Draft

EXISTENCE OF EQUILIBRIA IN AUCTIONS WITH INTERDEPENDENT VALUES

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1. Introduction

This paper studies a sealed-bid first-price auction of a single item. It introduces a new method to prove existence of a mixed-strategy equilibrium.[1] An indirect construction yields an equilibrium with nonatomic distributions of bids, thereby circumventing discontinuities due to tie-breaking rules.

First one establishes existence of equilibria for an auxiliary game with a novel specification of how bidders select optimal bids when ties are possible. For this auxiliary game there is a well-defined fixed-point problem for which equilibria are the solutions. Moreover, there exist essential sets of fixed points for which every small perturbation of the problem has a nearby fixed point. The second step considers such perturbations derived by perturbing payoffs in the auxiliary game. One then establishes that limit points of equilibria for these perturbed games are equilibria of the auxiliary game that have no atoms in the distributions of players' bids. The probability of tied bids is therefore zero, which implies that these limit equilibria are equilibria of the original auction regardless of the tie-breaking rule. For the proof here it suffices to consider perturbations in which each player anticipates that his submitted bid will be slightly distorted by noise before it is received by the auctioneer.

This method is applied to establish existence of equilibria in behavioral strategies when bidders' values are interdependent.[2] That is, each bidder n observes a private signal s_n, submits a bid $b_n(s_n)$, and then obtains a payoff that is nonzero only if his bid wins, in which case his payoff is $v_n(s_1, s_2, \ldots) - b_n(s_n)$, where his realized value v_n depends on the signals $s_1, s_2, \ldots$ observed by all bidders. Interdependent values occur, for example, when bidders' signals are informative about an unobserved common-value component. The existence theorem, Theorem 4.1 below, assumes that (a) the joint distribution of signals has a positive

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[1]References to the previous journal literature are omitted from this draft.

[2]For the special case of private values, [2] provides a simpler proof of existence.

and continuous density on a hypercube, (b) players' value functions are continuous and nondecreasing over the same domain and range, and (c) a family of conditional expectations of a player's value are strictly increasing in his signal (see Assumption 6 below).

For simplicity the proof in Section 4 addresses explicitly only the case of two bidders who are symmetric *ex ante* and establishes existence of a symmetric equilibrium. The extension to auctions with N asymmetric bidders will be provided in a revised version.

2. The Auction Game

We consider an N-player game G that represents a first-price sealed-bid auction for a single item. In the extensive form of the game, first Nature specifies a profile $s = (s_1, s_2, \dots, s_N)$ of signals, one for each player $n = 1, 2, \dots, N$, according to a distribution F. Then each player n observes his own signal s_n and chooses a bid $b_n(s_n)$. Finally, player n's payoff is $v_n(s) - b_n(s_n)$ if he wins the item and zero otherwise. He wins if his bid is strictly higher than others' bids, or when tied with others' bids, if he is selected by a tie-breaking rule.

We impose the following assumptions on the distribution of signals.

Assumption: [Distribution of Signals]

(1) The set of possible signal profiles is a product set $S = \prod_n S_n$. Each player's set of possible signals is the same real closed interval, say $S_n = [0, 1]$.
(2) The distribution F of signal profiles has a density f that is positive and continuous on S.[3]

For each n, let $\mathcal{S}_n$ be the Borel measurable subsets of S_n and let λ_n be the Lebesgue measure on S_n.

When considering player n, his opponents are denoted m. For each player n and his signal s_n, let $F_n(\cdot|s_n)$ be the conditional distribution of the signals s_m of n's opponents, and let $f_n(\cdot|s_n)$ be its density function. Assumption (2) ensures that player n's conditional densities $f_n(\cdot|s_n) : S_m \to \mathbb{R}$ indexed by his signal $s_n \in S_n$ are an equicontinuous family.

We impose the following assumptions on players' value functions.

Assumption: [Distributions of Values]

(3) The set of possible profiles of realized values is a product set $V = \prod_n V_n$. Each player's set of possible values is the same real closed interval, say $V_n = [v_*, v^*]$, where $v_* < v^*$.
(4) The joint valuation function $v : S \to V$ is continuous.

[3]This assumption can be weakened considerably: the distribution F need only be absolutely continuous with respect to the product of its marginal distributions.

(5) For each player n his value function $v_n : S \to V_n$ is nondecreasing in his opponents' signals s_m.

(6) For each player n and each measurable function $\beta_m : S_m \to [0,1]$ that is nonzero on a set of m's signals with positive measure, the conditional expectation of n's value given the realizations of β_m and n's signal s_n, namely[4]

$$\frac{\int_{S_m} v_n(s_n, s_m)\beta_m(s_m) f_n(s_m|s_n)\, ds_m}{\int_{S_m} \beta_m(s_m) f_n(s_m|s_n)\, ds_m},$$

is strictly increasing in n's signal s_n.

Assumption (6) is stronger than requiring that player n's value v_n is strictly increasing in his signal s_n. It requires that a higher signal implies a higher expected value even if player n conditions on an informative event about his opponents' signals for which β is the likelihood function, as for example when n evaluates his expected payoff conditional on the event that his bid exceeds opponents' bids.

Lastly, the present exposition requires that players are symmetric *ex ante*.

Assumption: [Symmetry of Players]

(6) The distribution function F and the joint valuation function v are symmetric with respect to players.

In view of Assumption (3) it suffices to assume that for each player the feasible set of bids is the interval $B = [v_*, v^*]$, the same as the interval of the player's possible values. Denote the Borel measurable subsets of B by $\mathbf{B}$.

Because the game has perfect recall, we specify a player's strategy in behavioral form as mixtures over bids conditional on his signals. A behavioral strategy for player n is a transition probability function $\sigma_n(\cdot|\cdot) : \mathbf{B} \times S_n \to [0,1]$ such that for each signal s_n, $\sigma_n(\cdot|s_n)$ is a probability measure on B; and for each event $A \in \mathbf{B}$, $\sigma_n(A|\cdot)$ is a measurable function on S_n.[5] Let Σ_n be the set of behavioral strategies of player n.

Endow behavioral strategies with the topology of weak convergence; i.e. a sequence σ_n^k of strategies in Σ_n converges to σ_n iff for every continuous function $\eta : B \to \mathbb{R}$ and each event $O_n \in \mathcal{S}_n$,

$$\int_{O_n} \int_B \eta(b)\, d\sigma_n^k(b|s_n)\, ds_n \to \int_{O_n} \int_B \eta(b)\, d\sigma_n(b|s_n)\, ds_n\,.$$

[4]Here and later the integral is computed using the Lebesgue measure on S_m.

[5]Strictly speaking a behavioral strategy is an equivalence class of transition probability functions, where σ_n is equivalent to σ_n' if $\sigma_n(\cdot|s_n) = \sigma_n'(\cdot|s_n)$ λ_n-a.e. on S_n.

With this topology, Σ_n is a compact (metrizable) space.[6] Instead of the Lebesgue measure λ_n on S_n, if one uses the measure $F_m(\cdot|s_m)$ for some signal $s_m \in S_m$ of player $m \neq n$ then it represents player m's interim belief after receiving his signal s_m. By Assumption (2) on F, for all m and s_m the induced topology on Σ_n is the same.

Remark: It is convenient here to work with behavioral strategies rather than distributional strategies. Note however that given a behavioral strategy σ_n of player n and a signal s_m of player $m \neq n$ there is for player m a well-defined conditional belief that is the distributional strategy $\varsigma_n(\sigma_n, s_m)$ defined on $\mathcal{S}_n \times \mathbf{B}$ by

$$\varsigma_n(\sigma_n, s_m)(O_n \times C) = \int_{O_n} \sigma_n(C|s_n) f_m(s_n|s_m)\, ds_n\,,$$

where $O_n \times C$ is the event that n's signal $s_n \in O_n \in \mathcal{S}_n$ and n's bid is in $C \in \mathbf{B}$. The family of distribution functions ς_n is continuous on $\Sigma_n \times S_m$; in particular, the expectation w.r.t. $\varsigma_n(\sigma_n, s_m)$ of any real-valued continuous (or u.s.c. or l.s.c) function on $S_n \times B$ is continuous (or u.s.c. or l.s.c., respectively) w.r.t. (σ_n, s_m). •

Say that a bid $b \in B$ is a point of continuity of n's strategy σ_n if $\sigma_n(\{\, b \,\}|s_n)$ is zero λ_n-a.e. on S_n, i.e. for almost no signal does n bid b with positive probability. The set of bids that are not points of continuity of σ_n is countable. Moreover, if σ_n^k is a sequence of strategies converging to σ_n and b is a point of continuity of σ_n then, since the indicator function for b is u.s.c., $\int_{S_n} \sigma_n^k(\{b\}|s_n)\, ds_n$ converges to zero. Therefore there exists a subsequence such that $\sigma_n^k(\{\, b \,\}|\cdot)$ converges λ_n-a.e. on S_n to zero.

For the auction game G the expected payoff to a player n when his signal is s_n, he bids b_n, and m uses strategy $\sigma_m \in \Sigma_m$ is

$$\pi_n(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b][\sigma_m([v_*, b_n)|s_m) + \frac{1}{N}\sigma_m(\{\, b_n \,\}|s_m)] f_n(s_m|s_n)\, ds_m\,.$$

Consistent with the assumed symmetry of the players, the coefficient $\frac{1}{N}$ assumes that the tie-breaking rule gives the players equal chances of winning in the event of a tie at the bid b_n, but actually the tie-breaking rule has no role in the sequel.

Hereafter we assume there are only two players, i.e. $N = 2$. When referring to player n, his opponent is player m.

[6]This definition is weaker than requiring (a.e. on S_n) pointwise weak convergence of the sequence of distributions $\sigma_n^k(\cdot|s_n)$. An equivalent definition requires that the displayed integral be u.s.c. (resp. l.s.c.) for each function η that is u.s.c. (resp. l.s.c.).

3. The Auxiliary Game

This section specifies an auxiliary game G^* that is exactly the same as the auction game G specified in Section 2 except for a change in the way a player responds to the possibility of tied bids.

Before specifying the game G^*, first define for each player n the following alternative payoff functions. For each $(s_n, b_n, \sigma_m) \in S_n \times B \times \Sigma_m$,

$$\hat{\pi}_n(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b_n]\sigma_m([v_*, b_n)|s_m) f_n(s_m|s_n)\, ds_m \,,$$

$$\bar{\pi}_n^+(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b_n]^+ \sigma_m(\{\, b_n \,\}|s_m) f_n(s_m|s_n)\, ds_m \,,$$

$$\bar{\pi}_n^-(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b_n]^- \sigma_m(\{\, b_n \,\}|s_m) f_n(s_m|s_n)\, ds_m \,.$$

Note that $\hat{\pi}_n(s_n, b_n, \sigma_m)$ envisions winning if n's bid b_n strictly exceeds m's bid, whereas $\bar{\pi}_n^+(s_n, b_n, \sigma_m)$ and $\bar{\pi}_n^-(s_n, b_n, \sigma_m)$ envision only the event that n's bid b_n is tied with m's bid. The latter two differ according to whether n receives the positive or negative part of the payoff $v_n(s_n, s_m) - b_n$, corresponding to the best and worst scenarios.

Now define $\pi_n^+ = \hat{\pi}_n + \bar{\pi}_n^+$ and $\pi_n^- = \hat{\pi}_n + \bar{\pi}_n^-$. Then π_n^+ and π_n^- represent the best and worst payoffs from ties. Note that if m's strategy σ_m does not generate any atoms in the distribution of m's bids then $\bar{\pi}_n^+ = \bar{\pi}_n^- = 0$ and therefore both π_n^+ and π_n^- agree with n's payoff function π_n in the auction game G. The following lemma establishes the continuity properties of the best and worst payoff functions π_n^+ and π_n^-.

Lemma 3.1. *The payoff functions π_n^+ and $\pi_n^- : S_n \times B \times \Sigma_m \to \mathbb{R}$ are upper and lower semi-continuous, respectively.*

Proof. Let $(s_n^k, b_n^k, \sigma_m^k)$ be a sequence converging to (s_n, b_n, σ_m). Let S_m^0 be the set of signals s_m^0 of m such that $v_n(s_n, s_m^0) = b_n$. S_m^0 is a closed interval $[\underline{s}_m^0, \overline{s}_m^0]$ since n's value v_n is continuous and nondecreasing in s_m. Fix $\varepsilon > 0$. Take the ε interval around S_m^0 and choose $0 < \delta \leqslant \varepsilon$ such that for k large enough, b_n^k belongs to the interval $b_n \pm \delta$ and $v_n(s_n^k, s_m) - b_n + \delta$ is negative if $s_m \leqslant \underline{s}_m^0 - \varepsilon$ and positive if $s_m \geqslant \overline{s}_m^0 + \varepsilon$. Then

$$\begin{aligned} \pi_n^+(s_n^k, b_n^k, \sigma_m^k) \;\leqslant\; & \int_{s_m < \underline{s}_m^0 - \varepsilon} [v_n(s_n^k, s_m) - b_n + \delta]\sigma_m^k([v_*, b_n - \delta)|s_m) f_n(s_m|s_n^k)\, ds_m \\ +\; & \int_{s_m \geqslant \overline{s}_m^0 + \varepsilon} [v_n(s_n^k, s_m) - b_n + \delta]\sigma_m^k([v_*, b_n + \delta]|s_m) f_n(s_m|s_n^k)\, ds_m \\ +\; & \int_{s_m^0 \pm \varepsilon} [v_n(s_n^k, s_m) - b_n + \delta)]^+ \sigma_m^k([v_*, b_n + \delta]|s_m) f_n(s_m|s_n^k)\, ds_m \,. \end{aligned}$$

Going to the limit, one sees that $\limsup_k \pi_n^+(s_n^k, b_n^k, \sigma_m^k)$ on the left side is no more than the right side of the above inequality with $(s_n^k, b_n^k, \sigma_m^k)$ replaced by its limit. Now choose a sequence of ε's and thus δ's decreasing to zero. Then one obtains $\pi_n^+(s_n, b_n, \sigma_m)$ as the limit of the right-hand side, which proves that π_n^+ is upper semi-continuous. The proof of the lower semi-continuity of π_n^- is similar. □

Now define the auxiliary game G^* as follows. Given m's strategy σ_m, say that n's bid b_n is an optimal reply when his signal is s_n if $\pi_n^+(s_n, b_n, \sigma_m) \geqslant \pi_n^-(s_n, c, \sigma_m)$ for every bid $c \in B$. That is, the best payoff from b_n must be as good as the worst payoff from any other bid.

For each (s_n, σ_m) let $\phi_n(s_n, \sigma_m)$ be the set of player n's optimal replies to σ_m when n's signal is s_n.

Lemma 3.2. *The correspondence $\phi_n : V_n \times \Sigma_m \twoheadrightarrow B$ is upper semi-continuous and has nonempty and compact images.*

Proof. For each signal s_n of player n and strategy σ_m of m, the function π_n^+ is upper semi-continuous in n's bid b and hence attains a maximum over B. Any maximizer of π_n^+ is trivially an optimal reply and hence ϕ_n has nonempty images. The other two properties follow from the fact that π_n^+ is u.s.c. while π_n^- is l.s.c. □

Let $\Phi_n : \Sigma_m \twoheadrightarrow \Sigma_n$ be the correspondence that assigns to each strategy σ_m of m the set of n's strategies σ_n such for each signal $s_n \in S_n$ the support of $\sigma_n(\cdot|s_n)$ is a nonempty subset of $\phi_n(s_n, \sigma_m)$. Then Φ_n is an upper semi-continuous correspondence with nonempty, compact, and convex images. And so too is the optimal-reply correspondence $\Phi^* : \Sigma \twoheadrightarrow \Sigma$ obtained as the product of Φ_n and Φ_m. The Fan-Glicksberg fixed-point theorem therefore implies that Φ^* has a fixed point. Hence the auxiliary game G^* has an equilibrium. The symmetry of the players implies further that there is a symmetric fixed point and thus a symmetric equilibrium of G^*.

An equilibrium of G^* is not necessarily an equilibrium of the auction game G. However, those equilibria of G^* with nonatomic distributions of bids are equilibria of G.

4. Existence of Equilibria for the Auction Game

This section establishes that a symmetric equilibrium exists for the auction game G, regardless of the tie-breaking rule. This is done by showing that the auxiliary game G^* has a symmetric equilibrium with nonatomic bid distributions that is then a symmetric equilibrium of the auction game G.

Theorem 4.1. *The auction game G has a symmetric equilibrium. In particular, it has a symmetric equilibrium with no atoms in the distribution of any player's bids.*

The remainder of this section is devoted to the proof of this existence theorem. Throughout, by a fixed point or an equilibrium we mean a symmetric one. One begins with the observation that the optimal-reply correspondence Φ^* defined above has essential sets of fixed points, i.e. a set such that every sufficiently small perturbation of Φ^* has a fixed point arbitrarily close to the set. To exploit this property we construct a sequence of perturbed auxiliary games G^k converging to G^* that induce a sequence of perturbed correspondences Φ^k converging to Φ^*. We then show that limit points of equilibria of the perturbed games, obtained as fixed points of the perturbed correspondences, have nonatomic bid distributions. Hence, these limits points are equilibria of both G^* and the auction game G. The perturbed games G^k are obtained simply by supposing that a player's bid is distorted by noise before it is received by the auctioneer.

The sequence of perturbed games G^k is constructed as follows. For each positive integer k, the strategy sets in the game G^k are the same as in the auction game G and in the auxiliary game G^*. However, when player n bids b the auctioneer perceives n's bid as the sum of b and a random variable uniformly distributed on the interval $[-1/k, 1/k]$. Thus, the payoff functions are defined as follows. First, for each bid $b \in B$ let μ_b^k be the uniform distribution over the interval $[b-1/k, b+1/k]$. Next, for each strategy σ_m^k of player m, define a transition probability distribution $\tilde{\sigma}_m^k$ from S_m to $B^k \equiv [v_* - 1/k, v^* + 1/k]$ via its distribution function

$$\tilde{\sigma}_m^k([v_* - 1/k, b]|s_m) = \int_B \int_{v_*-1/k}^{b} d\mu_c^k(b')\, d\sigma_m^k(c|s_m)\,.$$

Then $\tilde{\sigma}_m^k([v_* - 1/k, b]|s_m)$ is the probability that the bid from m received by the auctioneer is no more than b, given that bids are subject to noise. Finally, the payoff to player n when his signal is s_n, he bids $b_n \in B$, and his opponent plays the strategy σ_m is

$$\pi_n^k(s_n, b, \sigma_m) = \int_{B^k} \pi_n^+(s_n, c, \tilde{\sigma}_m^k)\, d\mu_b^k(c)\,,$$

where the domain of $\pi_n^+(s_n, c, \sigma_m)$ is extended now to include bids in $B^k \setminus B$ in the obvious way.[7]

The payoff function π_n^k is continuous. As in the previous section, therefore, the correspondence ϕ_n^k that assigns to each (s_n, σ_m) the set of bids that are optimal for n when his signal is s_n in reply to m's strategy σ_m in game G^k is an u.s.c. correspondence with nonempty and compact images. The induced optimal-reply correspondence $\Phi^k : \Sigma \twoheadrightarrow \Sigma$ satisfies the conditions for existence of a fixed point, which is then an equilibrium of G^k.

[7]In this definition, one could equivalently replace π_n^+ with π_n^- or, indeed, any function that agrees with π_n^+ at bids that are points of continuity of σ_m.

Select a subsequence of k's diverging to infinity for which a sequence σ^k of equilibria of the perturbed games G^k converges to some strategy profile $\sigma^* \in \Sigma$. Lemmas 4.4 and 4.5 below establish first that σ^* is an equilibrium of the auxiliary game G^*, and then that it is also an equilibrium of the auction game G because the probability of tied bids is zero. Preceding these are two preliminary lemmas that establish bounds on the equilibrium payoffs in G^k and G^*, and continuity of players' equilibrium payoff functions as functions of their signals. In the course of these lemmas, convergent subsequences are selected to ensure regularity properties.

Lemma 4.2. *Let (s_n^k, b_n^k) be a subsequence converging to (s_n, b_n). Then:*

$$\pi_n^+(s_n, b_n, \sigma_m^*) \geqslant \limsup_k \pi_n^k(s_n^k, b_n^k, \sigma_m^k) \geqslant \liminf_k \pi_n^k(s_n^k, b_n^k, \sigma_m^k) \geqslant \pi_n^-(s_n, b_n, \sigma_m^*)\,.$$

Proof. Fix any small $\varepsilon > 0$. Since π_n^+ is upper semi-continuous and $(s_n^k, b_n^k, \sigma_m^k)$ converges to (s_n, b_n, σ_m^*), there exists K large enough such that for all $k \geqslant K$, $\pi_n^+(s_n^k, b_n', \sigma_m^k) < \pi_n^+(s_n, b_n, \sigma_m^*) + \varepsilon$ for all $b_n' \in [b_n - 1/K, b_n + 1/K]$. The payoff $\pi_n^k(s_n^k, b_n^k, \sigma_m^k)$ is obtained as the average of values of $\pi_n^+(s_n^k, b_n', \sigma_m^k)$ for b_n' in a smaller interval, so when k is very large this implies that $\pi_n^+(s_n, b_n, \sigma_m^*) + \varepsilon \geqslant \limsup_k \pi_n^k(s_n^k, b_n^k, \sigma_m^k)$. Since ε was arbitrary, this establishes the first of the claimed inequalities. The proof for the inequality involving π_n^- is similar. □

For each k and each player n let $\theta_n^k : S_n \to \mathbb{R}$ be the function that assigns to each signal s_n the corresponding equilibrium payoff of player n from the equilibrium σ^k of game G^k.

Lemma 4.3. *The family of functions θ_n^k is bounded and equicontinuous, and has a subsequence that converges to a continuous function $\theta_n^* : S_n \to \mathbb{R}$.*

Proof. For each signal s_n, player n's equilibrium payoff is clearly in the interval $[-1/k, v^* - v_* + 1/k]$. Therefore, the θ_n^k's are uniformly bounded. Equicontinuity follows from the continuity of v_n and the equicontinuity property for the conditional densities $f_n(\cdot|s_n)$. Since the θ_n^k's are both bounded and equicontinuous, they are totally bounded; hence there is a Cauchy subsequence that is convergent. □

Assume hereafter that for each n the sequence θ_n^k is itself the Cauchy subsequence identified in the above lemma. The next lemma establishes that σ^* is an equilibrium of the auxiliary game G^*.

Lemma 4.4. *For a.e. signal s_n of player n, every bid b in the support of $\sigma_n^*(\cdot|s_n)$ is an optimal reply to σ_m^* in the auxiliary game G^*.*

Proof. Let X_n be the set of $(s_n, b) \in S_n \times B$ such that $\pi_n^+(s_n, b, \sigma_m^*) \geqslant \theta_n^*(s_n)$. By the upper semi-continuity of π_n^+ and the continuity of θ_n^*, X_n is a closed set. We claim that for each $(s_n, b) \notin X_n$, there exists an open neighborhood $W_n \times C$ such that $\int_{W_n} \sigma_n^*(C|s_n)\, ds_n = 0$. This suffices to prove the lemma since $\pi_n^-(s_n, c, \sigma_m^*) \leqslant \theta_n^*(s_n)$ by Lemma 4.2. To prove this claim, choose any open neighborhood $W_n \times C$ of (s_n, b) such that its closure is contained in $(S_n \times B) \setminus X_n$. We assert that for all large k, $\pi_n^k(s_n', c, \sigma_m^k) < \theta_n^k(s_n')$ for all $(s_n', c) \in W_n \times C$. Indeed, otherwise there exists a sequence (s_n^k, c_n^k) in $W_n \times C$ converging to some (s_n', c_n) in the closure of $W_n \times C$ and such that $\pi_n^k(s_n^k, c_n^k, \sigma_m^k) = \theta_n^k(s_n')$ for all k, and then $\lim_k \pi_n^k(s_n^k, c_n^k, \sigma_m^k) = \theta_n^*(s_n')$. But that is impossible since, by Lemma 4.2, $\lim_k \pi_n^k(s_n^k, c_n^k, \sigma_m^k) \leqslant \pi_n^+(s_n', c_n, \sigma_m^*)$, while by construction, $\pi_n^+(s_n', c_n, \sigma_m^*) < \theta_n^*(s_n')$. Therefore, for all large k, $\pi_n^k(s_n', c, \sigma_m^k) < \theta_n^k(s_n')$ for all $(s_n', c) \in W_n \times C$ as asserted. Since the bids in C are suboptimal against σ_m^k for signals in W_n for all large k, it follows that $\int_{W_n} \sigma_n^k(C|s_n)\, ds_n = 0$ for all such k. By the lower semi-continuity of the indicator function on the open set C, $\int_{W_n} \sigma_n^*(C|s_n)\, ds_n = 0$, which proves the claim. □

We now prove the key lemma, which asserts that in the equilibrium σ^* of G^* each bidder's strategy induces a nonatomic distribution of his bids.

Lemma 4.5. *Every bid b is a point of continuity of σ_m^*, i.e. $\int_{S_m} \sigma_m^*(\{ b \}|s_m)\, ds_m = 0$.*

Proof. Fix a bid $b \in B$. For each player m let $W_m \subset S_m$ be the set of signals s_m such that $\sigma_m^*(\{ b \}|s_m) > 0$. Suppose to the contrary that $\int_{W_m} \sigma_m^*(\{ b \}|s_m)\, ds_m \equiv \eta > 0$ for some player m. Since we are considering a symmetric equilibrium, the same holds for his opponent n as well. For each bid c, and for each $j = k$ or $*$ define the bid distribution functions

$$H^j(c) = \int_{S_m} \sigma_m^j([v_*, c]|s_m)\, ds_m \qquad \text{and} \qquad H^j(c^-) = \int_{S_m} \sigma_m^j([v_*, c)|s_m)\, ds_m\,.$$

In particular, $H^*(b) = H^*(b^-) + \eta$. Also for each k define the expectation

$$\bar{H}^k(c) = \int_{B^k} H^k(c')\, d\mu_c^k(c')\,.$$

Based on the Lebesgue measure of signals, H^j is the distribution function of bids by player m when he plays the strategy σ_m^j, and $\bar{H}^k$ is the expectation of H^k when m's received bids are perturbed via the k-th uniform distribution. Clearly $\bar{H}^k$ is a continuous function on B^k.

We now study the supposed atom of σ_m^* of size η at b by zooming in on small intervals around it that shrink as $k \to \infty$. On these intervals we study the equilibria σ^k of the perturbed games G^k to establish that their limit, the equilibrium σ^* of G^*, has no atoms in the distribution of bids. The gist of the proof is to show that if k is large then player

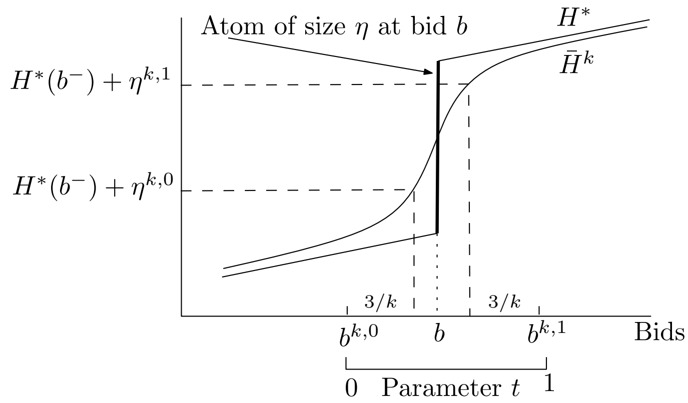

FIGURE 1

n's optimal replies avoid bidding b, instead preferring to bid more or less than b. For the following construction it may be helpful to refer to the illustrative example in Figure 1.

Consider a sequence η^k in the interior of the unit square. Represent each η^k as the pair $(\eta^{k,0}, \eta^{k,1})$. Let $\tilde{b}^{k,0}$ be the highest bid c such that $\bar{H}^k(c) = H^*(b^-) + \eta^{k,0}$, and similarly $\tilde{b}^{k,1}$ is the lowest bid c such that $\bar{H}^k(c) = H^*(b^-) + \eta^{k,1}$. Hereafter suppose that the sequence is such that η^k converges to $(0, \eta)$ and both $\tilde{b}^{k,0}$ and $\tilde{b}^{k,1}$ converge to b. Define $b^{k,0} = \tilde{b}^{k,0} - 3/k$ and $b^{k,1} = \tilde{b}^{k,1} + 3/k$, which also converge to b.

Define $\delta^k = b^{k,1} - b^{k,0}$ and $\varepsilon^k = 1/(k\delta^k)$. Then $\delta^k > 6/k$ and $0 < \varepsilon^k < 1/6$. Now select a subsequence such that ε^k converges to some limit point ε^* in $[0, 1/6]$. Observe that $\varepsilon^* = 0$ iff $k[b^{k,1} - b^{k,0}]$ diverges to infinity. Define the intervals $T^k = [2\varepsilon^k, 1 - 2\varepsilon^k]$ and $T^* = [2\varepsilon^*, 1 - 2\varepsilon^*]$.

Next rescale bids in the interval $[b^{k,0}, b^{k,1}]$, using instead the parameter $t \in T \equiv [0, 1]$. For each k, define the linear function $\zeta^k : T \to [b^{k,0}, b^{k,1}]$ via $\zeta^k(t) = b^{k,0} + \delta^k t$. In the following a bid $\hat{b}$ in the interval $[b^{k,0}, b^{k,1}]$ is represented by the parameter $\hat{t}$ for which $\hat{b} = \zeta^k(\hat{t})$ and we refer to $\hat{b}$ and $\hat{t}$ interchangeably.

For a bid parameter $t \in [\varepsilon^k, 1 - \varepsilon^k]$, the bid $\zeta^k(t)$ is in the interval $[b^{k,0} + 1/k, b^{k,1} - 1/k]$. Recall that if the bid $\zeta^k(t)$ is chosen then the bid received by the auctioneer is $\zeta^k(t)$ plus a noise term that is uniformly distributed on the interval $[-1/k, 1/k]$. Hence the uniform distribution of received bids in the interval $[\zeta^k(t) - 1/k, \zeta^k(t) + 1/k]$ induces via ζ^k a distribution ν_t^k over corresponding parameters t' in T, where the distribution ν_t^k is uniform over the interval $[t - \varepsilon^k, t + \varepsilon^k]$. If $t \in [0, \varepsilon^k]$, then define ν_t^k to be the distribution that places probability $(2\varepsilon^k)^{-1}(\varepsilon^k - t)$ on zero and assigns the rest of the probability uniformly on $(0, t + \varepsilon^k]$. The distribution ν_t^k for $t \in [1 - \varepsilon^k, 1]$ is similarly defined, by having a mass point at 1. Hereafter, ν_t^k plays the role of the distribution of the received bid t' conditional on the chosen bid t. For each $t \in T$, as k goes to infinity one obtains a well-defined limit distribution ν_t^* that is uniform over the interval $t \pm \varepsilon^*$ if $t \in I^*$.

The following definitions employ this rescaling.

For each k, from m's strategy σ_m^k construct the induced transition function $\tau_m^k : T \times S_m \to [0,1]$ via its conditional distribution function

$$\tau_m^k([0,t]|s_m) = \sigma_m^k([\zeta^k(0), \zeta^k(t)]|s_m)\,,$$

which is the conditional probability given his signal s_m that m's bid parameter is in the interval $[0,t]$. Also let $\bar{\tau}_m^k(\cdot|s_m) = \left(\tau_m^k(T|s_m)\right)^{-1}\tau_m^k(\cdot|s_m)$ be the normalized probability when $\bar{\tau}_m^k(T|s_m) > 0$, and otherwise let it be zero. Then $\bar{\tau}_m^k$ is a probability transition function on S_m, with the measure on S_m that assigns to each S'_m, the probability $\tau_m^k(T|S'_m)/\tau_m^k(T|S_m)$. Select a further subsequence of the k's such that m's induced bid distribution τ_m^k and the normalized version $\bar{\tau}_m^k$ converge to limits τ_m^* and $\bar{\tau}_m^*$ respectively.

Observe that $\int_{S_m} \tau_m^k([2\varepsilon^k, 1-3\varepsilon^k]|s_m)\, ds_m \geqslant \eta^{k,1} - \eta^{k,0}$, and since $\eta^{k,1} - \eta^{k,0}$ converges to η, also $\int_{S_m} \tau_m^*([2\varepsilon^*, 1-3\varepsilon^*]|s_m)\, ds_m \geqslant \eta$. This follows from the fact that for each k, $H^k(\tilde{b}^{k,0} - 1/k) \leqslant H^*(b^-) + \eta^{k,0}$ (since otherwise $\bar{H}^k(\tilde{b}^{k,0}) > H^*(b^-) + \eta^{k,0}$) and $H^k(\tilde{b}^{k,1}) \geqslant H^*(b^-) + \eta^{k,1}$.

For each $j = k$ or $*$, each parameter $\tilde{t} \in [\varepsilon^j, 1-\varepsilon^j]$, and each signal $s_m \in S_m$, define the conditional probability that m's received bid parameter is in the interval $[0,\tilde{t}]$ by its conditional distribution function:

$$\tilde{\tau}_m^j([0,\tilde{t}]|s_m) = \int_{[0,1]} \int_0^{\tilde{t}} d\nu_t(t')\, d\tau_m^j(t|s_m)\,.$$

Also define for each bid parameter $t \in T^j = [2\varepsilon^j, 1-2\varepsilon^j]$ and m's signal $s_m \in S_m$ the conditional expectation of $\tilde{\tau}_m^j$ with respect to the induced noise distribution ν_t^j of player n:

$$\hat{\tau}_m^j(t|s_m) = \int_{[0,1]} \tilde{\tau}_m^j([0,t']|s_m)\, d\nu_t^j(t')\,.$$

Note that $\tilde{\tau}_m^j([0,t]|s_m)$ and $\hat{\tau}_m^j(t|s_m)$ are weakly increasing in t for each signal s_m, since $\tau_m^j([0,t]|s_m)$ has that property and ν_t^j is a family of distributions for which higher t implies first-order stochastic dominance.

Observe that from player n's perspective, in the perturbed game G^k, if he chooses t and thus the bid $\zeta^k(t)$ then his probability of winning when m's signal is s_m and chooses a bid parameter in T is $\hat{\tau}_m^k(t|s_m)$. (Because of the noise, the probability is zero that both players' received bid parameters are any $t' \in T$.)

Finally, for each k and each pair (s_n, s_m) of the players' signals, denote by $p^k_n(s_n, s_m)$ the probability that n wins when he chooses a bid in $T^k = [2\varepsilon^k, 1-2\varepsilon^k]$ given his signal s_n and m chooses a bid parameter in T given his signal s_m. That is,

$$p^k_n(s_n, s_m) = \int_{T^k} \hat{\tau}^k_m(t|s_m)\, d\tau^k_n(t|s_n)\,.$$

Then

$$\prod_n \tau^k_n([2\varepsilon^k, 1-2\varepsilon^k]|s_n) \leqslant \sum_n p^k_n(s_n, s_m) \leqslant \prod_n \tau^k_n(T|s_n)\,.$$

Define $\bar{p}^k_n(s_n, s_m) \equiv \left(p^k_n(s_n, s_m) + p^k_m(s_m, s_n)\right)^{-1} p^k_n(s_n, s_m)$.

Analogously, for each pair of measurable subsets $(\tilde{S}_n, \tilde{S}_m)$ of the players' signals, define the probability

$$p^k_n(\tilde{S}_n, \tilde{S}_m) = \int_{\tilde{S}_n \times \tilde{S}_m} p^k_n(s_n, s_m)\, ds_n\, ds_m\,,$$

and $\bar{p}^k_n(\tilde{S}_n, \tilde{S}_m) \equiv \left(p^k_n(\tilde{S}_n, \tilde{S}_m) + p^k_m(\tilde{S}_m, \tilde{S}_n)\right)^{-1} p^k_n(\tilde{S}_n, \tilde{S}_m)$.

The remainder of the proof is broken into five steps. Step 1 shows that the limit of the parameterized version accurately represents a player's limit strategy σ^*_n.

Step 1. We claim that when viewed as sequences in $L_\infty(S_n, \lambda_n)$, the space of bounded λ_n-measurable functions on S_n, in the weak*-topology: (a) $\tau^k_n(T|\cdot)$ converges to $\sigma^*_n(\{\, b\,\}|\cdot)$, and (b) $\sigma^k_n([v_*, b^{k,0})|\cdot)$ converges to $\sigma^*_n([v_*, b)|\cdot)$. To prove this, given a measurable subset $\tilde{S}_n$ of S_n, let $\eta(\tilde{S}_n)$ be $\int_{\tilde{S}_n} \sigma^*_n(\{\, b\,\}|s_n)\, ds_n$ and let $\eta^-(\tilde{S}_n) = \int_{\tilde{S}_n} \sigma^*_n([v_*, b)|s_n)\, ds_n$. For each $\varepsilon > 0$, choose bids $\underline{b}, \overline{b}$ that are points of continuity of σ^*_n such that $\underline{b} < b < \overline{b}$ and such that

$$\int_{\tilde{S}_n} \sigma^*_n([v_*, \underline{b}]|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n) - \varepsilon \quad \text{and} \quad \int_{\tilde{S}_n} \sigma^*_n([v_* \overline{b}]|s_n)\, ds_n \leqslant \eta^-(\tilde{S}_n) + \eta(\tilde{S}_n) + \varepsilon\,.$$

For all large k, $\underline{b} < b^{k,0}$ and $b^{k,1} < \overline{b}$. Therefore,

$$\limsup_k \int_{\tilde{S}_n} \tau^k_n(T|s_n)\, ds_n \leqslant \eta(\tilde{S}_n) + 2\varepsilon \quad \text{and} \quad \liminf_k \sigma^k_n([v_*, b^{k,0})|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n) - \varepsilon\,.$$

Since ε was arbitrary,

$$\limsup_k \int_{\tilde{S}_n} \tau^k_n(T|s_n)\, ds_n \leqslant \eta(\tilde{S}_n) \quad \text{and} \quad \liminf_k \sigma^k_n([v_*, b^{k,0})|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n)\,.$$

For an arbitrary $\varepsilon' > 0$ and all large k,

$$\int_{S_n} \tau^k_n([2\varepsilon^k, 1-3\varepsilon^k]|s_n)\, ds_n \geqslant \eta - \varepsilon'/2\,,$$

while

$$\int_{S_n\setminus\tilde{S}_n} \tau_n^k(T|s_n)\,ds_n \leqslant \int_{S_n\setminus\tilde{S}_n} \tau_n^*(T|s_n)\,ds_n + \varepsilon'/2\,.$$

Therefore, $\int_{\tilde{S}_n} \tau_n^k([\varepsilon^k, 1-\varepsilon^k]|s_n)\,ds_n \geqslant \eta(\tilde{S}_n) - \varepsilon'$. Thus,

$$\lim_k \int_{\tilde{S}_m} \tau_n^k([\varepsilon^k, 1-\varepsilon^k]|s_n)\,ds_n = \int_{\tilde{S}_m} \sigma_n^*(\{\,b\,\}|s_n)\,ds_n\,.$$

Finally, for the ε chosen at the beginning of this step, for large k

$$\int_{\tilde{S}_n} [\sigma_n^k([v_*, b^{k,0})|s_n) + \tau_n^k(T|s_n)]\,ds_n \leqslant \eta^-(\tilde{S}_n) + \eta(\tilde{S}_n) + \varepsilon\,.$$

Hence, using the fact that τ_n^k converges to τ_n^*,

$$\limsup_k \int \sigma_n^k([v_*, b^{k,0})|s_n)\,ds_n \leqslant \eta^-(\tilde{S}_n) + \varepsilon\,.$$

Therefore, since ε was arbitrary,

$$\limsup_k \int \sigma_n^k([v_*, b^{k,0})|s_n)\,ds_n \leqslant \eta^-(\tilde{S}_n)\,.$$

As seen above, $\liminf_k \sigma_n^k([v_*, b^{k,0})|s_n)\,ds_n \geqslant \eta^-(\tilde{S}_n)$, so this completes Step 1. •

Recall that $W_n \subset S_n$ is the set of n's signals s_n such that $\sigma_n^*(\{\,b\,\}|s_n) > 0$ and by hypothesis W_n has positive measure η. From Step 1 it follows that $\tau_n^*(T|s_n) = 0$ for a.e. $s_n \in S_n \setminus W_n$, while $\tau_n^*([2\varepsilon^*, 1-3\varepsilon^*]|s_n) = \sigma_n^*(\{\,b\,\}|s_n)$ for a.e. $s_n \in W_n$. Also, by selecting a convergent subsequence, p_n^k converges to a function p_n^* and $\bar{p}_n^k$ converges to $\bar{p}_n^*$, with the property that for each pair $(\tilde{S}_n, \tilde{S}_m)$,

$$\prod_n \int_{\tilde{S}_n} \tau_n^*([2\varepsilon^*, 1-3\varepsilon^*]|s_n)\,ds_n = \sum_n p_n^*(\tilde{S}_n, \tilde{S}_m).$$

To prepare for the next steps, for each k, signal s_n, and a probability distribution μ over T^k, define

$$\rho_n^k(s_n, \mu) = \int_{[0,1]} \int_{S_m} [v_n(s_n, s_m) - b]\hat{\tau}_m^k(t|s_m) f_n(s_m|s_n)\,ds_m\,d\mu(t)\,,$$

which is approximately the portion of n's expected payoff from bidding close to b when his signal is s_n and his bid is matched against bids of his opponent that are also close to b. When n chooses the strategy $\bar{\tau}_n^k(\cdot|s_n)$, then this payoff is:

$$\rho_n^k(s_n, \bar{\tau}_n^k(\cdot|s_n)) = \big(\tau_n^k(T|s_n)\big)^{-1} \int_{S_m} [v_n(s_n, s_m) - b] p_n^k(s_n, s_m) f_n(s_m|s_n)\,ds_m.$$

Similarly, for a a distribution μ over T, define

$$\rho_n^*(s_n,\mu) = \sup_{(s_n^k,\mu^k)\to(s,\mu)} \limsup_k \rho_n^k(s_n^k,\mu^k).$$

Because of the equicontinuity of f and the continuity of v, we could take the sequence s_n^k in the above definition to be the constant sequence of s_n.

Finally, let

$$\xi_n^*(s_n) = \int_{S_m} [v(s_n,s_m) - b]\sigma_m^*([v_*,b)|s_m) f_n(s_m|s_n)\, ds_m,$$

which is n's expected payoff based on m's limit strategy σ_m^* from bidding b and winning only if m bids less.

Define $\vartheta_n^*(s_n) = \theta_n^*(s_n) - \xi_n^*(s_n)$. Thus $\vartheta_n^*(s_n)$ is the amount by which the equilibrium payoff $\theta_n^*(s_n)$ exceeds the payoff from bidding b and winning only if m's bid is strictly less. Hence $\vartheta_n^*(s_n)$ is n's incremental payoff from winning with the bid b when matched with the same bid b by m. Steps 2 and 3 show that the incremental payoff $\vartheta_n^*(s_n)$ bounds the limit $\rho_n^*(s_n,\mu)$ of his payoff from bids close to b when his signal is s_n and his bid is matched against m's bids close to b, and show that in fact these two payoffs are equal and positive if t is in the support of his limit strategy $\tau_n^*(\cdot|s_n)$ for those signals $s_n \in W_n$ for which n bids b with positive probability in the equilibrium σ^*.

Step 2. We claim that $\vartheta_n^*(s_n) \geqslant \rho_n^*(s_n,\mu)$ for each signal s_n and μ. More specifically, if (s_n^k,μ^k) is a sequence converging to (s_n,μ^k), then $\limsup \rho_n^k(s_n^k,\mu^k) \leqslant \vartheta_n^*(s_n)$; and this inequality holds as an equality if, letting $\zeta^k(\mu^k)$ be the distribution over bids induced by μ^k via ζ^k, $\pi_n^k(s_n^k,\zeta^k(\mu^k)) = \theta_n^k(s_n)$—which is true, for e.g., if s_n^k is generic and the support of μ^k is contained in the support of $\tau_n^k(\cdot|s_n^k)$. To prove this claim, consider a sequence (s_n^k,μ_n^k) converging to (s_n,μ_n). Analogous to the definition of ξ_n^* above, define

$$\xi_n^k(s_n^k) = \int_{S_m} [v_n(s_n^k,s_m) - b]\sigma_m^k([v_*,b^{k,0})|s_m) f_n(s_m|s_n^k)\, ds_m\,.$$

Since $\sigma_m^k([v_*,b^{k,0})|s_m)$ converges to $\sigma_m^*([v_*,b)|s_m)$, $\xi_n^k(s_n^k)$ converges to $\xi_n^*(s_n)$.

The difference $\pi_n^k(s_n^k,\zeta^k(\mu^k),\sigma_m^k) - \xi_n^k(s_n^k) - \rho_n^k(s_n^k,\mu^k)$ is

$$\int_{T^k}\int_{S_m}\int_{[0,1]} [b-\zeta^k(t')][\sigma_m^k([v_*,b^{k,0})|s_m) + \tilde{\tau}_m^k([0,t')|s_m)] f_n(s_m|s_n^k)\, d\nu_t(t')\, ds_m\, d\mu(t).$$

As k goes to infinity, $\zeta^k(t')$ converges to b for all $t' \in [0,1]$ and hence this difference converges to zero. Therefore, for each s_n and t, since $\pi_n^k(s_n^k,\zeta^k(\mu^k),b) \leqslant \theta_n^k(s_n^k)$, it follows that $\xi_n^*(s_n) + \limsup_k \rho_n^k(s_n^k,\mu^k) \leqslant \theta_n^*(s_n) = \xi_n^*(s_n) + \vartheta_n^*(s_n)$ and this inequality holds with equality under the given conditions as well. •

Step 3. For a.e. $s_n \in W_n$:

$$0 < \vartheta_n^*(s_n) = \big(\tau_n^*(T|s_n)\big)^{-1} \left(\int_{S_m} [v_n(s_n, s_m) - b] p_n^*(s_n, s_m) f(s_m|s_n)\, ds_m \right).$$

As in the previous step, it is easy to see that for each W_n',

$$\lim_k \int_{W_n'} \tau_n^k(T|s_n)[\pi_n^k(s_n, \zeta^k(\bar{\tau}_n^k(T|s_n)), \sigma_m^k) - \xi_n^k(s_n) - \rho_n^k(s_n, \bar{\tau}_n^k(\cdot|s_n))]\, ds_n = 0$$

where $\zeta^k(\bar{\tau}_n^k(T|s_n))$ is the distribution over bids induced by $\bar{\tau}_n^k$. Since the strategies in $\zeta^k(\bar{\tau}_n^k(T|s_n))$ are optimal for each s_n, $\pi_n^k(s_n, \zeta^k(\bar{\tau}_n^k([0,1]|s_n), \sigma_m^k)) = \theta_n^k(s_n)$. Also, the equicontinuity of θ_n^k and ξ_n^k imply that the left-hand side of the above equation equals:

$$\int_{W_n'} \left(\tau_n^*(T|s_n)[\theta_n^*(s_n) - \xi_n^*(s_n)] - \int_{S_m} [v_n(s_n, s_m) - b] p_n^*(s_n, s_m) \right) f_n(s_m|s_n)\, ds_n.$$

As $\theta_n^*(s_n) = \xi_n^*(s_n) + \vartheta_n^*(s_n)$, and W_n' is an arbitrary subset of W_n, the claimed equality follows.

To prove the positivity of θ_n^* a.e. on W_n, we first show that it is non-negative. Suppose, to the contrary, that $\vartheta_n^*(s_n)$ is negative for some s_n. Then $\xi_n^*(s_n) > \theta_n^*(s_n)$. Obviously now $b > v_*$, since otherwise $\xi_n^*(s_n) = 0$ and thus $\theta_n^*(s_n)$ would be negative, which is impossible. Observe then that $\xi_n^*(s_n)$ is the limit of $\pi_n^*(s_n, b^l, \sigma_m^*)$ for a sequence b^l of bids approaching b from the left that are points of continuity of σ_m^*. Therefore, one can choose $b' < b$ such that b' is a point of continuity of σ_m^* and $\pi_n^*(s_n, b', \sigma_m^*) > \theta_n^*(s_n)$, which is impossible for generic s_n. Thus, $\vartheta_n^*(s_n)$ is non-negative a.e.

Now fix a subset W_n' of W_n with positive measure. Then $p_n^*(W_n', S_m) > 0$ since by symmetry $p_n^*(W_n', W_n') = 1/2$. Also for each $s_n \in W_n'$, $\vartheta_n^*(s_n) \geqslant 0$. Therefore,

$$\frac{\int_{W_n'} \int_{S_m} v_n(s_n, s_m) p_n^*(s_n, s_m) f_n(s_m|s_n)\, ds_m\, ds_n}{\int_{W_n'} \int_{S_m} p_n^*(s_n, s_m) f_n(s_m|s_n)\, ds_m\, ds_n} \geqslant b\,.$$

By Assumption (6) therefore, for each s_n that is greater than the essential supremum of W_n':

$$\frac{\int_{W_n'} \int_{S_m} v_n(s_n, s_m) p_n^*(s_n', s_m) f_n(s_m|s_n)\, ds_m\, ds_n'}{\int_{W_n'} \int_{S_m} p_n^*(s_n', s_m) f_n(s_m|s_n)\, ds_m\, ds_n'} > b\,.$$

Therefore, $\lim_k \rho_n^k(s_n, \bar{\tau}_n^k(\cdot|W_n')) > 0$, i.e. by mimicking the strategy of W_n', in the limit s_n obtains a positive payoff. By Step 2, this implies that $\vartheta_n^*(s_n) > 0$ such an s_n. Since W_n' was an arbitrary set of W_n with positive measure, we have that for all s_n greater than the essential infimum of W_n, $\vartheta_n^*(s_n) > 0$. •

Step 4. We claim that for a.e. $s_n \in W_n$, $\bar{p}^*_n(s_n, \cdot)$ is weakly decreasing almost everywhere on W_m. To prove this claim, for a subset $\tilde{W}_n$ of W_n with positive measure and $x \in [0,1]$, let $\bar{s}_m(x, \tilde{W}_n)$ be the essential supremum of the set of m's signals s_m such that $\bar{p}^*_n(\tilde{W}_n, s_m) \geqslant x$. If the claim is not true then there exists a set $\tilde{W}_n$ of positive measure and $x \in (0,1]$ such that there is a positive measure of signals s_m in $[0, \bar{s}_m(x, \tilde{W}_n)] \cap W_m$ for which $\bar{p}^*_n(\tilde{W}_n, s_m) < x$. Choose two closed and disjoint subsets W^0_m and W^1_m of $[0, \bar{s}_m(x, \tilde{W}_n)] \cap W_m$ such that both sets have positive measure and: (i) for each $s^0_m \in W^0_m$, $s^1_m \in W^1_m$, one has $s^0_m < s^1_m$; (ii) $\bar{p}^*_n(\tilde{W}_n, W^0_m) < x$ and $\bar{p}^*_n(\tilde{W}_n, W^1_m) \geqslant x$. Therefore, $\bar{p}^*_m(W^0_m, \tilde{W}_n) - \bar{p}^*_m(W^1_m, \tilde{W}_n) > 0$.

Since for each $i = 0, 1$, $\bar{p}^k_m(W^i_m, \tilde{W}_n)$ converges to $\bar{p}^*_m(W^i_m, \tilde{W}_n)$, we have that for a subsequence there exists for all k a pair $(s^{k,i}_m, t^{k,i})$ converging to some $(s^{\infty,i}_m, t^{\infty,0})$ such that: (i) $s^{k,i}_m \in W^i_m$ for all k; (ii) $t^{k,i}$ is in the support of $\tau^k_m(\cdot|s^{k,i}_m)$ for all k; (iii) letting $\hat{\tau}^{*,i}_n(t^{\infty,i}|\cdot)$ be the limit of $\hat{\tau}^{k,i}_n(t|\cdot)$, $\int_{\tilde{W}_n} \hat{\tau}^{\infty,0}_n(t^{\infty,0}|s_n) - \hat{\tau}^{\infty,1}_n(t^{\infty,1}|s_n)\, ds_n > 0$. Obviously $t^{k,0} > t^{k,1}$ for all k and hence $\int_{S_n} \hat{\tau}^{\infty,0}_n(t^{\infty,0}|s_n) - \hat{\tau}^{\infty,1}_n(t^{\infty,1}|s_n)\, ds_n > 0$ as well.

Letting $s^{\infty,i}_m$ be the limit of $s^{k,i}_m$, we have from the previous step that $\rho^*_m(s^{\infty,i}_m, t^{\infty,i}_m) = \int_{S_n} [v_m(s^{\infty,i}_m, s_n) - b]\hat{\tau}^{\infty}_n(t^{\infty,i}_m|s_n) f_m(s_n|s^{\infty,i}_m)\, ds_n = \vartheta_m(s^{\infty,i}_m)$. This implies, for $i = 1$, that

$$\frac{\int_{S_n} v_m(s^{\infty,1}_m, s_n)\beta(s_n) f_m(s_n|s^{\infty,1}_m)\, ds_n}{\int_{S_n} \beta(s_n) f_m(s_n|s^{\infty,1}_m)\, ds_n} \leqslant b$$

where $\beta(s_n) = \hat{\tau}^{\infty}_n(t^{\infty,0}|s_n) - \hat{\tau}^{\infty}_n(t^{\infty,1}|s_n)$. By Assumption (6), the same expression using $s^{\infty,0}_m$ in place of $s^{\infty,1}_m$ would be a strict inequality since $s^{\infty,0}_m < s^{\infty,1}_m$. Thus, $\lim_k \rho^k_m(s^{k,0}_m, t^{k,1})$ is strictly greater than $\lim_k \rho^k_m(s^{k,0}_m, t^{k,0})$, which is the desired contradiction. •

It is easy to show using the previous step that the support of τ^*_n is a singleton for a.e. $s_n \in W_n$ and that this function is then weakly increasing, a fact not used below.

We come now to the final step. For those signals $s_n \in W_n$ for which n bids b in the equilibrium σ^* of G^*, Step 5 shows that the amount $\vartheta^*_n(s_n)$ by which the equilibrium payoff $\theta^*_n(s_n)$ exceeds the payoff $\xi^*_n(s_n)$ from bidding b and winning only if m's bid is strictly less, cannot be so large as to imply always winning when m also bids b, and that therefore player n prefers to bid slightly more than b. Thus Step 5 obtains a contradiction to the conclusion of Lemma 4.4 that σ^* is an equilibrium of G^*, and thereby a contradiction to the initial hypothesis that m's strategy σ^*_m induces an atom at the bid b.

Step 5. Fix a generic signal $s_n \in W_n$. By Steps 2 and 3, $\vartheta^*_n(s_n)$ and $p^*_n(s_n, W_m)$ are both positive. The incremental payoff is

$$\vartheta^*_n(s_n) = \int_{W_m} [v_n(s_n, s_m) - b]\tau^*_m(T|s_m)\bar{p}^*_n(s_n, s_m) f_n(s_m|s_n)\, ds_m\,.$$

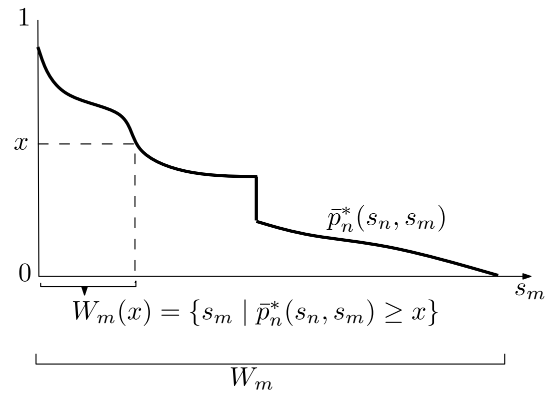

FIGURE 2

For each $x \in [0,1]$, define the set $W_m(x) \equiv \{\, s_m \in W_m \mid \bar{p}_n^*(s_n, s_m) \geqslant x \,\}$. Figure 2 illustrates a possible form of $\bar{p}_n^*(s_n, s_m)$ and $W_m(x)$ for the case that W_m is an interval.

By Step 3, $W_m(x)$ is monotonically nonincreasing in x. One can therefore express $\vartheta_n^*(s_n)$ as the double integral

$$\vartheta_n^*(s_n) = \int_{W_m} \int_{[0,1]} [v_n(s_n, s_m) - b] 1_{W_m(x)}(s_m) \tau_m^*(T|s_m) f_n(s_m|s_n)\, dx\, ds_m,$$

where $1_{W_m(x)}(\cdot)$ is the indicator function for $W_m(x)$. For each x and signal s_n, define

$$u_n(x, s_n) = \int_{W_m(x)} [v_n(s_n, s_m) - b] \tau_m^*(T|s_m) f_n(s_m|s_n)\, ds_m\,.$$

Then reversing the order of integration above yields the alternative formula

$$\vartheta_n^*(s_n) = \int_{[0,1]} u_n(x, s_n)\, dx\,.$$

If $u_n(x, s_n) > 0$ for some x then for all $x_1 \leqslant x \leqslant x_2$, $u_n(x_1, s_n) \geqslant u_n(x_2, s_n)$, with the inequality being strict if $W_m(x_1) \setminus W_m(x_2)$ has positive measure. To prove this, let $\overline{s}_m(x)$ be the essential supremum of $W_m(x)$. Since $u_n(x, s_n) > 0$, $W_m(x)$ has positive measure and $\overline{s}_m(x)$ is well-defined. Also, $v_n(s_n, \overline{s}_m(x)) > b$, since otherwise by Assumption (5), $v_n(s_n, s_m) \leqslant b$ for all $s_m \in W_m(x)$ and then $u_n(x, s_n) \leqslant 0$. Fix $x_1 < x \leqslant x_2$ and for each $i = 1, 2$, let $\overline{s}_m(x_i)$ be the essential supremum of $W_m(x_i)$. By the monotonicity of $W_m(\cdot)$, $\overline{s}_m(x_1) \geqslant \overline{s}_m(x_2)$, where the inequality is strict if $W_m(x_1) \setminus W_m(x_2)$ has positive measure. Observe that for each $\overline{s}_m(x) \leqslant s_m \leqslant \overline{s}_m(x_1)$, $v_n(s_n, s_m) > b$, and thus $u_n(x_1, s_n)$ is strictly positive. If $u_n(x_2, s_n)$ is non-positive then one is done. Otherwise, if $u_n(x_2, s_n) > 0$, then $v_n(s_n, \overline{s}_m(x_2)) > b$ and thus, again by Assumption (5), for all $\overline{s}_m(x_2) \leqslant s_m \leqslant \overline{s}_m(x_1)$, $v_n(s_n, s_m) > b$. Thus, $u_n(x_1, s_n) > u_n(x_2, s_n)$.

Since $\vartheta_n^*(s_n)$ is greater than zero, $u_n(x, s_n) > 0$ for some x. Therefore, the preceding paragraph implies that $u_n(x, s_n) \leqslant u_n(0, s_n)$ for all x, and hence

$$\vartheta_n^*(s_n) \leqslant \int_{W_m} [v_n(s_n, s_m) - b]\tau_m^*(T|s_m) f_n(s_m|s_n)\, ds_m \,.$$

Observe that this inequality must be strict for a.e. $s_n \in W_n$. Otherwise, when n's signal is s_n he wins with probability one against every signal $s_m \in W_m$ of player m, which in an equilibrium is not possible outside a set of measure zero in W_n.

To finish the proof, we show that when the above inequality is strict player n with signal s_n prefers some bid $b' > b$ for all large k. Indeed, first observe that $b < v^*$, since otherwise, the payoff $\theta_n^*(s_n) \leqslant 0$ (the maximum value is v^* and if it equals the bid b, then player n cannot make a profit in any event). Then, observe that the total payoff $\xi_n^*(s_n) + \int_{W_m} [v_n(s_n, s_m) - b]\tau_m^*([0, 1]|s_m) f_n(s_m|s_n)\, ds_m$ is the right-hand limit of $\pi_n^+(s_n, b, \sigma_m^*)$. So, if the inequality is strict then one can choose $b' > b$ that is a point of continuity and such that $\pi_n^+(s_n, b', \sigma_m^*) > \xi_n^*(s_n) + \vartheta_n^*(s_n)$. But that is impossible since by definition $\xi_n^*(s_n) + \vartheta_n^*(s_n) = \theta_n^*(s_n)$ and therefore $\pi_n^+(s_n, b', \sigma_m^*) > \theta_n^*(s_n)$, contradicting the initial supposition that $\theta_n^*(s_n)$ is n's equilibrium payoff when his signal is s_n. •

The contradiction obtained in Step 5 implies the falsity of the initial hypothesis that player m's strategy induces an atom at the bid b, and thereby concludes the proof of Lemma 4.5. □

In sum, Lemma 4.4 establishes that a limit point σ^* of the equilibria σ^k of the auxiliary games G^k is an equilibrium of G^*, and Lemma 4.5 establishes that σ^* does not induce atoms in the distribution of any player's bids. Hence the probability of tied bids is zero and therefore σ^* is also an equilibrium of the auction game G, regardless of the tie-breaking rule. This verifies Theorem 4.1.

Remark. Assumptions (5) and (6) are invoked only in Steps 2, 3, and 4 of Lemma 4.5. Without these assumptions, the proof shows how to generate an endogenous tie-breaking rule, as in [3]. For each bid b at which there is an atom, of which there is a countable number, the limiting probabilities $p_n^*(s_n, s_m)$ and $p_m^*(s_m, s_n)$ give the relative odds of s_n or s_m winning if both bid b and then report their signals truthfully. In the event of a tie at such a bid b, the incentive to truthfully report one's signal is obtained from the fact that the limit payoff $\vartheta_n^*(s_n)$ maximizes $\rho_n^*(s_n, t)$ and that misreporting one's signal, say $\tilde{s}_n \in W_n$ instead of s_n, obtains $\lim_k \rho_n^k(s_n, \tilde{t}^k)$ as the payoff, where $\tilde{t}^k$ is a sequence converging to a $\tilde{t}$ in the support of $\tau_n^*(\cdot|\tilde{s}_n)$, and for which $\tilde{t}^k$ is in the support of $\tau_n^k(\cdot|\tilde{s}_n^k)$ for a sequence of $\tilde{s}_n^k$ converging to $\tilde{s}_n$.

5. Concluding Remarks

In auction theory, most existence theorems focus on equilibria in pure strategies that are strictly increasing in bidders' signals. Like Theorem 4.1 here, monotone pure-strategy equilibria obviate discontinuities due to tie-breaking rules because they induce nonatomic distributions of bids. However, these theorems rely on unrealistically strong assumptions about the joint distribution of signals, such as affiliation.[8] In contrast, the theorem here establishes existence of an equilibrium in mixed strategies using weak assumptions about the distribution of bidders' signals and their value functions.[9] In general we see advantages to distinguishing between those assumptions sufficient for existence of equilibria, and those that ensure pure strategies and monotonicity.

The proof of Theorem 4.1 brings additional advantages. One is that the auxiliary game induces a well-defined fixed-point problem in the space of behavioral strategies, a feature absent from previous work. This problem necessarily has essential sets of fixed points, those for which every perturbation of the problem has a nearby fixed point. As shown by the proof in Section 4, limit points of these nearby fixed points are equilibria of both the auxiliary game and the auction game. Because essential fixed points are those for which the Leray-Schauder index is nonzero, there is the further possibility of distinguishing between those with positive and negative indices, since it is known that in finite games those with positive and negative indices are dynamically stable and unstable, respectively, under monotone adjustment processes.

References

[1] De Castro, L. (2009), Affiliation and Dependence in Economic Models, Economics Department, University of Illinois, 12 October 2009.

[2] Govindan, S., and R. Wilson (2009b), Existence of Equilibria in Private-Value Auctions, Stanford Business School, 1 February 2010.

[3] Jackson, M., L. Simon, J. Swinkels, W. Zame (2002), Communication and Equilibrium in Discontinuous Games of Incomplete Information, Econometrica, 70: 1711-1740.

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[8] De Castro [1, Theorem 3.1] proves that the subset of continuous density functions that are not affiliated is open and dense. He examines various plausible weaker properties and establishes for some that there need not exist equilibria in increasing pure strategies. He also shows that an auction with private values jointly distributed according to a density that is piecewise constant has pure-strategy equilibria, and an example for which no pure strategy equilibrium is increasing.

[9] We hope in later work to establish existence of an equilibrium in pure strategies.