Optimal Auctions with Financially Constrained Bidders [*]

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March 10, 2010

Abstract

We consider an environment where ex-ante symmetric potential buyers of an indivisible good have liquidity constraints, i.e. they cannot pay more than their 'budget' regardless of their valuation. A buyer's valuation for the good as well as her budget are her private information. We derive the symmetric constrained-efficient and revenue maximizing auctions for this setting. We show how to implement these via a standard auction (all pay) with a modified winning rule. In general, the optimal auction requires 'pooling' both at the top and in the middle despite the maintained assumption of a monotone hazard rate. Further, the auctioneer will never find it desirable in terms of revenue or social welfare to subsidize bidders with low budgets.

Keywords: optimal auction, budget constraints

JEL Classification Numbers: D44

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1 INTRODUCTION

This paper considers the sale of a single good to buyers whose valuations and budgets are their private information. We provide an analysis of both the revenue maximizing and constrained efficient symmetric auctions. Our analysis is for the case where the solution concept is Bayes-Nash. Individual rationality is imposed in an interim sense. In other words, we only require the auction to offer each buyer a non-negative expected surplus. We then show how our analysis can be applied when the auction is required to be dominant strategy implementable or ex-post individually rational.

Incorporating budget constraints allows one to relax a standard assumption in the auction literature that conflates a buyer's willingness to pay with her ability. The assumption is untenable in a variety of situations. For instance, in government auctions (privatization, license sales etc.), the sale price may well exceed a buyers' liquid assets, and she may need to rely on an imperfect (i.e. costly) capital market to raise funds. These frictions limit her ability to pay, but not her valuation (how much she would pay if she had the money). Indeed, these financial constraints are more palpable to bidders than valuations, which are relatively amorphous.

Here we assume that buyers have 'hard' budget constraints: no buyer can pay more than her budget regardless of her valuation.[1] We do not model the source of these constraints. The reader is referred to Che and Gale [8] for a discussion of possible explanations.

1.1 WHY BUDGET CONSTRAINTS MATTER IN OPTIMAL AUCTION DESIGN

It is natural to ask why not run an auction where agents simply report the minimum of the valuation v and budget b, $\min(v, b)$. We claim such a mechanism will be strictly suboptimal in terms of revenue. This is because it will (i) pool types 'too much' and, (ii) discourage competition. To see why, it is helpful to examine the behavior of a 'natural' mechanism where bidders report $\min(v, b)$.

For simplicity, suppose two buyers whose budgets are common knowledge. Buyer A has a budget of a 1. Buyer B is not budget constrained (or alternately has a large budget). Both bidders have valuations that are i.i.d. draws from a uniform distribution over $[0, 2]$.

Consider a standard sealed bid second price auction. In in this mechanism it is a dominant strategy for each bidder to bid $\min(v, b)$. Therefore, bidder 2 will bid his valuation; while A will bid his valuation if it is less than a 1, and 1 otherwise. However when A bids 1, he pays less than 1 when he wins.

By contrast consider a first price auction. In equilibrium, an A with value more than 1 will bid higher than A with a value of a 1 (but still below the budget of 1). The second price

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[1]The literature also considers the case of 'soft' budget constraints, where bidders may be able to get additional funds from the market at some cost.

mechanism is pooling bidders 'too much,' depressing revenues. Switching to a mechanism that causes A to bid closer to this expected payment is a better idea in terms of revenue. As a result, intuitively, first price auctions are better than second price auctions, and all pay mechanisms are better still.

However, a standard sealed bid all-pay auction (with a reserve) will not be revenue maximizing. There is no reason for the optimal mechanism to give the good to the agent with the highest bid. The intuition springs from the following idea: In the sealed bid second price auction, B's bids will be distributed uniformly over $[0, 2]$, while A's bids will be distributed uniformly over $[0, 1)$ with probability $\frac{1}{2}$, and be 1 with probability $\frac{1}{2}$. If we treat this distribution over bids as the distribution over buyer valuations, the revenue maximizing mechanism as characterized in Myerson [19] will **not** allocate the good to the agent with the highest valuation. In particular, a A with value 1 will be allotted the good over B who announces a value slightly larger than 1. This is because such A has a higher *virtual valuation* than B. The revenue maximizing auction will therefore require a modified winning rule.

We, show how to implement the optimal mechanism as a (modified) sealed bid all pay auction. Therefore, it is possible to design the auction so that in equilibrium each buyer has to report a single number. The constrained efficient mechanism will need a similar modification to the winning rule, where the highest bidder may win the good.

Further, it will be clear from our analysis that:

1. The revenue maximizing bayesian incentive compatible (IC) and Ex-post individually rational (IR) mechanism can be implemented by a similarly modified first price auction.

2. The revenue maximizing dominant strategy IC and Ex-post IR mechanism can be implemented by a similarly modified second price auction.

The modified all-pay auction will have a higher expected revenue than the (modified) first price auction. In turn, the modified first price auction will have a higher expected revenue than the (modified) second price auction.

1.2   Discussion of Main Results

In this section we describe the main qualitative features of the revenue maximizing auction subject to budget constraints.[2] In particular, we contrast with the optimal auction when buyers have no budget constraints (Myerson [19]). Our discussion is for the case where the distribution satisfies the monotone hazard rate condition.

When buyers are not budget constrained, the type of an agent is just her valuation, and Myerson [19] applies. In this case we know that at each realized profile of types, the optimal allocation rule allots to the highest valuation above the reserve $\underline{v}$. The reserve is the lowest

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[2]The constrained efficient auction is similar but has no reserve price.

valuation with a non-negative 'virtual valuation.' Assuming 2 bidders and valuations to be uniform in $[0, 1]$, the resulting interim allocation probabilities are as graphed in Figure 1(a). The optimal auction could be implemented as any standard auction (first price, second price, or all pay) with the appropriately chosen reserve.

COMMON KNOWLEDGE COMMON BUDGETS   Now suppose all buyers have the same (common knowledge) budget constraint b. The type of an agent is still just her valuation. Laffont and Robert showed that the revenue maximizing auction will 'pool' some types at the top. All types above some $\bar{v}$ will be treated as if they had valuation exactly $\bar{v}$: the budget constraint binds for precisely these types.

Laffont and Robert argued that the optimal allocation rule would allot the good to the highest valuation subject to this 'pooling', and subject to it being higher than an appropriately chosen reserve $\underline{v}$. They showed that the reserve was lower than the one in Myerson. The resulting interim allocation probabilities are as graphed in Figure 1(b). The optimal auction could be implemented as an all pay auction with appropriately chosen reserve. In equilibrium, all types $\bar{v}$ and above would bid exactly b and therefore get pooled.
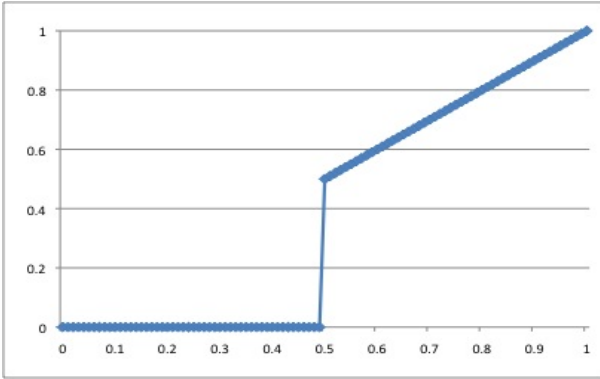
However, the interim allocation probabilities displayed in Figure 1(b) are optimal only under an additional condition: the density function of the valuations must be (weakly) decreasing.[3] If this condition is violated, our analysis shows that there can be additional pooling in the middle as displayed in Figure 1(c).

PRIVATE BDUGETS   Finally, suppose bidders have one of 2 budgets $b_H > b_L$. Here, the type of a bidder is 2 dimensional- his valuation, and his budget. As in Laffont and Robert, there will be pooling at the top. However there will be two cutoffs, $\bar{v}_H \geq \bar{v}_L$. All high budget bidders with valuation at least $\bar{v}_H$ will be pooled and all low budget bidders with valuation at least $\bar{v}_L$ will be pooled. A bidder with valuation $v < \bar{v}_L$ will get the same allocation whether he is of a high budget or low budget type.

Our key finding is the additional distortion the optimal allocation rule produces for high budget bidders with valuation only slightly higher $\bar{v}_L$. High budget bidders whose valuations are in the range $[\bar{v}_L, \bar{v}_L + \Delta]$ will be pooled with lower budget bidders with valuation in $[\bar{v}_L, 1]$. The resulting interim allocation probabilities are graphed in Figure 1(d). The case of $k > 2$ possible budgets will involve $k - 1$ such intervals in which the outcome is distorted, and k cutoffs corresponding to the budget constraints.

The auction is once again implemented as an all pay auction, but with a modified winning rule. As before there will be (an appropriately chosen) reserve price. Further the auction will commit to treating all bids between $[b_L, b_L + \delta)$ as b_L. In the resulting equilibrium, all buyers with a high budget and a valuation in $[\bar{v}_L, \bar{v}_L + \Delta]$ will bid b_L and be pooled low
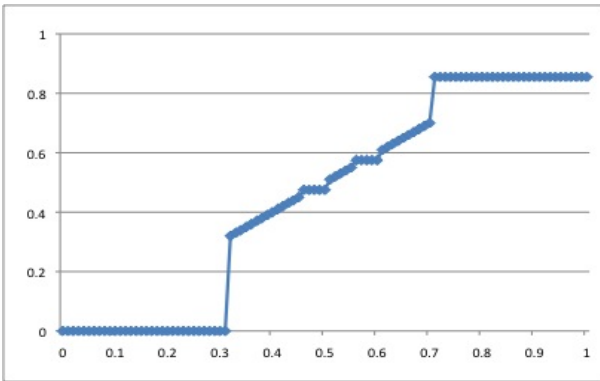
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[3]This was overlooked in the original papers by Laffont and Robert, and Maskin.
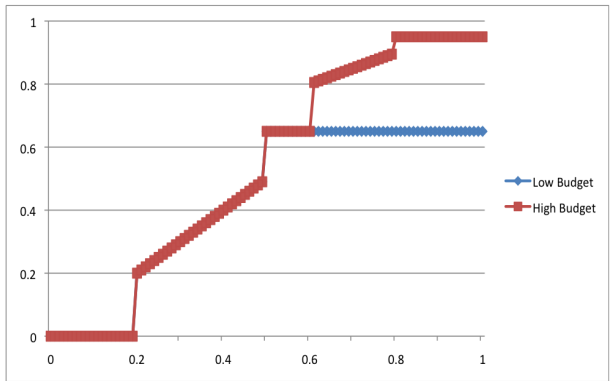
(a) No Budget Constraints
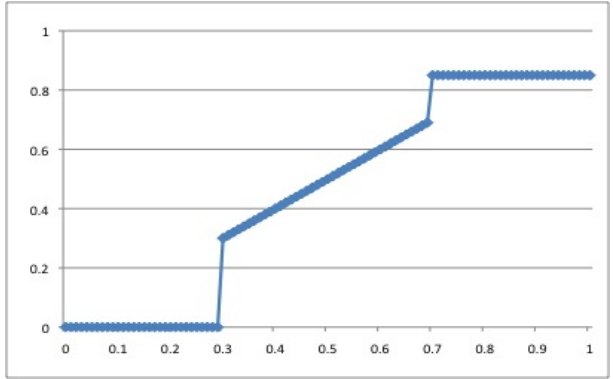
(b) Common Budget Constraint, Decreasing Density

(c) Common Budget Constraint

(d) Private Budgets

FIGURE 1: Optimal Interim Allocation Probability $a(v)$ plotted against v.

budget bidders with valuation larger than $\bar{v}_L$. As in the common knowledge budget case, buyers with low budget and valuation larger than $\bar{v}_L$ bid b_L and get pooled. Similarly, buyers with a high budget and valuation larger than $\bar{v}_H$ bid b_H and get pooled.

1.2.1 SUBSIDIES

Budget constraints depress revenues because low budget bidders cannot put competitive pressure on high budget bidders. Therefore, it is natural to inquire into instruments to foster competition. In other settings where some bidders are disadvantaged relative to others,

subsidies have been suggested. [4]

A subsidy is not the only instrument for encouraging competition. For this reason an analysis of the optimal auction is useful: it may suggest other instruments that are more effective. We find that the optimal mechanism rules out subsidies that are type dependent. In particular, we show that loosening the budget constraints with a lump-sum transfer does not improve revenue. Rather, as we described above, the optimal mechanism favors bidders with small budgets with a higher probability of winning, by distorting the allocation rule in their favor.

Given our symmetry assumption this no-subsidy result may not be surprising. However, our result applies also to the case when bidders valuations are private but budgets common knowledge and non-identical. In this case, one can distinguish between bidders.[5]

The method of analysis yields another insight regarding the design of auctions in such settings. Where prior work suggested there may be gains to subsidizing low budget bidders, our analysis shows that the auctioneer would decline to subsidize bidders if he was running the optimal auction. Thus, arguments in favor of subsidies depend on the analysis of specific (i.e. sub-optimal) auction mechanisms.

1.3 RELATED LITERATURE

REVENUE RANKING 'STANDARD' AUCTIONS   Theoretical investigations of auctions with budget constraints have mainly been confined to analyzing 'standard' auction formats when bidders are financially constrained. Che and Gale [8], for example, consider the revenue ranking of first price, second price and all pay auctions under financial constraints. Benoit and Krishna [4] look into the effects of budget constraints in multi-good auctions, and they compare sequential to simultaneous auctions. Brusco and Lopomo [7] study strategic demand reduction in simultaneous ascending auctions and show that inefficiencies can emerge even if the probability of bidders having budget constraints is arbitrarily small. This summary is by no means complete and for illustrative purposes only.

OPTIMAL DESIGN   Research focused on optimal design is more limited. Laffont and Robert [14] as well as Maskin[18] offer an incomplete analysis of the case when valuations are private information but budgets are common knowledge and identical. Malakhov and Vohra [15] consider the case when one bidder has a known budget constraint and the other does not.

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[4]In the FCC spectrum auctions, Ayres and Cramton [3] argued that subsidizing women and minority bidders actually increased revenues since it induced other bidders to bid more aggressively. Their argument was based on the assumption that minority bidders would typically assign lower valuations to the asset than large bidders. In a procurement context, Rothkopf et al [22] find that subsidizing inefficient competitors can be desirable. Zheng [24] studies a stylized setting where bidders are subject to a 'soft' budget constraint and shows that if the auctioneer in this setting has access to cheaper funds, he may wish to subsidize some bidders.

[5]We still require that buyers' valuations be ex-ante symmetric, i.e. i.i.d. draws from the same distribution.

Che and Gale [9] compute the revenue maximizing pricing scheme when there is a single buyer whose budget constraint and valuation are both his private information.[6] Borgs et al [6] study a multi-unit auction and design an auction that maximizes worst case revenue when the number of bidders is large. Nisan et al [10] show in a closely related setting that no dominant strategy incentive compatible auction can be Pareto-efficient when bidders are budget constrained. None of these papers considers the problem of design when budget *and* valuation are private information.

MULTI-DIMENSIONAL MECHANISM DESIGN   Ostensibly this paper is a contribution to the literature on mechanism design when agents' types are multidimensional. However, the difficulties associated with multidimensional types (see for example Rochet and Choné [20]) do not arise because the extra dimension of private information does not influence valuations.

The difficulty here is that budget constraints render the associated incentive compatibility constraints non-differentiable, despite the standard assumption of quasi-linear utility. Therefore the standard first-order techniques have no bite in this setting. We skirt this difficulty by considering a model of discrete types, i.e there are only a finite (if large) number of possible valuations and budgets.[7] This makes the problem of optimal design amenable to the use of tools from linear programming, which is less involved than its continuum of types counterpart. Regardless, a discussion characterizing the optimal mechanism in the continuum of types case by considering successively finer discrete type spaces is provided in Section 6.

1.4   ORGANIZATION OF THIS PAPER

In Section 2 we describe the model. In Section 3 we examine the special case when all bidders have the same common knowledge budget constraint. This helps build intuition for the more involved private information case. In Section 4 we examine the case when bidders' budgets are private information. In Section 5 we discuss the (im)-possibility of profitably subsidizing bidders. Section 6 discusses the characterization of the optimal auction in the case when bidder types are drawn from the continuum.

2   A DISCRETE FORMULATION

2.1   THE ENVIRONMENT

There are N risk neutral bidders interested in a single indivisible good.

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[6]Their definition of a financial constraint is akin to the soft constraints described earlier.

[7]Readers with long memories will recall that the 'original' optimal auction paper by Harris and Raviv [12] also assumed discrete types.

SPACE OF TYPES   Each has a private valuation for the good v in $V = \{\epsilon, 2\epsilon, \dots, m\epsilon\}$. [8] For notational convenience we take $\epsilon = 1$. Further, each bidder has a privately known budget constraint b in $B = \{b_1, b_2, \dots, b_k\}$, wlog $b_1 < b_2 < \dots < b_k$. The type of a bidder is a 2-tuple consisting of his valuation and his budget $t = (v, b)$; and the space of types is $T = V \times B$.

We assume that bidders' types are i.i.d. draws from a commonly known distribution π over T. We require that π satisfy a generalization of the monotone hazard rate condition. Define $f_b(v) = \pi(v|b) > 0$, i.e. the probability a bidder has valuation v conditional on her budget being b. Further, define $F_b(v) = \sum_1^v f_b(v)$. We require that:

$$(v, b) \geq (v', b') \implies \frac{1 - F_b(v)}{f_b(v)} \leq \frac{1 - F_{b'}(v')}{f_{b'}(v')}$$

For notational simplicity only we assume that the valuation and budget components of a bidder's type are independent, and that all budgets are equally likely:[9]

$$\mathbb{P}\big(t = (v, b)\big) = \pi(t) = \frac{1}{k} f(v). \tag{1}$$

BUYER PREFERENCES   An agent of type $t = (v, b)$ who is given the good with probability a and asked to make a payment p derives utility:

$$u(a, p|(v, b)) = \begin{cases} va - p & \text{if } p \leq b, \\ -\infty & \text{if } p > b. \end{cases}$$

In other words an agent has a standard quasi-linear utility up to his budget constraint, but cannot pay more than his budget constraint under any circumstances.

2.2   SELLER'S PROBLEM

By the Revelation Principle, we confine ourselves without loss of generality to direct revelation mechanisms. The seller must specify an allocation rule and a payment rule. The former determines how the good is to be allocated as a function of the profile of reported types and the latter the payments each agent must make as a function of the reported types. We denote the implied interim expected allocation and payment for a bidder of type t as $a(t)$ and $p(t)$ respectively.

To ensure participation of all agents we require interim individual rationality:

$$\forall t \in T, \, t = (v, b) : \quad va(t) - p(t) \geq 0. \tag{2}$$

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[8]The assumption that valuations are equally spaced is for economy of notation only.
[9]It will be clear from the proofs that these assumptions are not necessary.

The budget constraint and individual rationality require that no type's ex-post payments exceed their budget.

$$\forall t \in T,\ t = (v, b): \quad p(t) \leq b. \tag{3}$$

The budget constraint as imposed is an interim budget constraint, i.e. *expected* payments cannot exceed the budget. Since our solution concept is Bayes-Nash, we impose only interim individual rationality and incentive compatibility constraints. Therefore if expected payments do not exceed the budget, then there exists a profile-by-profile payment rule such that these payments never exceed the budget *ex-post*. For example, the 'all-pay' payment rule where type t pays $p(t)$ regardless of other agents' reports has the desired properties.

To ensure that agents truthfully report their types we require that Bayesian incentive compatibility hold. However, due to the budget constraint, the incentive constraints will only require that a type $t = (v, b)$ has no incentive to misreport as types t' such that $p(t') \leq b$. We can write this as:

$$\forall t, t' \in T,\ t = (v, b): \quad va(t) - p(t) \geq \chi\{p(t') \leq b\}\left(va(t') - p(t')\right), \tag{4}$$

where χ is the characteristic function. Note that the presence of this characteristic function renders the incentive compatibility constraints non-differentiable, and thus standard first order conditions do not apply.

FEASIBLE INTERIM ALLOCATION RULES   A key prior result we use in this paper is from Border [5]. Border provides a set of linear inequalities which characterize the space of feasible interim allocation probabilities given the distribution over types and the number of buyers. In other words, they characterize which interim allocation probabilities can be achieved by some feasible profile-by-profile allocation rule. These inequalities simplify our problem significantly, since we now search over the (lower dimensional) space of interim allocation probabilities, rather than concerning ourselves with the allocation rule profile by profile. The Border inequalities state that a set of interim allocation probabilities $\{a(t)\}_{t \in T}$ is feasible if and only if the $a(t)$'s are non-negative:

$$\forall t \in T : a(t) \geq 0, \tag{5}$$

and:

$$\forall T' \subseteq T : \sum_{t \in T'} \pi(t) a(t) \leq \frac{1 - \left(\sum_{t \notin T'} \pi(t)\right)^N}{N}. \tag{6}$$

The left hand side of (6) is the expected probability the good is allocated to an agent with a type in T', which must be less than the probability that at least one agent has a type in T'.

EQUIVALENT OPTIMIZATION PROGRAM   Therefore, the problem of finding the revenue maximizing auction can be written as:

$$\max_{\{a(t),p(t)\}_{t\in T}} \sum_t \pi(t)p(t) \qquad \text{(RevOpt)}$$

$$\text{Subject to: } (2-6).$$

Similarly, the problem of finding the constrained efficient auction can be written as:

$$\max_{\{a(t),p(t)\}_{t\in T}} \sum_t \pi(t)va(t) \qquad \text{(ConsEff)}$$

$$\text{Subject to: } (2-6).$$

2.3   STRENGTHENING INTERIM IC AND IR

The optimal auction we identify is implementable as an all-pay auction with a modified winning rule. All-pay auctions are generally considered 'unappealing.' In our opinion, this is a critique of the interim individual rationality constraint. One can impose ex-post IR, rather than interim IR. The optimal auction with this more stringent constraint can be characterized using similar techniques to ours. The resulting implementation will be similar to a first price auction, with a modified winning rule. This is because that any ex-post IR and interim IC mechanism can be supported by an ex-post payment rule that requires the bidder to pay some fixed price p (contingent on his report) if he wins, and 0 if he loses. The resulting expected revenue from requiring the ex-post individual rationality will be less than if we required interim individual rationality.

In a similar vein one can ask what the optimal auction is if one desires dominant strategy implementability rather than Bayes-Nash (interim IC). An analogous argument shows that the optimal auction can be implemented as a second price auction, with a similarly modified winning rule. The resulting expected revenue will be lower still.

2.4   OVERVIEW OF LINEAR PROGRAMMING APPROACH

To orient the reader, we give an overview of the approach taken. First, by using a discrete type space, we were able to formulate the problem of finding the revenue maximizing auction

as a linear program. Abstracting from this , it has the following form:

$$Z = \max_x mx$$
$$\text{s.t. } Cx \leq d$$
$$Ax \leq b$$
$$x \geq 0$$

The first set of constraints, $Cx \leq d$, corresponding to (2 - 4), are 'complicated.' The second set, $Ax \leq b$, correspond to the feasibility constraints (6).

This set is 'easy' in the sense that A is an upper triangular matrix. An upper triangular matrix in a linear program is easy because the corresponding dual constraints will also have a triangular component. Further, it is well known that a triangular system of equations is easy to solve by Gaussian elimination. As a result the solution of this linear program is easy to characterize by complementary slackness.

Alternately,

$$Z(\lambda) = \max_x mx + \lambda(d - Cx)$$
$$\text{s.t. } Ax \leq b$$
$$x \geq 0$$

For each $\lambda \geq 0$, $Z(\lambda)$ is easy to compute because A is upper triangular.

By the duality theorem of linear programming,

$$Z = \min_{\lambda \geq 0} Z(\lambda).$$

Thus our task reduces to identifying the non-negative λ that minimizes $Z(\lambda)$. Now, $Z(\lambda)$ is a piecewise linear function of λ with a finite number of breakpoints. We find an indirect way to enumerate the breakpoints without explicitly listing them. In this way we compute the value Z.

In the auction context, the coefficients of the x variables in the function $mx + \lambda(d - Ax)$, i.e. $(m - \lambda A)$, have an interpretation as 'virtual values'.

3   THE COMMON KNOWLEDGE BUDGET CASE

In this section, we analyze the case where all bidders have the same, commonly known budget. This helps us build intuition and familiarity with the proof methods used subsequently to analyze the general case. We examine the case of revenue maximization.

Since all bidders have the same budget constraint b, a bidder's type is just her valuation.

Further, we can drop the characteristic function in the IC constraints since, by individual rationality, all types must have a payment of at most b. Given these simplifications, problem(RevOpt) becomes:

$$\max_{\{a(v),p(v)\}_{v \in V}} \sum f(v)p(v) \qquad \text{(RevOptCK)}$$

s.t.

$$
\begin{array}{ll}
\forall v & p(v) \leq b \\
\forall v, v' & va(v) - p(v) \geq va(v') - p(v') \\
\forall v & va(v) - p(v) \geq 0 \\
\forall V' \subseteq V & \sum_{v \in V'} f(v)a(v) \leq \frac{1 - \left(\sum_{v \notin V'} f(v) \right)^N}{N} \\
\forall v & a(v) \geq 0
\end{array}
$$

First, add a 'dummy' type 0 to the space of types, and define $a(0) = p(0) = 0$. We can subsume the IR constraint, by requiring IC over the extended type space $V' = V \bigcup \{0\}$. Standard arguments imply that an allocation rule $a(\cdot)$ can be part of an incentive compatible mechanism if and only if $a(v)$ is non-decreasing in v. Further, the payment rule that maximizes revenue associated with this allocation rule is:

$$p(v) = va(v) - \sum_{1}^{v-1} a(v'). \qquad (7)$$

Note the absence of a constant. Implicitly, we have set this to zero, ruling out subsidies in the form of lump-sum transfers. In effect therefore we will be computing the optimal subsidy-free mechanism.[10] In Section 5 we show that any lump-sum transfer will necessarily reduce expected revenue.

Substituting (7) back into (RevOptCK), we can rewrite it as:

$$\max_{\{a(v)\}_{v \in V}} \quad \sum f(v)\nu(v)a(v) \qquad (8)$$

s.t.

$$\forall v \qquad va(v) - \sum_{v'=1}^{v-1} a(v') \leq b \qquad (9)$$

$$\forall V' \subseteq V \qquad \sum_{v \in V'} f(v)a(v) \leq \frac{1 - \left(\sum_{v \notin V'} f(v) \right)^N}{N} \qquad (10)$$

$$
\begin{array}{ll}
\forall v & a(v) - a(v+1) \leq 0 \\
\forall v & a(v) \geq 0
\end{array}
$$

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[10]Note that incentive compatibility rules out type dependent subsidies.

Here $\nu(v) = v - \frac{1-F(v)}{f(v)}$ is type v's 'virtual valuation', as in Myerson [19].

Monotonicity of the allocation rule makes many of the constraints in (10) redundant.

PROPOSITION 1 (BORDER) *Let $a : T \to [0,1]$ be the interim probability of allocation for a type space T. For each $\alpha \in [0,1]$, set*

$$E_\alpha = \{t : a(t) \geq \alpha\}.$$

Then a is feasible if and only if for each E_α:

$$\sum_{t \in E_\alpha} a(t) f(t) \leq \frac{1 - \left(\sum_{T - E_\alpha} f(t)\right)^N}{N}. \tag{11}$$

Note that this makes the problem by decreasing the number of constraints we need consider exponentially. The original Border constraints would require constraints that are exponential in the number of types. By Proposition 1, the number of constraints is at most the number types. Since IC constraints normally imply some sort of monotonicity of the allocation rule, the sets E_α are easy to characterize, and therefore these constraints are easy to use. For instance, in this setting:

COROLLARY 1 *If $a(\cdot)$ is monotonic, it is feasible if and only if, $\forall v \in V$:*

$$\sum_v^m f(v')a(v') \leq \frac{1 - F^N(v-1)}{N} \tag{12}$$

For notational convenience define $c_v = \frac{1-F^N(v-1)}{N}$ for each v.

The constraint matrix in (12) is upper triangular, which makes determining the structure of an optimal solution easy. In addition, a straightforward calculation shows that if $a(t)$ is the efficient allocation then all of the inequalities in (12) bind.

By inspection, $a(v+1) > a(v) \implies p(v+1) > p(v); a(v+1) = a(v) \implies p(v+1) = p(v)$. Therefore, if the budget constraint (9) binds for some valuation $\bar{v}$, it must bind for all valuations $v \geq \bar{v}$. If the budget constraint does not bind in the optimal solution, the solution must be the same as Myerson's. Hence we assume the budget constraint binds in the optimal solution. We summarize this in the following observation.

OBSERVATION 1 If a^* is an optimal solution to (RevOptCK), the budget constraint must bind for some types $\{\bar{v}, \bar{v}+1, \ldots, m\}$. Further,

$$a^*(v) = a^*(\bar{v}) \qquad \forall v \geq \bar{v}.$$

Suppose the lowest type for which the budget constraint binds in the optimal solution a^* is $\bar{v}$. Substituting into program (8); and dropping the redundant Border constraints by Lemma 1, we conclude that a^* must be a solution to problem (RevOptCK):

$$\max_{\{a(v)\}_{v \in V}} \left(\sum_1^{\bar{v}-1} f(v)\nu(v)a(v) \right) + (1 - F(\bar{v}-1))\bar{v}a(\bar{v})$$

s.t.

$$-\sum_1^{\bar{v}-1} a(v') + \bar{v}a(\bar{v}) = b$$

$$\forall v \leq \bar{v} \qquad \sum_v^{\bar{v}-1} f(v')a(v') + (1 - F(\bar{v}-1))a(\bar{v}) \leq c_v$$

$$\forall v \qquad a(v) - a(v+1) \leq 0$$

$$\forall v \qquad a(v) \geq 0$$

Denote the dual variable for the budget constraint by η, the dual variable for the Border constraint corresponding to type v by β_v and the dual variable for the monotonicity constraint corresponding to type v by μ_v. The dual program is:[11]

$$\min_{\eta, \{\beta_v\}_1^{\bar{v}}, \{\mu_v\}_1^{\bar{v}-1}} b\eta + \sum_1^{\bar{v}} c_v\beta_v \qquad \text{(DOPT)}$$

$$\bar{v}\eta + (1 - F(\bar{v}-1))\sum_1^{\bar{v}} \beta_v - \mu_{\bar{v}-1} \geq (1 - F(\bar{v}-1))\bar{v} \qquad (a(\bar{v}))$$

$$\forall v \leq (\bar{v}-1) \quad -\eta + f(v)\sum_1^v \beta_{v'} + \mu_v - \mu_{v-1} \geq f(v)\nu(v) \qquad (a(v))$$

$$\forall v \qquad \beta_v, \mu_v \geq 0$$

Let $\underline{v}$ be the lowest valuation for which $a^*(\underline{v}) > 0$. Complementary slackness implies that:

$$\bar{v}\eta + (1 - F(\bar{v}-1))\sum_1^{\bar{v}} \beta_v - \mu_{\bar{v}-1} = (1 - F(\bar{v}-1))\bar{v} \qquad (13)$$

$$\underline{v} \leq v \leq (\bar{v}-1) \qquad -\eta + f(v)\sum_1^v \beta_{v'} + \mu_v - \mu_{v-1} = f(v)\nu(v) \qquad (14)$$

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[11]The primal variable associated with each dual constraint is displayed in brackets next to the constraint.

Re-writing (13,14) yields:

$$\sum_{1}^{\bar{v}} \beta_v - \frac{\mu_{\bar{v}-1}}{1 - F(\bar{v} - 1)} = \bar{v} - \bar{v}\frac{\eta}{1 - F(\bar{v} - 1)},$$

$$\underline{v} \le v \le (\bar{v} - 1) \qquad \sum_{1}^{v} \beta_{v'} + \frac{\mu_v}{f(v)} - \frac{\mu_{v-1}}{f(v)} = \nu(v) + \frac{\eta}{f(v)}$$

Intuitively, these equations tell us that the 'correct' virtual valuation of a type v is $\nu(v) + \frac{\eta}{f(v)}$, where $\nu(v)$ is the Myersonian virtual valuation, and $\frac{\eta}{f(v)}$ corrects for the budget constraint: allocating to lower types reduces the payment of the high types, and hence 'relaxes' the budget constraint. As in Myerson, we require that the adjusted virtual valuation $\nu(v) + \frac{\eta}{f(v)}$ be increasing in v. A sufficient condition for this is that $f(v)$ is weakly decreasing and satisfies the monotone hazard rate condition. By analogy with Myerson, the lowest type that will be allotted is the lowest type ($\underline{v}$) whose adjusted virtual valuation is non-negative. Finally, the optimal allocation rule will be efficient between types $\bar{v} - 1$ and $\underline{v}$.

PROPOSITION 2 *Suppose $f(v)$ is weakly decreasing in v, and $f(\cdot)$ satisfies the monotone hazard rate condition, i.e. $\frac{1-F(v)}{f(v)}$ is decreasing in v. Then the solution of (RevOptCK) can be described as follows: there will exist two cutoffs $\bar{v}$ and $\underline{v}$. No valuation less than $\underline{v}$ will be allotted. All types $\bar{v}$ and above will receive the same interim allocation probability, and the budget constraint will bind for exactly those types. The allocation rule will be efficient between types $\bar{v} - 1$ and $\underline{v}$. Finally, $\underline{v}$ is the lowest type such that*

$$\nu(\underline{v}) + \frac{\eta}{f(\underline{v})} \ge 0,$$

where

$$\eta = \frac{(1 - F(\bar{v} - 1))(1 - F(\bar{v} - 2))}{\bar{v}f(\bar{v} - 1) + (1 - F(\bar{v} - 1))}.$$

If $f(v)$ is not weakly decreasing in v or does not satisfy the monotone hazard rate, the optimal solution may require pooling in the middle.

PROOF: The proof proceeds by constructing dual variables that complement the primal solution described in the statement of the proposition.

Since $a^*(v) = 0$ for $v < \underline{v}$ and $f(v) > 0$ for all v, the corresponding Border constraints (10) do not bind at optimality. Therefore $\beta_v = 0$ for all $v < \underline{v}$. Further $0 = a^*(\underline{v} - 1) < a^*(\underline{v})$ by definition of $\underline{v}$, and so, by complementary slackness, $\mu_{\underline{v}-1} = 0$. Similarly, since $\bar{v}$ is the lowest type for which the budget constraint binds, $a^*(\bar{v}) > a^*(\bar{v} - 1)$, implying that $\mu_{\bar{v}-1} = 0$.

Subtracting the dual constraints corresponding to types $\bar{v}$ and $\bar{v} - 1$ and using the fact

that $\mu_{\underline{v}-1} = 0$, we have:[12]

$$\beta_{\bar{v}} + \frac{\mu_{\bar{v}-2}}{f(\bar{v}-1)} = \bar{v} - \bar{v}\frac{\eta}{1 - F(\bar{v}-1)} - \nu(\bar{v}-1) - \frac{\eta}{f(\bar{v}-1)} \tag{15}$$

Subtracting the dual constraints corresponding to v and $v - 1$, where $\underline{v} + 1 \leq v \leq \bar{v} - 1$, we have:

$$\beta_v + \frac{\mu_v}{f(v)} - \frac{\mu_{v-1}}{f(v)} - \frac{\mu_{v-1}}{f(v-1)} + \frac{\mu_{v-2}}{f(v-1)} = \nu(v) + \frac{\eta}{f(v)} - \nu(v-1) - \frac{\eta}{f(v-1)} \tag{16}$$

Finally, the dual constraint corresponding to type $\underline{v}$ reduces to:

$$\beta_{\underline{v}} + \frac{\mu_{\underline{v}}}{f(\underline{v})} = \nu(\underline{v}) + \frac{\eta}{f(\underline{v})} \tag{17}$$

It suffices to identify a non-negative solution to the system (15-17) such that $\beta_v = 0$ for all $v < \underline{v}$ and $\mu_{\underline{v}-1} = 0$.

Consider the following solution.

$$\beta_{\bar{v}} = 0$$

$$\underline{v} + 1 \leq v \leq \bar{v} - 1 \quad \beta_v = \nu(v) - \nu(v-1) + \eta\Big(\frac{1}{f(v)} - \frac{1}{f(v-1)}\Big)$$

$$\underline{v} \leq v \leq \bar{v} - 1 \quad \mu_v = 0$$

$$\eta = \frac{(1 - F(\bar{v}-1))(1 - F(\bar{v}-2))}{\bar{v}f(\bar{v}-1) + (1 - F(\bar{v}-1))}$$

Direct computation verifies that the given solution satisfies (15-17). In fact it is the unique solution to (15-17) with all μ's equal to zero. All variables are non-negative. In particular, β_v for $\underline{v} + 1 \leq v \leq \bar{v} - 1$ is positive. This is because $f(\cdot)$ satisfies the monotone hazard rate and (weakly) decreasing density conditions, for any v, $\nu(v) - \nu(v-1) + \eta\big(\frac{1}{f(v)} - \frac{1}{f(v-1)}\big) > 0$. Furthermore, it complements the primal solution described in the statement of the proposition. This concludes the case where our regularity condition on the distribution of types (monotone hazard rate, decreasing density) are met.

Now suppose our sufficient condition is violated, i.e. $\nu(v) - \nu(v-1) + \eta\big(\frac{1}{f(v)} - \frac{1}{f(v-1)}\big) < 0$ for some v. The dual solution identified above will be infeasible since $\beta_v < 0$. More generally, there can be no dual solution that satisfies (15-17) with all $\mu_v = 0$. Hence, there must be at least one v between $\underline{v}$ and $\bar{v} - 1$ such that $\mu_v > 0$. This implies, by complementary slackness, that the corresponding primal constraint, $a(v) - a(v+1) \leq 0$ binds at optimality, implying pooling. $\qquad\square$

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[12]This step is where the upper triangular constraint matrix is helpful.

In fact one can further restrict the set of optimal dual solutions.

LEMMA 1 *In any solution to the primal problem (8), at most one of the Border constraint (10) corresponding to type v, and the monotonicity constraint corresponding to type $v-1$ can bind. Further, by complementary slackness:*

$$\forall v: \qquad \beta_v \mu_{v-1} = 0 \qquad (18)$$

PROOF: See Appendix B. □

The solution to the system of equations (15- 17), (18) constitutes the optimal dual solution. It is easily seen that this solution is unique- therefore even in the case where ironing is required, there is a unique solution. Further the μ's in the solution are the 'ironing' multipliers a la Myerson.

A comment on implementation of the revenue maximizing auction is in order. First consider what one can call the regular case- i.e. that f satisfies both the monotone hazard rate and weakly decreasing density conditions. In this case, the implementation will be as described by Laffont and Robert: the auction will be implemented as an all pay auction, with an appropriately chosen reserve price.[13] If f is not regular- i.e. it violates either the monotone hazard rate or decreasing density conditions, then additional pooling may be required. In particular the all-pay auction described above may not be optimal. The pooling identified by the optimal auction must then be implemented by modifying the winning rule in the all-pay auction.

We are also in a position to describe the constrained efficient auction for this setting. The proof is very similar to that of Proposition 2, and therefore omitted.

PROPOSITION 3 *Suppose $f(v)$ is weakly decreasing in v. Then the constrained efficient auction in this setting can be described as follows: there will exist a cutoff $\bar{v}$. All types $\bar{v}$ and above will receive the same interim allocation probability, and the budget constraint will bind for exactly those types. The allocation rule will be efficient for types below $\bar{v}-1$. If the sufficient conditions are not met, the optimal solution may require pooling in the middle.*

In other words, if the distribution of types has decreasing density, the constrained efficient mechanism is an all-pay auction (with no reserve).

4 THE GENERAL CASE

Recall the original program (RevOpt). For economy of notation, we deal with the case where a bidder's valuation and budget are determined independently, and all budgets are equally

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[13]The reserve price p will be the payment of type $\underline{v}$ in the auction described above.

likely, i.e. for any type $t = (v, b)$,

$$\pi(v, b) = \frac{1}{k}f(v).$$

$$\max_{a,p} \quad \sum_{j=1}^{k} \sum_{v=1}^{m} \frac{1}{k} f(v) p(v, b_j)$$

$\forall (v, b) \in T$	$va(v, b) - p(v, b) \geq 0$
$\forall (v, b) \in T$	$p(v, b) \leq b$
$\forall (v, b), (v', b') \in T$	$va(v, b) - p(v, b) \geq \chi\{p(v', b') \leq b\}[va(v', b') - p(v', b')]$
$\forall T' \subseteq T$	$\sum_{t \in T'} \pi(t) a(t) \leq \dfrac{1 - (\sum_{t \notin T'} \pi(t))^N}{N}$
$\forall t \in T$	$a(t) \geq 0$

The incentive compatibility constraints can be separated into 3 categories:

1. Misreport of value only:

$$va(v, b) - p(v, b) \geq va(v', b) - p(v', b). \tag{19}$$

2. Misreport of budget only:

$$va(v, b) - p(v, b) \geq \chi\{p(v, b') \leq b\}[va(v, b') - p(v, b')]. \tag{20}$$

3. Misreport of both:

$$va(v, b) - p(v, b) \geq \chi\{p(v', b') \leq b\}[va(v', b') - p(v', b')]. \tag{21}$$

Standard arguments imply that the IC constraints corresponding to a misreport of value, (19), can be satisfied by some pricing rule if and only if $v \geq v'$ implies that $a(v, b) \geq a(v', b)$. Incentive compatibility and individual rationality imply

$$p(v, b) \leq va(v, b) - \sum_{1}^{v-1} a(v', b).$$

The difficulty stems from the IC constraints relating to misreport of budget, (20) and (21). In particular, we need (further) constraints on the allocation rule such that there exists an incentive compatible pricing rule. The following lemmata identify the space of interim

allocations such that each type's payment is the maximum possible, i.e.

$$p(v, b) = va(v, b) - \sum_{1}^{v-1} a(v', b). \tag{22}$$

LEMMA 2 *For any budget b, an allocation rule a is incentive compatible only if:*

$$p(v, b) = b \implies a(v', b) = a(v, b) \quad \forall v' \geq v. \tag{23}$$

PROOF: It is easy to see that for any v, b, incentive compatibility implies that:

$$a(v + 1, b) \geq a(v, b) \implies p(v + 1, b) \geq p(v, b).$$

Further, by observation, (22) implies that:

$$\begin{aligned} a(v + 1, b) > a(v, b) &\implies p(v + 1, b) > p(v, b), \\ a(v + 1, b) = a(v, b) &\implies p(v + 1, b) = p(v, b). \end{aligned}$$

Equation (23) follows. $\qquad\square$

LEMMA 3 *Fix an allocation rule a such that a is incentive compatible and individually rational with pricing rule (22). Fix two budgets $b' > b$. Let $\underline{v}_b$ be the largest v such that $p(\underline{v}_b, b') \leq b$. Then,*

$$\forall v \leq \underline{v}_b \quad a(v, b') = a(v, b).$$

Further, $a(\underline{v}_b + 1, b') > a(m, b)$.

PROOF: By assumption, $p(v, b') \leq b$ for any $v \leq \underline{v}_b$. By individual rationality, $p(v, b) \leq b$ for any v. Therefore the incentive compatibility constraints (20) corresponding to type (v, b) misreporting as (v, b') and type (v, b') misreporting as (v, b) for any $v \leq v_b$ imply that:

$$\forall v \leq \underline{v}_b \qquad va(v, b) - p(v, b) = va(v, b') - p(v, b')$$

$$\implies \forall v \leq \underline{v}_b \qquad \sum_{1}^{v-1} a(v', b) = \sum_{1}^{v-1} a(v', b')$$

$$\implies \forall v \leq (\underline{v}_b - 1) \quad a(v, b) = a(v, b').$$

To see that $a(\underline{v}_b, b) = a(\underline{v}_b, b')$, first consider the IC constraint corresponding to type $(\underline{v}_b + 1, b)$ misreporting as type $(\underline{v}_b, b')$. By assumption $p(\underline{v}_b, b') \leq b$, therefore we can drop the

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characteristic function and write the IC constraint as:

$$(\underline{v}_b + 1)a(\underline{v}_b + 1, b) - p(\underline{v}_b + 1, b) \geq (\underline{v}_b + 1)a(\underline{v}_b, b') - p(\underline{v}_b, b')$$

$$\implies \sum_1^{\underline{v}_b} a(v, b) \geq \sum_1^{\underline{v}_b} a(v, b')$$

$$\implies a(\underline{v}_b, b) \geq a(\underline{v}_b, b').$$

The last inequality follows since $\sum_1^{\underline{v}_b - 1} a(v, b) = \sum_1^{\underline{v}_b - 1} a(v, b')$. Similarly one can show that $a(\underline{v}_b, b) \leq a(\underline{v}_b, b')$.

Finally, we need to show that $a(\underline{v}_b + 1, b') > a(m, b)$. By assumption,

$$p(\underline{v}_b + 1, b') > b \geq p(m, b)$$

$$\implies (\underline{v}_b + 1)a(\underline{v}_b + 1, b') - \sum_1^{\underline{v}_b} a(v, b') > ma(m, b') - \sum_1^{m-1} a(v, b)$$

$$\implies (\underline{v}_b + 1)a(\underline{v}_b + 1, b') > ma(m, b') - \sum_{\underline{v}_b + 1}^{m-1} a(v, b)$$

$$> (\underline{v}_b + 1)a(m, b).$$

The last inequality follows since for any v, $a(v + 1, b) \geq a(v, b)$. $\qquad\square$

Lemma 2 shows that for each b_i there is a cutoff $\bar{v}_i \in V$, the lowest valuation such that $p(\bar{v}_i, b_i) = b_i$, and $a(v, b_i) = a(\bar{v}_i, b_i)$ for all $v \geq \bar{v}_i$. Lemma 3 shows that for each b_i there exists a cutoff $\underline{v}_i$, the highest valuation such that $p(\underline{v}_i, b_{i+1}) \leq b_i$; and that $a(v, b_i) = a(v, b_{i+1})$ for all $v \leq \underline{v}_i$. We summarize this in the following definition:

DEFINITION 1 *Given an allocation rule a that is incentive compatible and individually rational with pricing rule (22), define cutoffs:*

$$\bar{v}_i = \arg\min\{v : p(v, b_i) = b_i\} \qquad \forall i \leq k,$$
$$\underline{v}_i = \arg\max\{v : p(v, b_{i+1}) \leq b_i\} \qquad \forall i \leq k - 1.$$

Note that $\underline{v}_i < \bar{v}_{i+1}$. Further, define :

$$\bar{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\},$$
$$\underline{V} = \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{k-1}\}.$$

A word on our choice of notation is in order here. $\underline{v}_i$ is the highest valuation with budget greater than b_i that pays at most b_i. On the other hand $\bar{v}_i$ is the lowest valuation with budget
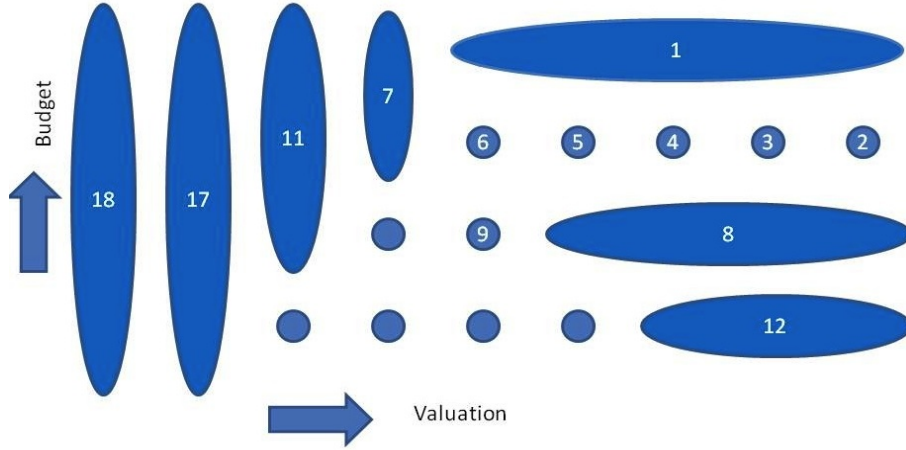
FIGURE 2: AN IC ALLOCATION RULE

b_i for whom the budget constraint binds. In particular, we do not require that $\underline{v}_i \leq \bar{v}_i$ (nor will this generally be true).

Lemmas 2 and 3 imply:

OBSERVATION 2 An allocation rule $a : T \rightarrow [0,1]$ is consistent with cutoffs $\bar{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ and $\underline{V} = \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{k-1}\}$, where $\underline{v}_i \leq \bar{v}_{i+1}$ for all i, and pricing rule (22), incentive compatible and individually rational if and only if:

$$\forall v, b \quad a(v, b) \leq a(v+1, b) \tag{24}$$
$$\forall i \quad a(\bar{v}_i - 1, b_i) < a(\bar{v}_i, b_i) \tag{25}$$
$$\forall i \quad p(\bar{v}_i, b_i) = b_i \tag{26}$$
$$\forall i, v \geq \bar{v}_i \quad a(v, b_i) = a(\bar{v}_i, b_i) \tag{27}$$
$$\forall i, v \leq \underline{v}_i \quad a(v, b_i) = a(v, b_{i+1}) \tag{28}$$
$$\forall i \quad p(\underline{v}_i + 1, b_i + 1) > b_i \tag{29}$$
$$\forall i \quad a(\underline{v}_i + 1, b_i + 1) > a(m, b_i) \tag{30}$$

Figure 2 depicts an incentive compatible allocation rule for a type space with 10 possible valuations and 4 possible budgets. The blobs represent types that are pooled. The numbering reflects decreasing allocations: i.e. the blob numbered 1 has the highest interim probability of getting the good, 2 has the second highest and so on.

Given a collection of cut-offs we describe how to find an allocation rule consistent with those cutoffs that maximizes revenue. By Observation 2 we can drop the individual ratio-nality, budget, and incentive compatibility constraints in (RevOpt) and substitute instead

([24](#)-[30](#)). Therefore, we have:

$$\max_a \quad \sum_{i=1}^{k}\sum_{v=1}^{m} \frac{f(v)}{k}\nu(v)a(v,b_i) \qquad\qquad \text{(REVOPT)}$$

$$\text{s.t.} \quad (24-30), (5), (6).$$

To ensure a well defined program the strict inequalities in [(25)](#) and [(30)](#) have to be relaxed to a weak inequality. If for a given set of cutoffs, the optimal solution to ([REVOPT](#)) binds at inequality [(25)](#) or [(30)](#), we know that the set of cutoffs being considered cannot be feasible. Hence we can restrict attention to cut-offs where (the weak version of) the inequalities do not bind at optimality.

Therefore, having fixed the cutoffs $\underline{V}$, by [(24)](#) and [(30)](#), most of the Border constraints are rendered redundant by Proposition [1](#). In particular consider type (v,b_i); $v \le \bar{v}_i$. Then, by Observation [2](#), $E_{(v,b_i)}$, the set of all types t such that $a(t) \ge a(v,b_i)$ is:

$$E_{(v,b_i)} = \bigcup_{j=i+1}^{k} \{(v',b_j) : v' \ge \min(v, \underline{v}_i+1,\dots,\underline{v}_{j-1}+1)\}\bigcup\{(v',b_i) : v' \ge v\}.$$

It follows from Proposition [1](#) that the relevant Border constraints to be considered are:

$$\forall i, v \le \bar{v}_i \qquad \sum_{t\in E_{(v,b_i)}} a(t)f(t) \le \frac{1-\left(\sum_{T-E_{(v,b_i)}} f(t)\right)^N}{N}. \qquad (31)$$

The next lemma further restricts the configurations of cutoffs in a revenue maximizing rule.

LEMMA 4 *Let a^* solve ([REVOPT](#)). Then the cutoffs $\bar{V}, \underline{V}$ as defined in Definition [1](#) must satisfy:*

$$\forall i \le k-1 \qquad \underline{v}_i \ge \bar{v}_i - 1.$$

SKETCH OF PROOF: Suppose instead that in some profit maximizing allocation rule a; for some i, $\underline{v}_i < \bar{v}_i - 1$. We outline how to construct a rule a' with cutoff $\underline{v}'_i = \underline{v}_i + 1$ that achieves weakly more revenue. Since $\underline{v}_i < \bar{v}_i - 1$, $a(\underline{v}_i + 1, b_{i+1}) > a(v, b_i)$ for all v. Consider decreasing $a(\underline{v}_i+1, b_{i+1})$ by δ and increasing each $a(v,b_i), v \ge \bar{v}_i$ by δ'. If $\delta f(\underline{v}_i + 1) = \delta'(1-F(\underline{v}_i))$, we will maintain feasibility with respect to the Border conditions. Pick δ such that $a(\underline{v}_i+1, b_i) = a(\underline{v}_i+1, b_{i+1}) - \delta$. This modified allocation rule corresponds to the cutoff $\underline{v}'_i = \underline{v}_i+1$. The net change in revenue is $(\bar{v}_i - \nu(\underline{v}_i+1))\delta\frac{f(\underline{v}_i+1)}{k}$, which is clearly non-negative. However this simple procedure will violate the budget constraints.

Appendix [B.2](#) provides a formal proof, i.e. it shows that there exists a similar revenue increasing construction, where the resulting allocation is feasible in ([REVOPT](#)). □

With this added restriction on cutoffs; the set of incentive compatible and individually rational rules are summarized in Observation 3. Since $\underline{v}_i \geq \bar{v}_i - 1$, (29) and (30) are satisfied automatically; $\underline{v}_i \equiv \arg\min\{v : a(v + 1, b_{i+1}) > a(\bar{v}_i, b_i)\}$.

Lemma 4 further implies that the budget constraint corresponding to budget b_i can bind in an optimal solution only if the budget constraints corresponding to each $b_j < b_i$ bind. Therefore, in any optimal solution, there must be a largest budget b_i such that the budget constraints corresponding to $b \leq b_i$ bind, and the budget constraints corresponding to $b > b_i$ are slack. For notational simplicity we assume that in the optimal solution, all budget constraints bind.

OBSERVATION 3 An allocation rule $a : T \to [0, 1]$, consistent with the cutoffs $\bar{V} = \{\bar{v}_1 \leq \bar{v}_2 \leq \ldots \leq \bar{v}_k\}$ and pricing rule (22) is incentive compatible and individually rational if and only if there exist, $x : V \to [0, 1]$ and $y : \bar{V} \to [0, 1]$ such that:

$$\forall i \leq k, v \geq \bar{v}_i \qquad a(v, b_i) = y(\bar{v}_i), \tag{32}$$

$$\forall i \leq k, \bar{v}_{i-1} \leq v \leq \bar{v}_i - 1, j \geq i \qquad a(v, b_j) = x(v), \tag{33}$$

$$\forall i \leq k \qquad \bar{v}_i y(\bar{v}_i) - \sum_{1}^{\bar{v}_i - 1} x(v) = b_i, \tag{34}$$

$$\forall v \qquad x(v) \leq x(v + 1), \tag{35}$$

$$\forall i \leq k \qquad x(\bar{v}_i - 1) < y(\bar{v}_i), \tag{36}$$

$$\forall i < k \qquad y(\bar{v}_i) \leq x(\bar{v}_i). \tag{37}$$

Figure 3 displays an allocation rule whose cutoffs satisfy Lemma 4. As before, the blobs represent types that are pooled. The numbering reflects decreasing allocations: i.e. the blob numbered 1 has the highest interim probability of getting the good, 2 has the second highest and so on.

Substituting (32) and (33) into (31), the Border constraints to be considered are:

$\forall i \leq k, \ \bar{v}_{i-1} + 1 \leq v \leq \bar{v}_i$

$$\sum_{v'=v}^{\bar{v}_i - 1} \frac{k - i + 1}{k} f(v') x(v') + \sum_{j=i+1}^{k} \sum_{\bar{v}_{j-1}}^{\bar{v}_j - 1} \frac{k - j + 1}{k} f(v') x(v') + \sum_{j=i}^{k} \frac{(1 - F(\bar{v}_j - 1))}{k} y(\bar{v}_j) \leq c_v, \tag{38}$$

$\forall i \leq k$

$$\sum_{j=i+1}^{k} \frac{(1 - F(\bar{v}_j - 1))}{k} y(\bar{v}_j) + \sum_{j=i+1}^{k} \sum_{\bar{v}_{j-1}}^{\bar{v}_j - 1} \frac{k - j + 1}{k} f(v) x(v) \leq \frac{1 - (1 - \frac{k - i}{k}(1 - F(\bar{v}_i - 1)))^N}{N}. \tag{39}$$
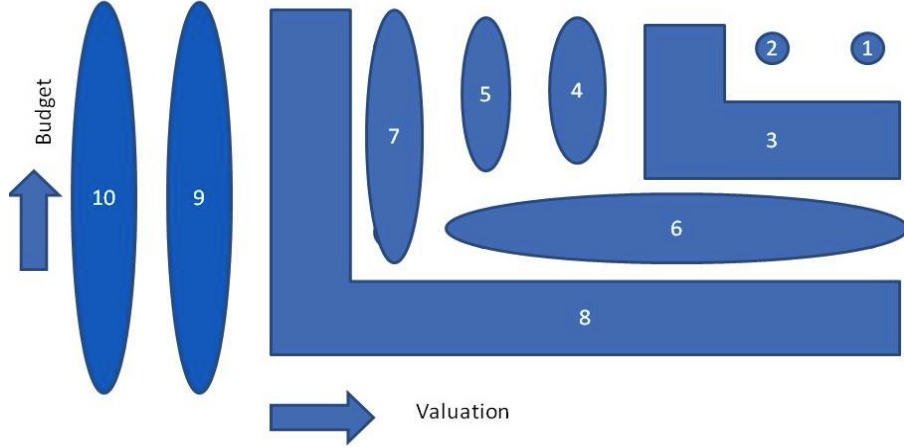
FIGURE 3: A POTENTIALLY OPTIMAL IC RULE

where the c's are the right hand side of the appropriate Border inequality (31), i.e. for $(\bar{v}_{i-1} + 1) \leq v \leq \bar{v}_i$

$$c_v = \frac{1 - (1 - \frac{k-i}{k}(1 - F(\bar{v}_i - 1)) - \frac{1}{k}(1 - F(v - 1)))^N}{N}.$$

Making the appropriate substitutions, (REVOPT) becomes:

$$\max_{x,y} \quad \sum_{j=1}^{k} \sum_{\bar{v}_{j-1}}^{\bar{v}_j - 1} \frac{k - j + 1}{k} f(v)\nu(v)x(v) + \frac{\sum_{i=1}^{k} \bar{v}_i(1 - F(\bar{v}_i - 1))y(\bar{v}_i)}{k} \quad \text{(REVOPT2)}$$

s.t. $\quad (5), (34 - 39).$

As before, we conjecture an optimal solution and verify optimality with a suitably chosen dual solution. Hence we flip to the dual and examine its properties.

Let η_i be the dual variable associated with the budget constraint (34). Since we assume constraint (36) does not bind at optimality, the corresponding dual variable will be 0, and therefore is dropped. Let μ_v be the dual variable associated with the monotonicity constraint (35), and $\bar{\mu}_{\bar{v}_i}$ the dual variable associated with the constraint (37). Denote by β_v the dual variable associated with (38), and $\bar{\beta}_{\bar{v}_i}$, the dual variable associated with (39). The dual

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program is:

$$\min_{\eta,\mu,\beta} \quad \sum_{i=1}^{k} b_i \eta_i + \sum_{v=1}^{\bar{v}_k} c_v \beta_v + \sum_{i=1}^{k} \bar{c}_{\bar{v}_i} \bar{\beta}_{\bar{v}_i} \qquad \text{(DOPT2)}$$

$$\forall i \leq k, (\bar{v}_{i-1} + 1) \leq v \leq (\bar{v}_i - 1)$$

$$-\sum_{j=i}^{k} \eta_j + \frac{k-i+1}{k} f(v) \left(\sum_{1}^{v} \beta_v + \sum_{j=1}^{i-1} \bar{\beta}_{\bar{v}_j} \right) + \mu_v - \mu_{v-1} \geq \frac{k-i+1}{k} f(v)\nu(v), \quad (40)$$

$$\forall i \leq k \quad -\sum_{j=i+1}^{k} \eta_j + \frac{k-i}{k} f(v) \left(\sum_{1}^{\bar{v}_i} \beta_v + \sum_{j=1}^{i} \bar{\beta}_{\bar{v}_j} \right) + \mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i} \geq \frac{k-i}{k} f(\bar{v}_i)\nu(\bar{v}_i), \qquad (41)$$

$$\forall i \leq k \quad \bar{v}_i \eta_i + \frac{(1 - F(\bar{v}_i - 1))}{k} \left(\left(\sum_{1}^{\bar{v}_i-1} \beta_v + \sum_{j=1}^{i} \bar{\beta}_{\bar{v}_j} \right) + \bar{\mu}_{\bar{v}_i} \geq \frac{(1 - F(\bar{v}_i - 1))}{k} \bar{v}_i, \qquad (42)$$

$$\eta, \beta, \mu \geq 0.$$

Here, (40) is the dual inequality corresponding to primal variable $x(v)$ where $(\bar{v}_{i-1} + 1) \leq v \leq (\bar{v}_i - 1)$, (41) the dual inequality corresponding to $x(\bar{v}_i)$ and (42) the dual inequality corresponding to $y(\bar{v}_i)$. Fix an optimal primal solution(x^*, y^*) and let $\underline{v}$ be the lowest valuation which gets allotted in that solution. Therefore any type (v, b) where $v \geq \underline{v}$ gets allotted. It is easy to see that $\underline{v} \leq \bar{v}_1$. Complementary slackness implies that the inequalities (40) bind for all $v \geq \underline{v}$, as do (41, 42) for all i. Rewriting (40-42) as in the common knowledge case:

$$\left(\sum_{1}^{v} \beta_v + \sum_{j=1}^{i-1} \bar{\beta}_{\bar{v}_j} \right) + \frac{k(\mu_v - \mu_{v-1})}{(k-i+1)f(v)} = \nu(v) + \frac{k \sum_{j=i}^{k} \eta_j}{(k-i+1)f(v)}, \qquad (43)$$

$$\left(\sum_{1}^{\bar{v}_i} \beta_v + \sum_{j=1}^{i} \bar{\beta}_{\bar{v}_j} \right) + \frac{k(\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i})}{(k-i)f(\bar{v}_i)} = \nu(\bar{v}_i) + \frac{k \sum_{j=i+1}^{k} \eta_j}{(k-i)f(\bar{v}_i)}, \qquad (44)$$

$$\left(\sum_{1}^{\bar{v}_i-1} \beta_v + \sum_{j=1}^{i} \bar{\beta}_{\bar{v}_j} \right) + \frac{k\bar{\mu}_{\bar{v}_i}}{(1 - F(\bar{v}_i - 1))} = \bar{v}_i - \frac{k\bar{v}_i\eta_i}{(1 - F(\bar{v}_i - 1))}. \qquad (45)$$

Subtracting the equation (43) corresponding to $v - 1$ from the equation corresponding to v for $\bar{v}_{i-1} + 2 \leq v \leq \bar{v}_i - 1$, we have:

$$\beta_v + \frac{k(\mu_v - \mu_{v-1})}{(k-i+1)f(v)} - \frac{k(\mu_{v-1} - \mu_{v-2})}{(k-i+1)f(v-1)} = \nu(v) - \nu(v-1) + \frac{k \sum_{j=i}^{k} \eta_j}{k-i+1} \left(\frac{1}{f(v)} - \frac{1}{f(v-1)} \right). \quad (46)$$

Subtracting the equation (44) corresponding to $\bar{v}_i$ from equation (43) corresponding to $\bar{v}_i+1$, we have:

$$\beta_{\bar{v}_i+1} + \frac{k(\mu_{\bar{v}_i+1} - \mu_{\bar{v}_i})}{(k-i+1)f(\bar{v}_i+1)} - \frac{k(\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i})}{(k-i)f(\bar{v}_i)} = \nu(\bar{v}_i+1) - \nu(\bar{v}_i) + \frac{k\sum_{j=i}^{k}\eta_j}{k-i}\left(\frac{1}{f(\bar{v}_i+1)} - \frac{1}{f(\bar{v}_i)}\right).$$

Similarly, subtracting the equation (45) corresponding to $\bar{v}_i$ from (44) corresponding to $\bar{v}_i$ we have:

$$\beta_{\bar{v}_i} + \frac{k(\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i})}{(k-i)f(\bar{v}_i)} - \frac{k\bar{\mu}_{\bar{v}_i}}{(1-F(\bar{v}_i-1))} = \nu(\bar{v}_i) - \bar{v}_i + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\bar{v}_i)} + \frac{k\bar{v}_i\eta_i}{(1-F(\bar{v}_i-1))} \quad (47)$$

Finally, subtracting (43) corresponding to $\bar{v}_i - 1$ from (45) corresponding to $\bar{v}_i$, we have:

$$\bar{\beta}_{\bar{v}_i} + \frac{k\bar{\mu}_{\bar{v}_i}}{(1-F(\bar{v}_i-1))} - \frac{k\mu_{\bar{v}_i-2}}{(k-i)f(\bar{v}_i-1)} = \bar{v}_i - \nu(\bar{v}_i-1) - \frac{k\bar{v}_i\eta_i}{(1-F(\bar{v}_i-1))} - \frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)} \quad (48)$$

If the optimal solution a^* is strictly monotone, the inequalities (35-37) do not bind. Complementary slackness implies all the μ's are 0. As in the common knowledge budget case we set $\bar{\beta}_{\bar{v}_i} = 0$ for all i since this will satisfy complementary slackness. Therefore, from (48), we have that:

$$\eta_k = \frac{1}{k}\frac{(1-F(\bar{v}_k-1))(1-F(\bar{v}_k-2))}{\bar{v}_k f(\bar{v}_k-1) + (1-F(\bar{v}_k-1))},$$

$$\eta_i = \frac{1}{k}\frac{(1-F(\bar{v}_i-1))(1-F(\bar{v}_i-2))}{(k-i+1)\bar{v}_i f(\bar{v}_i-1) + (1-F(\bar{v}_i-1))} - \frac{(1-F(\bar{v}_i-1))\sum_{i+1}^{k}\eta_j}{(k-i+1)\bar{v}_i f(\bar{v}_i-1) + (1-F(\bar{v}_i-1))}$$

It is easily verified that the η's as specified are non-negative and therefore dual feasible. Further, one can show that $i \leq j \implies \eta_i \geq \eta_j$, in other words, as one would suspect, smaller budgets have larger shadow prices. Substituting into (46) we have, $\forall v : \bar{v}_{i-1} < v < \bar{v}_i$,

$$\beta_v = \nu(v) - \nu(v-1) + \frac{k\sum_{j=i}^{k}\eta_j}{k-i+1}\left(\frac{1}{f(v)} - \frac{1}{f(v-1)}\right).$$

Note that β_v for all v such that $\bar{v}_{i-1} < v < \bar{v}_i$ will be positive if f is weakly decreasing. Finally, substituting the η's into (47), we have:

$$\beta_{\bar{v}_i} = \nu(\bar{v}_i) - \nu(\bar{v}_i-1) + \frac{k\sum_{i+1}^{k}\eta_j}{(k-i)f(\bar{v}_i)} - \frac{k\sum_{i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)}.$$

Observe that $\beta_{\bar{v}_i}$ can be negative. The adjusted virtual value of valuation $\bar{v}_i$ is $\nu(\bar{v}_i) + \frac{k \sum_{i+1}^{k} \eta_j}{(k-i)f(\bar{v}_i)}$ which may be larger than the adjusted virtual valuation of $\nu(\bar{v}_i - 1) + \frac{k \sum_{i}^{k} \eta_j}{(k-i+1)f(\bar{v}_i-1)}$ even if f is weakly decreasing and satisfies the monotone hazard rate. This is because allocating to valuation $\bar{v}_i - 1$ also 'relaxes' the budget constraint corresponding to b_i (in addition to the budget constraints for larger budgets), which allocating to $\bar{v}_i$ does not.

In this instance, therefore, the allocation rule for the revenue maximizing rule will require ironing. As described in the introduction, for each budget b_i there will be an additional cutoff $\underline{v}_i$. Types (v, b) where $\bar{v}_i \leq v \leq \underline{v}_i$ and $b > b_i$ will be pooled with the types (v, b_i), $v \geq \bar{v}_i$ (i.e. the types for whom the budget constraint binds). The allocation rule will be efficient between $\underline{v}_i$ and $\bar{v}_{i+1}$.

Finally, the lowest valuation to be allotted will be $\underline{v}_0$, which is the lowest valuation whose adjusted virtual valuation is non-negative. To summarize:

PROPOSITION 4 *Suppose $f(v)$ is weakly decreasing in v, and $\frac{1-F(v)}{f(v)}$ is increasing in v. Then, there is an optimal solution $a^*(v, b)$ to (RevOpt) that can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \ldots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0$. No valuation less than $\underline{v}_0$ will be allotted. The allocation rule will satisfy (32-37). The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Further, for all $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$, $a^*(v, b) = y(\bar{v}_i)$. If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.*

To help understand the above proposition, we describe the implementation of the optimal auction. Consider the 'regular' case, i.e. f satisfies the monotone hazard rate and decreasing density conditions. The implementation is akin to an all-pay auction, the discreteness of types makes it slightly convoluted.[14]

There will be a reserve price r, and a (finite) set of valid bids larger than r. As in a standard all-pay auction, the highest bid will win, ties are broken randomly, and all bidders pay their bid. The set of valid bids is exactly the set of $p(v, b)$'s, $(v, b) \in T$, identified in the statement of Proposition 4.

To see how this corresponds to the description of the proposition, it is instructive to consider the bids submitted by various types in equilibrium. The reserve price r is such that no-type with a valuation less than $\underline{v}_0$ would find it profitable to submit a bid other than 0 in equilibrium. The type $(\bar{v}_i, b_i)$ is the lowest valuation with budget b_i to bid her budget. Critically, the lowest valid bid larger than b_i is such that types with valuations $\bar{v}_i$ through $\underline{v}_i - 1$ and budgets larger than b_i prefer to bid b_i (and potentially be pooled with other bidders), rather than bid higher and win the good outright. All other valid bids, i.e. bids that are not equal to any of the budgets b_1 through b_k, are separating. In other words, for any other valid bid, there is exactly one valuation that makes that bid in equilibrium.

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[14]See also the description of the continuous case in 6.2 for further intuition.

In the event that f violates either the monotone hazard rate or decreasing density condition, the optimal auction may require further pooling (over and above the pooling already required by the regular case). These extra pooling intervals can be identified by explicitly constructing feasible solutions to the dual(REVOPT) and its dual (DOPT2) which complement each other, and also satisfy the constraints identified in Lemma 1.

PROOF: As before, our proof proceeds by constructing a dual solution that complements the primal solution described in the statement of the proposition. Since $a^*(v, b) = 0$ for all $v < \underline{v}_0$, the corresponding Border constraints must be slack, and therefore $\beta_v = 0$ for all $v < \underline{v}_0$. Since $x^*(\underline{v}_i + 1) > y^*(\bar{v}_i)$, $\mu_{\underline{v}} = 0$.

The β_v for $\underline{v}_i + 2 \leq v \leq \bar{v}_{i+1} - 1$ is as specified in (46), with the corresponding μ's set to 0. By Lemma 1, β_v for $\bar{v}_i \leq v \leq \underline{v}_i$ is 0 since, by the statement of the proposition, $a^*(v, b) = y(\bar{v}_i)$ for all $b > b_i$. The relevant μ's can be calculated from the relevant equations.

Instead of computing these μ's, we can instead suppose that the types which have been ironed, $\{(v, b_i)$ for $v \geq \bar{v}_i\} \bigcup \{(v, b_j)$ for $j > i, \bar{v}_i \leq v \leq \underline{v}_i\}$, all correspond to one 'artificial' type, t_i. The probability of t_i is

$$\pi(t_i) = \frac{(k-i)}{k}(F(\underline{v}_i) - F(\bar{v}_i - 1)) + \frac{1}{k}(1 - F(\bar{v}_i - 1)).$$

Further, its adjusted virtual valuation is:

$$\nu(t_i) = \bar{v}_i - \frac{(k-i)(\underline{v}_i - \bar{v}_i + 1)(1 - F(\underline{v}_i))}{k\pi(t_i)} + \frac{(\underline{v}_i - \bar{v}_i + 1)\sum_{j=i+1}^{k}\eta_j}{\pi(t_i)} - \frac{\bar{v}_i\eta_i}{\pi(t_i)}$$

Since the budget constraint for budget b_i binds at $\bar{v}_i$, analogous to the proof of Proposition 2, $\bar{\beta}_{\bar{v}_i}$ is 0, and therefore we can solve for η_i from:

$$\nu(t_i) - \nu(\bar{v}_i - 1) - \frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} = 0. \tag{49}$$

Note that the adjusted virtual valuation of $\bar{v}_i - 1$ can be written as:

$$\nu(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} + \frac{k\eta_i}{(k-i+1)f(\bar{v}_i - 1)}.$$

To see that the η_i that solves (49) is positive, we need to show that:

$$\nu(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \bar{v}_i - \frac{(k-i)(1 - F(\underline{v}_i))}{k\pi(t_i)} + \frac{(\underline{v}_i - \bar{v}_i + 1)\sum_{j=i+1}^{k}\eta_j}{\pi(t_i)} \tag{50}$$

The right hand side of this inequality equals

$$\frac{1}{\pi(t_i)}\Big(\bar{v}_i\frac{(1-F(\bar{v}_i-1))}{k}+\sum_{v=\bar{v}_i}^{\underline{v}_i}\big(\frac{(k-i)f(v)}{k}\nu(v)+\sum_{j=i+1}^{k}\lambda_j\big)\Big).$$

However,

$$\nu(\bar{v}_i-1)+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)}<\bar{v}_i,$$

since $\sum_{j=i+1}^{k}\eta_j$ is appropriately small (see Proposition 6). Further, for $\bar{v}_i\leq v\leq\underline{v}_i$,

$$\nu(\bar{v}_i-1)+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)}<\nu(v)+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(v)},$$

follows from the monotone hazard rate and decreasing density conditions. (50) follows since its right hand side is a weighted average of the right hand side of the latter two inequalities.

Further, we have that:

$$\beta_{\underline{v}_i+1}=\nu(\underline{v}_i+1)+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\underline{v}_i+1)}-\nu(t_i). \tag{51}$$

To ensure that $\beta_{\underline{v}_i+1}\geq 0$ it suffices by inequality (49) that cutoffs $\bar{v}_i$ and $\underline{v}_i$ satisfy:

$$\nu(\underline{v}_i+1)+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\underline{v}_i+1)}\geq\nu(\bar{v}_i-1)+\frac{k\sum_{i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)}.$$

Finally, note that $\underline{v}_0$ will be the lowest valuation such that $\nu(v)+\frac{\sum_1^k\eta_j}{f(v)}\geq 0$; and

$$\beta_{\underline{v}_0}=\nu(v)+\frac{\sum_1^k\eta_j}{f(v)}.$$

The partial solution identified above, with all other dual variables set to 0, is an optimal dual solution. Since $\beta_v>0$ for all $\underline{v}_i+1\leq v\leq\bar{v}_{i+1}-1$, by complementary slackness, the corresponding Border constraints (10) bind. This concludes the case where our regularity condition on the distribution of types (monotone hazard rate, decreasing density) are met. If the monotone hazard rate or decreasing density assumptions are not satisfied then the dual solution identified may be infeasible, and therefore additional pooling will be required due to Lemma 1. □

We can also describe the constrained efficient auction for this setting. The proof is similar, and omitted.

PROPOSITION 5 *Suppose $f(v)$ is decreasing in v. Then the solution of (ConsEff) can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \dots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0 = 0$. The allocation rule will satisfy (32-37). The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Further, for all $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$, $a(v, b) = y(\bar{v}_i)$. If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.*

5 SUBSIDIES

Since budget constrained bidders are unable to effectively compete in the auction, this will depress auction revenues. To get around this problem, prior work has examined various kinds of subsidies (lump sum transfer, discounts) and their effect in a particular auction setting.

In our setting, there is only one possible (incentive compatible) means of subsidy- a lump sum transfer from the auctioneer to the agents. This is because, given an allocation rule, incentive compatibility determines prices up to a constant:

$$p(v, b) = va(v, b) - \sum_{1}^{v-1} a(v', b) + c.$$

Let us consider a subsidy via lump sum payment of some amount ϵ. This costs the auctioneer ϵ per agent. The effect of this subsidy is to relax the budget constraints by ϵ. Therefore the gain in revenue is (at most) $\epsilon \sum_i \eta_i$.[15] We show below that $\sum_i \eta_i \leq 1$, and thus $\epsilon \sum_i \eta_i \leq \epsilon$. As a result, if the auctioneer were running the optimal auction, he should not offer subsidies. This result remains true even when bidders' budgets are common knowledge.

PROPOSITION 6 *For all i,*

$$\sum_i^k \eta_j \leq \frac{(k-i+1)}{k}(1 - F(\bar{v}_i - 1)). \tag{52}$$

PROOF: We prove by induction on i. For $i = k$, we know that

$$\begin{aligned}
\eta_k &= \frac{1}{k}\frac{(1 - F(\bar{v}_k - 1))(1 - F(\bar{v}_k - 2))}{\bar{v}_k f(\bar{v}_k - 1) + (1 - F(\bar{v}_k - 1))} \\
&= \frac{(1 - F(\bar{v}_k - 1))}{k}\frac{(1 - F(\bar{v}_k - 2))}{(\bar{v}_k - 1)f(\bar{v}_k - 1) + (1 - F(\bar{v}_k - 2))} \\
&\leq \frac{(1 - F(\bar{v}_k - 1))}{k}.
\end{aligned}$$

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[15]Recall that η_i is the shadow price of the budget constraint.

For the induction hypothesis, assume that

$$\sum_{i+1}^{k} \eta_j \leq \frac{(k-i)}{k}(1 - F(\bar{v}_{i+1} - 1)).$$

Therefore we are left to show (52).

Recall from the proof of Proposition 4 that at optimality, η_i, the dual variable corresponding to the budget constraint corresponding to b_i, solves

$$\nu(t_i) - \nu(\bar{v}_i - 1) - \frac{k \sum_{j=i}^{k} \eta_j}{(k-i+1)f(\bar{v}_i - 1)} = 0,$$

where

$$\nu(t_i) = \bar{v}_i - \frac{(\underline{v}_i - \bar{v}_i + 1)}{\pi(t_i)}\left(\frac{k-i}{k}(1 - F(\underline{v}_i)) - \sum_{i+1}^{k} \eta_j\right) - \frac{\bar{v}_i}{\pi(t_i)}\eta_i,$$

and,

$$\pi(t_i) = \frac{1}{k}(1 - F(\bar{v}_i - 1)) + \frac{k-1}{k}(F(\underline{v}_i) - F(\bar{v}_i - 1)).$$

By the induction hypothesis,

$$\nu(t_i) \leq \bar{v}_i - \frac{\bar{v}_i}{\pi(t_i)}\eta_i,$$

and therefore

$$\bar{v}_i - \nu(\bar{v}_i - 1) \geq \frac{\bar{v}_i}{\pi(t_i)}\eta_i + \frac{k \sum_{j=i}^{k} \eta_j}{(k-i+1)f(\bar{v}_i - 1)}$$

$$\implies \frac{1 - F(\bar{v}_i - 2)}{f(\bar{v}_i - 1)} \geq \frac{\bar{v}_i}{\pi(t_i)}\eta_i + \frac{k \sum_{j=i}^{k} \eta_j}{(k-i+1)f(\bar{v}_i - 1)}.$$

Rearranging terms, we have

$$\frac{k-i+1}{k}\frac{(1 - F(\bar{v}_i - 2))\pi(t_i) + \bar{v}_i f(\bar{v}_i - 1)\sum_{i+1}^{k} \eta_j}{\frac{k-i+1}{k}\bar{v}_i f(\bar{v}_i - 1) + \pi(t_i)} \geq \sum_{i}^{k} \eta_j. \tag{53}$$

Once again, by the induction hypothesis,

$$\sum_{i+1}^{k} \eta_j \leq \frac{k-i}{k}(1 - F(\underline{v}_i)) = \frac{k-i+1}{k}(1 - F(\bar{v}_i - 1)) - \pi(t_i). \tag{54}$$

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Substituting (54) into (53),

$$\sum_i^k \eta_j \le \left(\frac{k-i+1}{k}\right)\frac{(1-F(\bar{v}_i-2))\pi(t_i) + \bar{v}_i f(\bar{v}_i-1)\left(\frac{k-i+1}{k}\left(1-F(\bar{v}_i-1)\right) - \pi(t_i)\right)}{\frac{k-i+1}{k}\bar{v}_i f(\bar{v}_i-1) + \pi(t_i)}$$

$$\equiv \phi(\bar{v}_i, \pi(t_i)).$$

Observation 4 in Appendix B shows that $\phi(\cdot)$ is decreasing in its second argument. Given $\bar{v}_i$, the lowest possible value for $\pi(t_i)$ is $\frac{1}{k}(1-F(\bar{v}_i-1))$, at which the left hand side of the bound will be maximized. Therefore, substituting $\pi(t_i) = \frac{1}{k}(1-F(\bar{v}_i-1))$,

$$\sum_i^k \eta_j \le \frac{k-i+1}{k}\frac{(1-F(\bar{v}_i-2))(1-F(\bar{v}_i-1)) + (k-i)\bar{v}_i f(\bar{v}_i-1)(1-F(\bar{v}_i-1))}{(k-i+1)\bar{v}_i f(\bar{v}_i-1) + (1-F(\bar{v}_i-1))}$$

$$\le \frac{k-i+1}{k}(1-F(\bar{v}_i-1))$$

$\square$

How then does the optimal auction encourage competition? Recall that for each i, types $\{(v, b_i) \text{ for } v \ge \bar{v}_i\} \bigcup \{(v, b_j) \text{ for } j > i, \bar{v}_i \le v \le \underline{v}_i\}$ are pooled. The pooling serves to allot the good to disadvantaged bidder types (v, b_i), $v \ge \bar{v}_i$ even in profiles where there are bidders with higher valuations and budgets present. Intuitively, favoring bidders in this way is better than lump-sum transfers because there are more degrees of freedom: a lump-sum transfer must be given to a bidder regardless of his type in order to maintain incentive compatibility.

6 CONTINUOUS VALUATIONS

Our analysis in the previous sections has concerned the case of discrete types. This has allowed us to use directly the tools of linear programming, without concerning ourselves with technical asides such as dual measures and various caveats that apply to LP duality when there are an infinite number of variables and/or constraints. However, since the standard models in mechanism design concern themselves with the case when bidder types are drawn from a continuum, there may be interest in how our characterization applies to this case when bidder types. We shed some light on this. The development mirrors the development of our main results- we first show that our results when bidders have common, commonly known budgets converge to the Laffont-Robert/Maskin characterizations. We then sketch the structure of the solution for the private budget case.

6.1 Common Knowledge Budgets

Suppose bidders all have a common knowledge budget b. Their valuations belong to the interval $[0, V]$, and are drawn i.i.d. from a distribution with density f and cumulative distribution F. Further, assume that f satisfies the decreasing density and increasing hazard rate conditions.

Suppose we now discretize this space- we assume that all bidder valuations belong to the set $\{\epsilon, 2\epsilon, \ldots, m\epsilon\}$, where $m\epsilon = V$. The 'density' of any type, $k\epsilon$ is $f^\epsilon(k\epsilon) = \int_{(k-1)\epsilon}^{k\epsilon} f(v)dv$. The cumulative distribution F^ϵ is defined analogously. Note that f^ϵ will satisfy both the decreasing density and monotone hazard rate conditions.

Our solution in this discrete type space can be summarized thus: there will exist two cutoffs $\underline{v}$ and $\bar{v}$. No valuation less than $\underline{v}$ will be allotted. Types $\bar{v}$ and above will be pooled, and the budget constraint will bind for those types. The allocation rule will be efficient between $\underline{v}$ and $\bar{v} - 1$. Therefore the optimal allocation rule $a(v)$ satisfies: [16]

$$\bar{v}a(v) - \epsilon \sum_{j=1}^{\bar{k}-1} a(j\epsilon) = b, \ \bar{k}\epsilon = \bar{v},$$

and $\underline{v}$ is the lowest type such that:

$$\underline{v} - \epsilon \frac{1 - F^\epsilon(\underline{v})}{f^\epsilon(\underline{v})} + \frac{\epsilon \eta}{f^\epsilon} \geq 0,$$

where

$$\eta = \frac{(1 - F(\bar{v} - \epsilon))(1 - F(\bar{v} - 2\epsilon))}{\bar{v}\frac{f(\bar{v}-\epsilon)}{\epsilon} + (1 - F(\bar{v} - \epsilon))}.$$

Note that as $\epsilon \to 0$:

$$p(v) \to va(v) - \int_0^v a(v')dv'$$

$$\eta \to \frac{(1 - F(\bar{v}))^2}{\bar{v}f(\bar{v}) - (1 - F(\bar{v}))}.$$

As a result as we go to finer and finer discretizations, the solution identified in Proposition 2 will converge to the solution identified in Laffont and Robert.

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[16] To see this, note that the payment rule will be $p(v) = va(v) - \epsilon \sum_{v' < v} a(v')$, and virtual valuations etc. must be updated accordingly.

6.2 Private Information Budgets

On the basis of this we are now able to identify the structure of the optimal solution in the case where valuations and budgets are both private information. Suppose valuations are from some interval $[0, V]$, while budgets are one of a finite number $b_1, \ldots b_k$. Valuations and budgets are independently determined, and each bidder's valuation is an i.i.d. draw from some distribution F with density f, and the budget is drawn i.i.d. according to a uniform distribution. Further suppose f satisfies the decreasing density and increasing hazard rate conditions.

The problem of maximizing the revenue in the continuous case can be written as:

$$\max_{a,p} \sum_{i=1}^{k} \frac{1}{k} \int_0^V p(v, b_i) f(v) dv \qquad \text{(Cont Rev Opt)}$$

$$\text{s.t. Incentive Compatibility}$$

$$\text{Individual Rationality}$$

$$p(v, b_i) \leq b_i$$

$$a \text{ feasible}$$

Standard arguments imply that Incentive compatibility and individual rationality pin down the pricing rule to the standard rent extraction formula, i.e. :

$$p(v, b) = va(v, b) - \int_0^v a(x, b) dx \qquad (55)$$

Therefore, substituting (55) into Cont Rev Opt, we can eliminate the variables p, leading to:

$$\max_{a \in C} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \nu(v) a(v, b_i) f(v) dv. \qquad \text{(Cont Rev Opt2)}$$

Once again, we can discretize this space- we assume that all bidder valuations belong to the set $\{\epsilon, 2\epsilon, \ldots, m\epsilon\}$, where $m\epsilon = V$. The 'density' of any type, $k\epsilon$ is $f^\epsilon(j\epsilon) = \int_{(j-1)\epsilon}^{j\epsilon} f(v) dv$. The cumulative distribution F^ϵ is defined analogously. Note that f^ϵ will satisfy both the decreasing density and monotone hazard rate conditions if f does.

This discretization can be thought of as an extra constraint, i.e. that each type $v \in [(k-1)\epsilon, k\epsilon)$ must be pooled for all $k = 1, 2, \ldots m$. Let us denote the feasible region with this extra constraint as C_ϵ. Consider the revenue maximizing mechanism for this discretization(as identified in Proposition 4), let us denote it (a^ϵ, p^ϵ). Note that a^ϵ solves:

$$\max_{a \in C_\epsilon} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \nu(v) a(v, b_i) f(v) dv,$$

and p^ϵ is then defined by (55).

Note that $C_\epsilon \subseteq C$, and that both are compact subsets of the L_1 space defined by the measure f. Further, the operator $T(a) = \sum_{i=1}^k \frac{1}{k} \int_0^V \nu(v)a(v,b_i)f(v)dv$ is a bounded linear operator from the L_1-space of allocation rules to $\Re$. Therefore T achieves its maximum on each set C_ϵ and C.

Since $C_\epsilon \uparrow C$ pointwise, it must be that:

$$\lim_{\epsilon \to 0} \max_{a \in C_\epsilon} \sum_{i=1}^k \frac{1}{k} \int_0^V \nu(v)a(v,b_i)f(v)dv$$

$$= \max_{a \in C} \sum_{i=1}^k \frac{1}{k} \int_0^V \nu(v)a(v,b_i)f(v)dv$$

Finally, recalling that $T(a)$ is a bounded linear operator: [17]

$$lim_{\epsilon \to 0} a^\epsilon \in \{\arg \max_{a \in C} \sum_{i=1}^k \frac{1}{k} \int_0^V \nu(v)a(v,b_i)f(v)dv\}.$$

We can now describe (the qualitative features of) an implementation of $\lim_{\epsilon \to 0}(a^\epsilon, p^\epsilon)$. Akin to the discrete case, the implementation will be an all-pay auction with a modified winning rule. There will be a reserve price r. Further for each budget $b_i < b_k$,there will be a $\triangle_i > 0$. [18] If there are multiple bids in the interval $[b_i, b_i + \triangle_i)$, and no bid exceeding $b_i + \triangle_i$, the auctioneer will award the good randomly to one of these bidders(rather than to the highest bid). In all other cases, the good will be awarded to the highest bidder.

As in the discrete case, due to the extra pooling being implemented, there will be 2 sorts of bidders who bid b_i in equilibrium:

1. Types with budget b_i and valuation larger than a cutoff valuation $\bar{v}_i$- these are the type for whom the budget constraint binds.

2. Types with a budget larger than b_i, and valuation in the interval $[\bar{v}_i, \underline{v}_i)$: these types would prefer to bid b_i and risk being pooled, rather than bid $b_i + \triangle_i$ and win the good outright if at all.

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[17]Technically we should also formally show that the sequence $\{a^\epsilon\}$ has a pointwise limit, but this follows by inspection of the a^ϵ's characterized by Proposition 4.

[18]These can be computed by solving the continuous versions of the appropriate equations in the proof of Proposition 4.

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A COUNTEREXAMPLES

A.1 LAFFONT AND ROBERT'S SOLUTION

In this section, we examine the classical formulation, with a continuum of types. We show by counterexample that the Laffont and Robert solution is not optimal for all distributions that meet the monotone hazard rate condition.

Suppose that, as in the original Laffont and Robert paper (herein LR), we have valuations belonging to the continuum, say interval $[0, 1]$; distributed with density $f(v)$, $F(v) = \int_0^v f(v) dv$. Their virtual valuation is defined as $\nu(v) = v - \frac{1-F(v)}{f(v)}$.

Further, suppose we have (as per their solution) 2 cutoffs, v_1, v_2. The allocation rule does not allot types below v_1; and pools all types above v_2. This will make the allocation rule:

$$a(v) = \begin{cases} \frac{1-F^N(v_2)}{N(1-F(v_2))} & v \geq v_2 \\ F^{N-1}(v) & v \in [v_1, v_2] \\ 0 & o.w. \end{cases} \tag{56}$$

At the optimal solution, the budget constraint must bind for all types v_2 and above, [19]

$$v_2 a(v_2) - \int_{v_1}^{v_2} a(v) dv = b. \tag{57}$$

Choose v_1 and v_2 to solve:

$$\max_{v_1, v_2} a(v_2)(1 - F(v_2))v_2 + \int_{v_1}^{v_2} \nu(v) f(v) a(v) dv$$

$$\text{s.t. } v_2 a(v_2) - \int_{v_1}^{v^2} a(v) dv = b$$

The first order conditions for optimality imply

$$f(v_1)\nu(v_1) + \frac{(1 - F(v_2))^2}{(1 - F(v_2)) + v_2 f(v_2)} = 0. \tag{58}$$

Therefore v_1 and v_2 are the solutions to (58) and the budget equation (57). Further note that this is the L-R solution. We are now in a position to state without proof the 'correct' version of Laffont-Robert's theorem.

THEOREM 1 *Suppose the distribution on types is such that the density is decreasing. Further, suppose that the monotone hazard rate condition is met. The allocation described by (56), where v_1, v_2 jointly satisfy (57) and (58), and the associated pricing rule*

$$p(v) = v.a(v) - \int_{\underline{v}}^{v} a(v) dv,$$

constitute the expected revenue maximizing mechanism.

A proof of this theorem requires taking the dual of an infinite dimensional linear program (see for example Anderson and Nash [1]), and defining the appropriate measure on the dual space. We can now use the intuition gleaned from Section 3 to identify a flaw in the L-R solution in the event that densities are not decreasing. Pick a $v_3 \in (v_1, v_2)$; and 'iron' some small interval of types $[v_3, v_3 + \epsilon]$. The new allotment rule is therefore:

$$a'(v) = \begin{cases} \frac{1 - F^N(v_2)}{N(1 - F(v_2))} & v \geq v_2 \\ \frac{F^N(v_3 + \epsilon) - F^N(v_3)}{N(F(v_3 + \epsilon) - F(v_3))} & v \in [v_3, v_3 + \epsilon] \\ F^{N-1}(v) & v \in [v_1, v_3) \bigcup (v_3 + \epsilon, v_2] \\ 0 & o.w. \end{cases} \tag{59}$$

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[19]If not, the solution of the overall program would be the same as Myerson's solution [19].

By Lemma 5 (see Appendix A.3), if $f(v)$ is increasing in the interval; then

$$g_\epsilon \equiv \epsilon \frac{F^N(v_3 + \epsilon) - F^N(v_3)}{N(F(v_3 + \epsilon) - F(v_3))} - \int_{v_3}^{v_3+\epsilon} F^{N-1}(v)dv > 0.$$

Let us assume that $f(v)$ is increasing in the range $[0, 1]$. As a result, the budget constraint is now slack. We can now potentially improve on the revenue by 'un-pooling' v_2 to $v_2 + \delta$.

First, δ solves the implicit equation:

$$\delta \frac{F^N(v_2 + \epsilon) - F^N(v_2)}{N(F(v_2 + \epsilon) - F(v_2))} - \int_{v_2}^{v_2+\delta} F^{N-1}(v)dv = g_\epsilon. \tag{60}$$

The change in revenue from ironing types $[v_3, v_3 + \epsilon]$ is:

$$\Delta(v_3, \epsilon) \equiv \int_{v_3}^{v_3+\epsilon} \nu(v)f(v)\left[\frac{F^N(v_3 + \epsilon) - F^N(v_3)}{N(F(v_3 + \epsilon) - F(v_3))} - F^{N-1}(v)\right]dv.$$

Similarly, the change in revenue from 'un-pooling' $[v_2, v_2 + \delta]$ is:

$$\Delta(v_2, \epsilon) \equiv -\int_{v_2}^{v_2+\delta} \nu(v)f(v)\left[\frac{F^N(v_2 + \delta) - F^N(v_2)}{N(F(v_2 + \delta) - F(v_2))} - F^{N-1}(v)\right]dv.$$

Therefore the total change in revenue is:

$$\Delta = \Delta(v_3, \epsilon) + \Delta(v_2, \epsilon)$$

Since $\nu(\cdot)$ and $f(\cdot)$ are both increasing; $\Delta(v_2, \epsilon) \geq 0 \geq \Delta(v_3, \epsilon)$. Potentially, $\Delta \geq 0$ for some suitable parameter choices. In other words, our perturbation of the L-R solution can increase expected revenue, therefore the L-R solution is not optimal. We flesh out a numerical example below.

A.1.1 AN EXAMPLE

There are 2 bidders, i.e. $N = 2$. Both have valuations in the interval $[0, 1]$ which are drawn i.i.d. with density $f(v) = 2v$; $F(v) = v^2$. Both have a common budget constraint $b = 0.5$. The 'virtual value' of a bidder of valuation v, $\nu(v) = \frac{3v^2 - 1}{2v}$, which is increasing on the interval $[0, 1]$. If there was no budget constraint, the optimal auction would be a second price auction with reserve price $v_0 = \frac{1}{\sqrt{3}}$, i.e. $\nu(v_0) = 0$.

Recall that the L-R solution would require us to compute v_1 and v_2 which jointly solve

(57) and (58). Making appropriate substitutions, v_1 and v_2 solve:

$$\frac{v_2}{2} + \frac{v_2^3}{6} + \frac{v_1^3}{3} = 0.5$$

$$3v_1^2 + \frac{(1-v_2^2)^2}{1+v_2^2} = 1$$

Solving, we get $v_1 = 0.5415$ and $v_2 = .7523$. Therefore; $v_1 < v_0 < v_2$. For the perturbation we outlined above, select $v_3 = \frac{1}{\sqrt{3}}(= v_0)$; and $\epsilon = 10^-4$. Our functional forms lend themselves to easy analytic calculation. It can be shown that:

$$g_\epsilon = \frac{\epsilon^3}{6}$$

$$\delta = \epsilon$$

$$\Delta(v_3, \epsilon) = -\frac{(3v_3^2 + 1)\epsilon^3}{6} - \frac{v_3\epsilon^4}{2} + \epsilon^5$$

$$\Delta(v_2, \epsilon) = +\frac{(3v_2^2 + 1)\epsilon^3}{6} + \frac{v_2\epsilon^4}{2} - \epsilon^5$$

Substituting we see that net change in revenue

$$\Delta \approx \frac{(v_2^2 - v_3^2)\epsilon^3}{2}$$

$$> 0$$

where the final inequality follows from the fact that $v_2 > v_3$.

A.2 MASKIN

Recall that Maskin [18] considered the same environment as Laffont and Robert, the only difference being he was interested in specifying the constrained efficient auction for this setting. Analogous to our analysis for Laffont and Robert, we can state the correct version of Maskin's main theorem:

THEOREM 2 *Suppose the distribution on types is such that the density is decreasing. The allocation described by (56), where $v_1 = 0$ and v_2 satisfies (57), and the associated pricing rule*

$$p(v) = v.a(v) - \int_{\underline{v}}^{v} a(v)dv,$$

constitute the expected revenue maximizing mechanism.

A.2.1 A COUNTER-EXAMPLE

There are 2 bidders, i.e. $N = 2$. Both have valuations in the interval $[0, 1]$ which are drawn i.i.d. with density $f(v) = 2v$; $F(v) = v^2$. Both have a common budget constraint $b = 0.5$. The Maskin solution would require us to pick a cutoff $\bar{v}$ to solve:

$$\frac{\bar{v}}{2} + \frac{\bar{v}^3}{6} = 0.5,$$

i.e. $\bar{v} = 0.8177$. Let us now pick $\epsilon \ll 1$, and iron $[0, \epsilon]$. It can be shown that the budget constraint for type $\bar{v}$ is relaxed by $\frac{\epsilon^3}{6}$. Therefore we can now have the efficient allocation for types $[\bar{v}, \bar{v} + \epsilon)$ and still satisfy the budget constraint. Further, one can show the expected loss of efficiency from ironing the interval $[0, \epsilon]$ is $O(\epsilon^5)$, while the gain in efficiency from unpooling the types $[\bar{v}, \bar{v} + \epsilon)$ is roughly $\frac{1}{3} v_1^2 \epsilon^3$.

A.3 IRONING

Let $f(.)$ be the density function for some distribution on $\Re$, and let $F(.)$ be the associated cumulative distibution function.

LEMMA 5 *If $f(.)$ is (strictly) increasing on some interval $[v_1, v_2]$, then for any $N > 1$, we have:*

$$(v_2 - v_1) \frac{F^N(v_2) - F^N(v_1)}{N(F(v_2) - F(v_1))} > \int_{v_1}^{v_2} F^{N-1}(v) dv$$

PROOF: Rewriting, we have to show that

$$\frac{\int_{v_1}^{v_2} f(v) F^{n-1}(v) dv}{\int_{v_1}^{v_2} f(v) dv} > \frac{\int_{v_1}^{v_2} F^{n-1}(v) dv}{\int_{v_1}^{v_2} dv}$$

This is true if and only if

$$\int_{v_1}^{v_2} dv \int_{v_1}^{v_2} f(v) F^{n-1}(v) dv > \int_{v_1}^{v_2} F^{n-1}(v) dv \int_{v_1}^{v_2} f(v) dv$$

Note that both sides are equal (to zero) at $v_2 = v_1$. Therefore we have the desired inequality if the derivative w.r.t v_2 of the left hand side is greater than the right hand side. Differentiating both sides w.r.t. v_2 and rearranging we have that this is true if and only if:

$$F^{N-1}(v_2) \left[f(v_2)(v_2 - v_1) - \int_{v_1}^{v_2} f(v) dv \right] + \int_{v_1}^{v_2} (f(v) - f(v_2)) F^{N-1}(v) dv > 0$$

The inequality now follows by observing that $f(v)$ is increasing in v, therefore

$$\int_{v_1}^{v_2} (f(v) - f(v_2))F^{N-1}(v)dv > F^{N-1}(v_2)\int_{v_1}^{v_2}(f(v) - f(v_2))dv$$

$\square$

B Miscellaneous Proofs

B.1 Cutoffs

This section provides a proof of Lemma 4. The Lemma states that in any solution to (REVOPT), the cutoffs as defined in Definition 1 are such that

$$\underline{v}_i \geq \bar{v}_i - 1 \quad \forall i \leq k - 1. \tag{61}$$

As the intuition outlined in the main body points out, this result is not surprising- if this condition is violated for some i, roughly speaking, decrease the allocation of types $(\underline{v}_i+1, b_{i+1})$ (and types pooled with it); and increase the allocation of types $\{(\bar{v}_i, b_i), \ldots, (m, b_i)\}$. This will clearly increase revenue since the virtual valuation of the latter is $\bar{v}_i > \nu(\bar{v}_i) \geq \nu(\underline{v}_i)$ which is the virtual valuation of the former. The trouble is that this simple change can violate the budget constraints.

Below we show how to perturb allocation rules not satisfying (61). This particular construction relies critically on the assumption that the distribution over types is such that valuation and budget are independent- the assumption that all budgets are equally likely is however only for notational convenience. It will be clear, however, that as long as the distribution f satisfies the generalized monotone hazard rate condition identified in (2.1), a similar construction will be feasible.

Proof: Suppose not, i.e. suppose that allocation rule a solves (REVOPT), with cutoffs $\bar{V}$ and $\underline{V}$ such that $\underline{v}_i < \bar{v}_i - 1$ for some i.

To this end, let $j \equiv \max\{i : \underline{v}_i < \bar{v}_i - 1\}$. Therefore $\underline{v}_i \geq \bar{v}_i - 1$ for all $i > j$. We show how to construct an allocation rule a' is feasible in (REVOPT) that achieves weakly more revenue, such that $\underline{v}'_i \geq \bar{v}'_i - 1$ for all $i > j - 1$.

For ease of notation assume that $\underline{v}_{j-1} \leq \underline{v}_i$ and define $\hat{v} \equiv \underline{v}_j + 1$. Consider the following perturbation of a^*:

1. Reduce the allocation of all types $(\hat{v}, b_{j+1}), \ldots, (\hat{v}, b_k)$ by ϵ each.

2. Reduce the allocation of all types in $\{(v, b) : v > \hat{v}, b \geq b_{j+1}\}$ by $\epsilon/(\hat{v} + 1)$ each.

3. Increase the allocation of type $(\hat{v}, b_j)$ by $(k - j)\epsilon$.

4. Increase the allocation of types in $\{(v, b_j) : v > \hat{v}\}$ by $(k - j)\epsilon/(\hat{v} + 1)$.

Firstly note that this perturbation is revenue neutral. Next we show that the resulting allocation is feasible in the optimization program. Feasibility with respect to the Border constraints is clear by construction. Next note that the payment of type (v, b), $v > \hat{v}$, $b > b_{j+1}$ changes by

$$-v\frac{\epsilon}{\hat{v}+1} + \sum_{\hat{v}+1}^{v-1}\frac{\epsilon}{\hat{v}+1} + \epsilon = 0.$$

Similarly the payment of type (v, b_j), $v > \hat{v}$ changes by

$$v\frac{(k-j)\epsilon}{\hat{v}+1} - \sum_{\hat{v}+1}^{v-1}\frac{(k-j)\epsilon}{\hat{v}+1} + (k-j)\epsilon = 0.$$

Therefore the budget constraints for all types are still satisfied. Further, the payment of type $(\hat{v}+1, b_{j+1})$,

$$\begin{aligned}
p'(\hat{v}+1, b_{j+1}) &= p(\hat{v}+1, b_{j+1}) \quad \text{(By Construction)} \\
&> b_j \quad \text{(By definition of $\hat{v}$)} \\
&\geq p'(m, b_j) \quad \text{(By budget constraint)}
\end{aligned}$$

Finally, set ϵ such that

$$a(\hat{v}, b_{j+1}) - \epsilon = a(\hat{v}, b_j) + (k-j)\epsilon,$$

.

Let us assume that $a'(\hat{v}, b_j) \leq a'(\hat{v}+1, b_j)$. We show that a' is incentive compatible and individually rational. By Observation 2 it is enough to show that a' satisfies (24- 29) (with $\underline{v}_i' = \underline{v}_i + 1$).

Recall that a would have satisfied (24- 29). Verifying that (25- 29) are satisfied with the new cutoff is straightforward. Inequality (24), i.e. that $a'(v, b) \geq a'(v-1, b)$ for all v, b, for $b < b_j$ follows from the fact that $a(v, b) \geq a(v-1, b)$. For $b = b_j$, it follows from our assumption that $a'(\hat{v}, b_j) \leq a'(\hat{v}, b_j)$. For $b > b_j$ we are done if $a'(\hat{v}, b) \geq a'(\hat{v}-1, b)$. But note that $a'(\hat{v}, b) = a'(\hat{v}, b_j) \geq a'(\hat{v}-1, b_j) = a'(\hat{v}-1, b)$ (here the first equality follows from our choice of ϵ, the second by construction, and the third since $a(\hat{v}-1, b) = a'(\hat{v}-1, b)$ for any b).

Now suppose instead that $a'(\hat{v}, b_j) > a'(\hat{v}+1, b_j)$. In this case our perturbation of a proceeds in two steps: the first step is the same as before with ϵ such that

$$a(\hat{v}, b_j) + (k-j)\epsilon = a(\hat{v}+1, b_j) + (k-j)\frac{\epsilon}{\hat{v}+1}.$$

Call the resulting allocation rule a''. Clearly, this perturbation will be revenue neutral; and will satisfy (25- 28) with the same cutoffs as a. Further $a''(\hat{v}, b_{j+1}) > a''(\hat{v}, b_j) = a''(\hat{v}+1, b_j)$.

Next consider the following perturbation of a'' :

1. Reduce the allocation of all types $(\hat{v}, b_{j+1}), \ldots, (\hat{v}, b_k)$ by ϵ each.

2. Reduce the allocation of all types in $\{(v, b) : v > \hat{v}, b \geq b_{j+1}\}$ by $\epsilon/(\hat{v} + 1)$ each.

3. Increase the allocation of type $(\hat{v}, b_j)$ and $(\hat{v} + 1, b_j)$ by $(k - j)\epsilon'$.

4. Increase the allocation of types in $\{(v, b_j) : v > \hat{v}\}$ by $(k - j)\epsilon/(\hat{v} + 1)$.

Pick ϵ, ϵ' to jointly solve:

$$\epsilon'(f(\hat{v}) + f(\hat{v})) = (k - j)\epsilon f(\hat{v})$$
$$a''(\hat{v}, b_{j+1}) - \epsilon = a''(\hat{v}, b_j) + (k - j)\epsilon'$$

Denote the resulting allocation rule a'. By construction, a' feasible with respect to the Border conditions and (weakly) revenue increasing. Further, given the decreasing density assumption; as long as $a'(\hat{v}+1, b_j) \leq a'(\hat{v}+2, b_j)$, a' will satisfy (24- 29) with cutoff $\underline{v}'_j = \underline{v}_j + 1$. If $a'(\hat{v} + 1, b_j) > a'(\hat{v} + 2, b_j)$, this second perturbation will have to be analogously modified- it should be clear how this can be done.

Note that this construction will increase $\underline{v}_j$, and (weakly) decrease $\bar{v}_j$. Therefore it can be continued until $\underline{v}_j \geq \bar{v}_j - 1$, and therefore $\underline{v}_i \geq \bar{v}_i - 1$ for all $i > j - 1$. $\qquad \square$

B.2   SUBSIDIES

This section proves a technical result needed in the proof of Proposition 6

OBSERVATION 4 The function

$$\phi(\pi) = \frac{(1 - F(v - 2))\pi + vf(v - 1)(\frac{k-i+1}{k}(1 - F(v - 1)) - \pi)}{\frac{k-i+1}{k}vf(v - 1) + \pi}$$

is decreasing in π.

PROOF: We are done if we can show that $\phi'(\pi) \leq 0$. Writing $\phi(\pi) = \frac{N(\pi)}{D(\pi)}$ with $N(\cdot), D(\cdot)$ appropriately defined,

$$\phi'(\pi) = \frac{N'(\pi)D(\pi) - D'(\pi)N(\pi)}{D^2(\pi)}.$$

Therefore we are done if we can show that $N'(\pi)D(\pi) - D'(\pi)N(\pi) < 0$. Note that

$$D'(\pi) = 1,$$
$$N'(\pi) = (1 - F(v - 2)) - vf(v - 1).$$

Therefore

$$N'(\pi)D(\pi) - D'(\pi)N(\pi)$$

$$= ((1 - F(v-2)) - vf(v-1))\left(\frac{k-i+1}{k}vf(v-1) + \pi\right)$$

$$\quad - \left((1 - F(v-2))\pi + vf(v-1)\left(\frac{k-i+1}{k}(1 - F(v-1)) - \pi\right)\right)$$

$$= ((1 - F(v-2)) - vf(v-1))\left(\frac{k-i+1}{k}vf(v-1)\right) - \frac{k-i+1}{k}vf(v-1)(1 - F(v-1))$$

$$= (-(v-1)f(v-1))\left(\frac{k-i+1}{k}vf(v-1)\right)$$

$$\leq 0.$$

$\square$