c) - c$ is weakly increasing at c^*. ∎

XXX This goes around here, but some work needs to happen

We prove this through a series of lemmas that comprise the balance of this section.[32] This discussion can be skipped without loss of continuity.

First, we show that $\limsup \phi'(c)$ cannot be in $(0, \infty)$.

[32] The proof is more straightforward if $\lim \phi'$ is known to exist. See the discussion at the end of this section.

Lemma 49 $\limsup \phi'(c) \in \{0, \infty\}$.

The proof of this is fairly dense, but hinges on using an extension of l'Hôpital's rule to the lim sup operator to derive a contradiction if $\limsup \phi'(c)$ is finite but positive. At its heart, the key is that S_1 and S_2 are very different near $c_1 = 1$ and $c_2 = 1 - A$.

Next, we show that $\phi'(c)$ does not tend to 0.

Lemma 50 $\limsup \phi'(c) > 0$.

Proof of Lemma 50: If $\limsup \phi'(c) = 0$ then $\phi'(c) \to 0$. So, for any small t, there is a last $c(t)$ at which $\phi'(c) = t$ (this is well defined since ϕ is continuously differentiable and $[0, 1 - A]$ is compact). But, by a change of variables,

$$\begin{aligned} S_1(\phi(c(t))) &= \int_{\phi(c(t))}^{1} \bar{F}_1(\psi(s))\, ds \\ &= \int_{c(t)}^{1-A} \bar{F}_1(s)\, \phi'(s)\, ds, \\ &< t(1 - A - c(t)), \end{aligned}$$

since $\phi'(s) < t$, and $\bar{F}_1 < 1$.

Thus,

$$\begin{aligned} \frac{1}{S_1(\phi(c(t)))\, \frac{f}{\bar{F}_1^2}(c(t))} &> \frac{t}{t(1 - A - c(t))\, \frac{f}{\bar{F}_1^2}(c(t))} \\ &> \frac{1}{\frac{f}{\bar{F}_1^2}(1 - A)} \frac{1}{t(1 - A - c(t))}. \end{aligned}$$

This diverges as $t \to 0$ and $c(t) \to 1 - A$. But then by Lemma 19 for small t, $\phi''(c(t)) > 0$, contradicting that $c(t)$ was the last moment at which $\phi' = t$. ■

We thus have

Corollary 51 $\limsup \phi'(c) = \infty$.

To conclude from this that $\phi'(c) \to \infty$, we would need to tie down $\liminf \phi'(c)$. A partial characterization key to our later results is

Lemma 52 *There is $\hat{c} < 1 - A$ such that for all $c > \hat{c}$, $\phi'(c) > 1$.*

We use (**??**) to show that when $\phi' = 1$, but c is close to $1-A$, $\phi'' > 0$ and so there are eventually no more crossings of $\phi' = 1$. Since $\limsup \phi'(c) = \infty$, $\phi'(c) > 1$ after some point.

A corollary that is mostly in the name of tidiness (and sunk costs) is that β_1 is indeed continuously differentiable at 1.

Corollary 53 $\beta_1'(c) \to 0$. *Hence,* β_1 *is continuously differentiable on* $[0,1]$.

The point is that since $\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))}$ stays above 1 near $1-A$, and since $\beta_2'(c) \to 0$, it must also be that $\beta_1'(\phi(c)) \to 0 = \beta_2'(1)$.

We conjecture that $\phi'(c)$ is sufficiently well behaved that $\phi'(c) \to \infty$. But, as $c \to 1-A$, and $\phi(c) - c \to A$, one can show that

$$\frac{\phi''(c)}{\phi'(c)} = Y(c,\delta) - \phi'(c) X(c,\delta),$$

where as $c \to 1-A$, and $\phi(c) - c \to A$, both X and Y tend to $-\infty$, and, for any finite $\phi'(c)$, there are points (c_2, δ) arbitrarily near $(1-A, A)$ where this expression takes on either sign at arbitrary magnitude. So, this is far from trivial. See Mares and Swinkels (2008b).

XXX End of random insertion.

Proof of Lemma 16**:** Since β_1 is increasing,

$$\liminf \frac{\beta_1(1) - \beta_1(c)}{1-c} \geq 0.$$

Assume

$$\limsup \frac{\beta_1(1) - \beta_1(c)}{1-c} = \alpha > 0. \tag{46}$$

For any c, $\beta_1(c)$ earns $\bar{F}(\psi(c))(\beta_1(c) - c)$, while a bid of 1 earns $\bar{F}(1-A)(1-c)$. Since $\beta_1(c)$ is a best response,

$$\bar{F}(\psi(c))(\beta_1(c) - c) \geq \bar{F}(1-A)(1-c),$$

and so for $c < 1$,

$$\frac{\beta_1(c) - c}{1-c} \geq \frac{\bar{F}(1-A)}{\bar{F}(\psi(c))}.$$

But, as $c \to 1$, $\psi(c) \to 1-A$, and so

$$\liminf \frac{\beta_1(c) - c}{1-c} \geq 1. \tag{47}$$

By (46), along a subsequence c_t, $c_t \to 1 - A$,

$$\frac{\beta_1(1) - \beta_1(c_t)}{1 - c_t} \geq \frac{\alpha}{2}.$$

But, since

$$\frac{\beta_1(1) - \beta_1(c_t)}{1 - c_t} + \frac{\beta_1(c_t) - c_t}{1 - c_t} = 1,$$

this means that $\frac{\beta_1(c_t) - c_t}{1 - c_t} < 1 - \frac{\alpha}{2}$ for all t, contradicting (47). So,

$$\liminf \frac{\beta_1(1) - \beta_1(c)}{1 - c} = \limsup \frac{\beta_1(1) - \beta_1(c)}{1 - c} = 0,$$

and β_1 is differentiable at 1, with $\beta_1'(1) = 0$.

Player 1's profit from mimicking $\tilde{c}$ is $\bar{F}(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. Taking the first order condition at $\tilde{c} = c$,

$$\bar{F}(\psi(c))\,\beta_1'(c) = f(\psi(c))\,\psi'(c)\,(\beta_1(c) - c).$$

But,

$$\psi'(c) = \frac{\beta_1'(c)}{\beta_2'(\psi(c))},$$

and so

$$\bar{F}(\psi(c))\,\beta_1'(c) = f(\psi(c))\,\frac{\beta_1'(c)}{\beta_2'(\psi(c))}\,(\beta_1(c) - c).$$

Cancelling $\beta_1'(c) > 0$, and rearranging,

$$\begin{aligned} \beta_2'(\psi(c)) &= \frac{f(\psi(c))}{\bar{F}(\psi(c))}(\beta_1(c) - c) \\ &< \frac{f(1 - A)}{\bar{F}(1 - A)}(1 - c). \end{aligned}$$

Thus, as $c \to 1$, $\beta_2'(\psi(c)) \to 0$. But then, $\beta_2'(1 - A)$ exists and equals 0. ■

Proof of Lemma 17: From Lemma 16, we have $\beta_1'(1) = 0$, and hence

$$\frac{1 - \beta_1(\phi(c))}{1 - \phi(c)} \to 0$$

as $c \to 1 - A$. But then, since for all c

$$\frac{1 - \beta_1(\phi(c))}{1 - \phi(c)} + \frac{\beta_1(\phi(c)) - \phi(c)}{1 - \phi(c)} = 1,$$

$$\frac{\beta_1\left(\phi\left(c\right)\right)-\phi\left(c\right)}{1-\phi\left(c\right)}\to 1.$$

The proof for β_2 is identical. ■

In what follows we shall need

Lemma 54 *For all f log-concave* $\limsup_{s\to 1}\left(\frac{\bar{F}(s)}{f(s)}\right)'\in[-1,0]$.

Proof of Lemma 54: Since f is log-concave, f is unimodal, and hence monotone on $[c,1]$ for some c. Assume first that f is non-decreasing on $[c,1]$. Then, $\frac{f'(s)}{f(s)}\geq 0$ for all $s\in[c,1]$, and so since $\frac{f'}{f}$ is non-increasing by log-concavity, $\lim_{s\to 1}\left(\frac{f'(s)}{f(s)}\right)$ is well defined, finite, and positive. Hence, since $\frac{\bar{F}(s)}{f(s)}\to 0$,

$$\begin{aligned}
\lim_{s\to 1}\left(\frac{\bar{F}\left(s\right)}{f\left(s\right)}\right)' &= \lim_{s\to 1}\left(\frac{-f^2\left(s\right)-\bar{F}\left(s\right)f'\left(s\right)}{f^2\left(s\right)}\right)\\
&= -1-\lim_{s\to 1}\left(\frac{\bar{F}\left(s\right)f'\left(s\right)}{f^2\left(s\right)}\right)\\
&= -1-\lim_{s\to 1}\left(\frac{\bar{F}\left(s\right)}{f\left(s\right)}\right)\lim_{s\to 1}\left(\frac{f'\left(s\right)}{f\left(s\right)}\right)\\
&= -1.
\end{aligned}$$

Assume f is decreasing on $[c,1]$ (this includes the case where $f\left(1\right)=0$). Then, for $s\in(c,1]$,

$$\begin{aligned}
0 &\geq \left(\frac{\bar{F}\left(s\right)}{f\left(s\right)}\right)'=\lim_{x\uparrow s}\left(\frac{\frac{\bar{F}(s)}{f(s)}-\frac{\bar{F}(x)}{f(x)}}{s-x}\right)\\
&\geq \lim_{x\uparrow s}\frac{1}{f(x)}\left(\frac{\bar{F}(s)-\bar{F}(x)}{s-x}\right)\\
&= \lim_{x\uparrow s}\frac{1}{f(x)}\left(\frac{-\int_x^s f(t)dt}{s-x}\right)\\
&\geq \lim_{x\uparrow s}\frac{1}{f(x)}\left(-\frac{f(x)\int_x^s dt}{s-x}\right)=-1. \quad ■
\end{aligned}$$

Proof of Lemma 49: Assume that $\limsup\phi'\left(c\right)=\alpha\in(0,\infty)$. Now,

$$
\begin{aligned}
\phi'(c) &= \frac{S_1(\phi(c))\frac{f_2(c)}{\bar{F}_2^2(c)}}{S_2(c)\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} \\
&= \frac{(\beta_1(\phi(c))-\phi(c))\frac{f_2(c)}{\bar{F}_2(c)}}{(\beta_2(c)-c)\frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))}} \\
&= \frac{(\beta_1(\phi(c))-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}}{(\beta_2(c)-c)\frac{\bar{F}_2(c)}{f_2(c)}} \\
&= \frac{\frac{(\beta_1(\phi(c))-\phi(c))}{1-\phi(c)}(1-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}}{\frac{\beta_2(c)-c}{1-A-c}(1-A-c)\frac{\bar{F}_2(c)}{f_2(c)}}.
\end{aligned}
$$

But then, by Lemma 17

$$
\begin{aligned}
\limsup \phi'(c) &= \lim \frac{\frac{(\beta_1(\phi(c))-\phi(c))}{1-\phi(c)}}{\frac{\beta_2(c)-c}{1-A-c}} \limsup \frac{(1-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}}{(1-A-c)\frac{\bar{F}_2(c)}{f_2(c)}} \\
&= \limsup \frac{(1-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}}{(1-A-c)\frac{\bar{F}_2(c)}{f_2(c)}} \\
&= \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup \frac{\left((1-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}\right)}{((1-A-c))}.
\end{aligned}
$$
[33]

Since the top and bottom of

$$
\frac{(1-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}}{(1-A-c)}
$$

go to 0 as $c \to 1-A$, a generalization of l'Hôpital's rule (Lee (1977)) gives

$$\begin{aligned}
&\limsup \phi'(c) \\
\leq\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup \frac{\frac{\partial}{\partial c}\left((1-\phi(c))\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}\right)}{\frac{\partial}{\partial c}((1-A-c))} \\
=\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup \frac{-\phi'(c)\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))} + (1-\phi(c))\left(\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)}\phi'(c)}{-1} \\
=\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \left(\limsup \phi'(c)\left(\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))} + (1-\phi(c))\left(-\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)}\right)\right) \\
\leq\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \left(\limsup \phi'(c)\right) \limsup \left(\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))} + (1-\phi(c))\left(-\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)}\right)
\end{aligned}$$

Since $\limsup \phi'(c) \in (0, \infty)$ by assumption, cancel to obtain

$$\begin{aligned}
1 \leq\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup \left(\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))} + (1-\phi(c))\left(-\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)}\right) \\
\leq\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \left(\underbrace{\limsup\left(\frac{\bar{F}_1(\phi(c))}{f_1(\phi(c))}\right)}_{=0} + \limsup\left((1-\phi(c))\left(-\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)}\right)\right) \\
=\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup \left((1-\phi(c))\left(-\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)}\right).
\end{aligned}$$

But, by Lemma 54 $\limsup \left(-\frac{\bar{F}_1(s)}{f_1(s)}\right)'_{s=\phi(c)} \in [0,1]$. The term $(1-\phi(c))$ is bounded as well. Thus,

$$\begin{aligned}
1 \leq\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup(1-\phi(c)) \limsup \left(-\frac{\bar{F}_2(s)}{f_2(s)}\right)'_{s=\phi(c)} \\
\leq\ & \frac{f_2(1-A)}{\bar{F}_2(1-A)} \limsup(1-\phi(c)) \\
=\ & 0,
\end{aligned}$$

a contradiction. ■

Proof of Lemma 52: Recall from Lemma 19 that

$$\frac{\phi''(c)}{\phi'(c)} = \phi'(c)\frac{\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)}}{S_1(\phi(c))\,g(c)} + \frac{g'(c)}{g(c)} - \phi'(c)\frac{g'(\phi(c))}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}}.$$

If $\phi'(c) = 1$, this reduces to

$$\begin{aligned}
\frac{\phi''(c)}{\phi'(c)} &= \frac{\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)}}{S_1(\phi(c))\, g(c)} + \frac{g'(c)}{g(c)} - \frac{g'(\phi(c))}{\frac{f_1(\phi(c))}{\bar{F}_1^2(\phi(c))}} \\
&> \left(\frac{1}{S_1(\phi(c))\, g(c)} - 2\right)\left(\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)}\right) \\
&= {}_s\left(\frac{1}{S_1(\phi(c))\, g(c)} - 2\right).
\end{aligned}$$

But,

$$\begin{aligned}
S_1(\phi(c))\, g(c) &= \int_{\phi(c)}^{1} \bar{F}(\psi(s))\, ds g(c) \\
&< \int_{\phi(c)}^{1} 1 ds g(c) \\
&= (1 - \phi(c))\, g(c)
\end{aligned}$$

which goes to 0. So, there is c^* such that for $c > c^*$, any point where $\phi' = 1$ has $\phi'' > 0$, and so there is at most one (upward) crossing of $\phi' = 1$ after that. Let $\hat{c}$ be that upward crossing if it exists, and $\hat{c} = c^*$ otherwise. ■

Proof of Theorem 45: Since $\phi'(c) \geq 1$,

$$\begin{aligned}
\frac{\partial}{\partial c} \ln \frac{\bar{F}(\phi(c))}{\bar{F}(c)} &= \phi'(c) \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=\phi(c)} - \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=c} \\
&\leq \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=\phi(c)} - \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=c} \\
&\leq 0
\end{aligned}$$

where the first inequality holds since $\phi'(c) \geq 1$, and $\frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=\phi(c)} < 0$ since $\bar{F}$ is decreasing, and the second inequality holds since $\bar{F}$ is log-concave and decreasing and $\phi(c) \geq c$. Thus, $\frac{\bar{F}(\phi(c))}{\bar{F}(c)}$is decreasing in c.

By Cauchy's theorem for all $c \in [0, 1 - A)$, there is $\xi_c \in [c, 1 - A)$ such that

$$\frac{\int_c^{1-A} \bar{F}(\phi(s)) ds}{\int_c^1 \bar{F}(s) ds} = \frac{\bar{F}(\phi(\xi_c))}{\bar{F}(\xi_c)} \leq \frac{\bar{F}(\phi(c))}{\bar{F}(c)},$$

or, rearranging,

$$\frac{\int_c^1 \bar{F}(s) ds}{\bar{F}(c)} \geq \frac{\int_c^{1-A} \bar{F}(\phi(s)) ds}{\bar{F}(\phi(c))}.$$

Adding c to each side yields

$$\beta_2(c) = c + \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))} \leq c + \frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} = \beta_s(c),$$

where $\beta_s(c)$ is the symmetric equilibrium bid function. Similarly,

$$\beta_1(c) = c + \frac{\int_c^1 \bar{F}(\psi(s))ds}{\bar{F}(\psi(c))} \geq c + \frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} = \beta_s(c). \blacksquare$$

16.3 Proofs for Section 13

Proof of Lemma 35: Consider an incentive compatible mechanism Ξ in which the buyer always buys. As noted in the text, Ξ is characterized by γ, and, adapting Myerson (1981) in the obvious ways to the setting, $BS(\gamma)$ is

$$\begin{aligned} & \int\int (\gamma(c_1, c_2)(v_1 - \omega(c_1)) + (1 - \gamma(c_1, c_2))(v_2 - \omega(c_2))) f(c_1) f(c_2) dc_1 dc_2 \\ = & \underbrace{\int\int (v_2 - \omega(c_2)) f(c_1) f(c_2) dc_1 dc_2}_{\text{Term 1}} \\ & + \int\int \gamma(c_1, c_2) \underbrace{(v_1 - v_2 - (\omega(c_1) - \omega(c_2)))}_{\eta(c_1, c_2)} f(c_1) f(c_2) dc_1 dc_2. \end{aligned}$$

The lemma follows since (recalling $\omega(c_2) = c_2 + \frac{F}{f}(c_2)$, and integrating out c_1), Term 1 equals

$$\begin{aligned} \int \left(v_2 - c_2 - \frac{F}{f}(c_2) \right) f(c_2) dc_2 &= v_2 - E(c_2) - \int F(c_2) dc_2 \\ &= v_2 - E(c_2) - 1 + \int (1 - F(c_2)) dc_2 \\ &= v_2 - E(c_2) - 1 + E(c_2) \\ &= v_2 - 1. \blacksquare \end{aligned}$$

Proof of Lemma ??: Since $\Delta = \omega(\phi_{M,\Delta}(c_2)) - \omega(c_2)$ for all Δ,

$$1 = \omega'(\phi_M(c_2)) \frac{\partial}{\partial \Delta} \phi_{M,\Delta}(c_2),$$

and hence

$$\frac{\partial}{\partial \Delta} \phi_{M,\Delta}(c_2) = \frac{1}{\omega'(\phi_M(c_2))} < 1,$$

since $\omega'(c) = 1 + \left(\frac{F(c)}{f(c)}\right)' > 1$ by log-concavity. ■

Proof of Lemma 39: From (34) ϕ_M is given by

$$\omega(\phi_M(c_2)) = \omega(c_2) + \Delta$$

and so, since ω is increasing, $\phi_M(c_2) > c_2$. Differentiating,

$$\omega'(\phi_M(c_2))\,\phi_M'(c_2) = \omega'(c_2),$$

and so

$$\phi_M'(c_2) = \frac{\omega'(c_2)}{\omega'(\phi_M(c_2))}.$$

If $\frac{F}{f}$,and thus ω, is convex, then, since $\phi_M(c_2) > c_2$, the bottom is bigger than the top, and so $\phi_M'(c_2) \leq 1$ (and strictly if $\frac{F}{f}$ is strictly convex). Similarly, if $\frac{F}{f}$ is concave, $\phi_M'(c_2) \geq 1$. ■

Proof of Lemma 41: Note that

$$\left(\frac{F}{f}\right)' = \frac{f^2 - f'F}{f^2} = 1 + \left(-\frac{f'}{f}\right)\left(\frac{F}{f}\right).$$

Since $f' \leq 0$, $-\frac{f'}{f}$ and $\frac{F}{f}$ are increasing and positive by log-concavity. ■

17 References

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