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EXISTENCE OF EQUILIBRIA IN PRIVATE-VALUE AUCTIONS

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This paper introduces a new method to establish existence of an equilibrium of an auction. Here it is applied to equilibria in behavioral strategies for a first-price sealed-bid auction of a single item for which bidders have privately known values drawn from a joint distribution.[1] The only restriction is that this distribution has a density that is positive and continuous on a hypercube.

The method circumvents discontinuities due to tie-breaking rules. First one establishes existence of equilibria for a modified auction in which every bidder offering the highest bid wins a copy of the item. For this modified auction there is a well-defined fixed-point problem for which equilibria are solutions. Then one establishes that equilibria obtained as limit points of equilibria of payoff perturbations of the modified auction have no atoms in the distributions of bids.[2] Because the probability of tied bids is zero, these equilibria of the modified auction are equilibria of the original auction, regardless of the tie-breaking rule. For the proof here, it suffices to consider perturbations of the modified auction in which a bidder anticipates that his submitted bid will be slightly distorted by noise before it is received by the auctioneer.

1. The Auction Game

We consider an N-player game G that represents a first-price sealed-bid auction for a single item. Each bidder $n = 1, \dots, N$ learns privately his value v_n for the item and then submits a bid b_n. He wins the auction if his bid is highest, or when tied for highest, if he is selected by a tie-breaking rule. If he wins the auction then he is awarded the item and his payoff is $v_n - b_n$. Otherwise his payoff is zero.

The joint probability distribution F of all bidders' values is supposed to be common knowledge among the bidders. Our basic assumption is the following.

Assumption: [Distribution of Values]

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[1]In [2] we apply this method to auctions with interdependent values. Here we address the special case of private values because proofs are simpler.

[2]That is, essential equilibria [1] of the modified auction have atomless bid distributions.

(1) The set of possible profiles of values is $V = \prod_n V_n$, where for each bidder n the set of possible values is the same interval $V_n = [v_*, v^*]$, where $v_* < v^*$.
(2) The distribution function F has a density f that is positive and continuous on V.[3]

For each bidder n and his value v_n, let $F_n(\cdot|v_n)$ be the conditional distribution over the values of n's opponents, and let $f_n(\cdot|v_n)$ be its density function. The assumption about the density f guarantees that for each n, v_{-n} and v_n, the two families of functions $f_n(v_{-n}|\cdot)$ and $f_n(\cdot|v_n)$ are equicontinuous in the parameters v_{-n} and v_n respectively.

We assume that the feasible set of bids is the interval $B = [v_*, v^*]$, the same as the interval of each bidder's possible values. To avoid confusion, we use $\mathbf{V}_n$ to denote the Borel measurable subsets of V_n and we use $\mathbf{B}$ to denote the corresponding collection when referring to B. Let λ be the Lebesgue measure on $[v_*, v^*]$.

2. Behavioral Strategies and Beliefs

A behavioral strategy for bidder n is a transition-probability function $\sigma_n(\cdot|\cdot) : \mathbf{B} \times V_n \to [0, 1]$ such that: for each value v_n, $\sigma_n(\cdot|v_n)$ is a probability measure on bids in B; and for each event $A \in \mathbf{B}$, $\sigma_n(A|\cdot)$ is a measurable function of n's value v_n.[4] Let Σ_n be the set of behavioral strategies of bidder n equipped with the topology of weak convergence, viz., a sequence σ_n^k in Σ_n converges to σ_n iff for every continuous function $\eta : B \to \mathbb{R}$ and each event $W_n \in \mathbf{V}_n$,[5]

$$\int_{W_n} \int_B \eta(b)\, d\sigma_n^k(b|v_n)\, dv_n \to \int_{W_n} \int_B \eta(b)\, d\sigma_n(b|v_n)\, dv_n\,.$$

With this topology, Σ_n is a compact (metrizable) space.[6] Let $\Sigma = \prod_n \Sigma_n$ be the space of all behavioral strategy profiles with the product topology.

The next definition deals with the belief of a bidder about the distribution of the highest bid among his opponents. Like a behavioral strategy, a belief is generally a transition function $\mathrm{H}_n(\cdot|\cdot) : \mathbf{B} \times V_n \to [0, 1]$. The intended interpretation is that $\mathrm{H}_n(\cdot|v_n)$ is the conditional distribution function of the highest bid among n's opponents, given that n's value is v_n. Thus $\mathrm{H}_n(A|v_n)$ is the probability that the highest bid among n's opponents is in A. In the sequel, however, it suffices to focus on events of the form $A = [v_*, b]$ or $A = [v_*, b)$. Thus,

[3]This assumption can be weakened considerably: the distribution F need only be absolutely continuous w.r.t. the product of its marginal distributions.

[4]Strictly speaking a behavioral strategy is an equivalence class of functions, where σ_n is equivalent to σ_n' if $\sigma_n(\cdot|v_n) = \sigma_n'(\cdot|v_n)$ λ-a.e. on V_n.

[5]Here and elsewhere the outer integral is computed with respect to the Lebesgue measure.

[6]This definition is weaker than requiring (a.e. on V_n) pointwise weak convergence of the sequence of distributions $\sigma_n^k(\cdot|v_n)$. An equivalent definition requires that the displayed integral be u.s.c. (resp. l.s.c.) for each η that is u.s.c. (resp. l.s.c.).

for each bid $b \in B$ and n's belief H_n, we write $H_n(b|v_n)$ and $H_n(b^-|v_n)$ for $\mathrm{H}_n([v_*, b]|v_n)$ and $\mathrm{H}_n([v_*, b)|v_n)$ respectively. Recall that a cumulative distribution function is necessarily upper semi-continuous, so as a function on B, $H_n(\cdot|v_n)$ has this property for each value v_n. Let $\mathcal{H}_n$ be the set of beliefs of bidder n equipped with the topology of weak convergence, and let $\mathcal{H} = \prod_n \mathcal{H}_n$ be the space of all belief profiles with the product topology.

We now describe the belief map $\beta : \Sigma \to \mathcal{H}$ that assigns to each strategy profile σ a belief profile $\beta(\sigma) = H$, defined as follows. For each bidder n, his value $v_n \in V_n$ and each subset $A \in \mathbf{B}$ of bids of the form $A = [v_*, b]$ or $[v_*, b)$ for some bid $b \in B$,

$$H_n(A|v_n) = \int_{V_{-n}} (\prod_{m \neq n} \sigma_m(A|v_m)) f(v_{-n}|v_n)\, dv_{-n} \,,$$

where m indexes the bidders other than n. Assumption 1 implies the following properties of beliefs.

Lemma 2.1. *For each strategy profile $\sigma \in \Sigma$, if $\beta(\sigma) = H$ then for each bidder n:*

1. *For each bid b, $H_n(b|v_n)$ is continuous in v_n.*
2. *$H_n(b|v_n)$ is upper semi-continuous in (b, v_n).*
3. *b is point of continuity of H_n for some v_n iff it is a point of continuity for all v_n.*

Proof. Since f is positive and continuous, the conditional density $f(v_{-n}|v_n)$ is continuous in v_n. Hence $H_n(b|v_n)$ is continuous in v_n. Suppose now that (b_n^k, v_n^k) is a sequence converging to (b_n, v_n). For each $c > b_n$, $b_n^k < c$ for all large k. Therefore $H_n(b_n^k|v_n^k) \leqslant H_n(c|v_n^k)$ for all large k. This implies that $\limsup_k H_n(b_n^k|v_n^k) \leqslant \lim_k H_n(c|v_n^k) = H_n(c|v_n)$, where the equality follows from point (1) of this lemma. Because c is an arbitrary bid higher than b_n, (2) follows from the right-continuity of distributions. Finally, (3) is a consequence of the assumption that the density function is positive. □

The space $\mathbf{H}$ of belief profiles consistent with some strategy profile is the range of β, i.e. $\mathbf{H} = \beta(\Sigma)$. $\mathbf{H}$ is a subspace of $\prod_n \beta_n(\Sigma)$ with the product of the weak topologies on each factor. The following lemma establishes the continuity of β and hence the compactness of $\mathbf{H}$.

Lemma 2.2. *The belief map β is continuous. In particular, if σ^k is a sequence of strategy profiles in Σ converging to σ then for each bidder n and value $v_n \in V_n$ the distribution $\beta_n(\sigma^k)(\cdot|v_n)$ converges weakly to $\beta_n(\sigma)(\cdot|v_n)$.*

Proof. Let $H_n^k = \beta_n(\sigma^k)$ and $H_n = \beta_n(\sigma)$. Since the indicator functions on the sets $[0, b]$ and $[0, b)$ are u.s.c. and l.s.c. on the set B of bids, convergence of the sequence of strategy profiles implies that for each v_n and each point b of continuity of the conditional distribution function $H_n(\cdot|v_n)$:

$$H_n(b^-|v_n) = H_n(b|v_n) \geqslant \limsup_k H_n^k(b|v_n) \geqslant \liminf_k H_n^k(b^-|v_n) \geqslant H_n(b^-|v_n)\,.$$

Thus, for each n and v_n the sequence $H_n^k(\cdot|v_n)$ converges weakly to $H_n(\cdot|v_n)$. Therefore $\lim_k H_n^k = H_n$. □

This lemma implies that convergence in $\mathbf{H}$ for the specified topology is the same as requiring that for each n and v_n the corresponding sequence of distributions converges.

The next lemma establishes two other key properties of the belief map β.

Lemma 2.3. *For each bidder n, $\beta_n(\sigma)(b_n|v_n)$ is an upper semi-continuous function of (σ, v_n, b_n). If b_n is a point of continuity of $\beta_n(\sigma)$ then the function is continuous at (σ, v_n, b_n).*

Proof. Let (σ^k, v_n^k, b_n^k) be a sequence converging to (σ, v_n, b_n). Let $H^k = \beta(\sigma^k)$ and let H be the limit. Fix any small $\varepsilon > 0$. There exists $c > b_n$ that is a point of continuity of $H_n(\cdot|v_n)$ and such that $H_n(c|v_n) \leqslant H_n(b|v_n) + \varepsilon/2$. Since c is a point of continuity of H_n there exists K large enough such that for all $k \geqslant K$, $H_n^k(c|v_n)$ is within $\varepsilon/4$ of $H_n(c_n|v_n)$. Since f is continuous and positive, the conditional densities $f_n(\cdot|w_n)$ are an equicontinuous family parameterized by $w_n \in V_n$. Therefore, for large k, $H_n^k(c|v_n^k)$ is within $\varepsilon/4$ of $H_n^k(c|v_n)$. Hence $H_n^k(c|v_n^k) \leqslant H_n(b|v_n) + \varepsilon$. Since b_n^k converges to b_n, $b_n^k \leqslant c$ for large k. Thus, $H_n^k(b_n^k|v_n^k) \leqslant H_n(c|v_n^k) \leqslant H_n(b|v_n) + \varepsilon$ for large k. Since ε is arbitrary, this establishes the upper semi-continuity property. If $\beta_n(\sigma)$ is continuous at b_n then one can apply the same argument as above by choosing $c < b_n$ that is a point of continuity to obtain for each ε that for large k, $H_n^k(b_n^k|v_n^k) \geqslant H_n(b_n|v_n) - \varepsilon$. □

3. A Fixed-Point Map on the Space of Behavioral Strategies

For each bidder n define a payoff function $\pi_n : V_n \times B \times \Sigma \to \mathbb{R}$ by

$$\pi_n(v_n, b_n, \sigma) = [v_n - b_n]\beta_n(\sigma)(b_n|v_n)\,.$$

If the belief $\beta_n(\sigma)(\cdot|v_n)$ has no mass points then the probability of tied bids is zero and therefore π_n is indeed n's payoff function in the auction game G. More generally, π_n is generated by an allocation rule that stipulates that all among the highest bidders receive copies of the item. In other words, to surely win an item a bidder needs only to match the highest bid among his opponents, not necessarily to exceed it. We assume initially that this is indeed the allocation rule, which thus defines a related game G^*. Later we demonstrate that there is an equilibrium of G^* in which there are no mass points in the distribution of opponents' highest bids and thus this equilibrium of G^* is also an equilibrium of the auction game G.

In the modified game G^*, bidder n's optimal replies to a profile σ of bidders' strategies are obtained by solving the following maximization problem, conditional on his value v_n:

$$\max_{b_n \in B(v_n)} \pi_n(v_n, b_n, \sigma)\,,$$

where $B(v_n)$ is the set of bids $b_n \leqslant v_n$. Let $\phi_n : V_n \times \Sigma \twoheadrightarrow B$ be the correspondence that maps each (v_n, σ) to the set $\phi_n(v_n, \sigma) = \arg\max_{b_n \in B(v_n)} \pi_n(v_n, b_n, \sigma)$ of n's optimal replies when his value is v_n.

Lemma 3.1. *The correspondence ϕ_n is upper semi-continuous and has nonempty compact images.*

Proof. Let $H_n = \beta_n(\sigma)$ be n's belief. By Lemma 2.1, the payoffs are u.s.c. in the bid b for each value v_n and strategy profile σ, provided that bids satisfy the individual rationality condition $b \leqslant v_n$ as above. Therefore, $\phi_n(v_n, \sigma)$ is nonempty and compact for each v_n and σ. Suppose (v_n^k, σ^k) is a sequence converging to (v_n, σ) and b_n^k is a corresponding sequence of optimal replies converging to the bid b_n. Observe that $\pi_n(\sigma, v_n, b) \geqslant \limsup_k \pi_n(\sigma^k, v_n^k, b_n^k)$, again using $b \in B(v_n)$. To show that b_n is an optimal reply, pick a bid c that is a point of continuity H_n. By Lemma 2.3, $H_n^k(c|v_n^k)$ converges to $H_n(c|v_n)$. Therefore, the payoffs $\pi_n(\sigma^k, v_n^k, c)$ converge to $\pi_n(\sigma, v_n, c)$. Since each b_n^k is an optimal reply, b_n is as good a reply as c. The result follows for an arbitrary bid c since there exists a sequence c^l converging to c such that $\pi_n(\sigma, v_n, c^l)$ converges to $\pi_n(\sigma, v_n, c)$. □

For each bidder n define the optimal-reply correspondence $\Phi_n : \Sigma \twoheadrightarrow \Sigma_n$ by specifying for each strategy profile σ that $\Phi_n(\sigma)$ is the set of n's behavioral strategies $\sigma_n \in \Sigma_n$ such that for each value $v_n \in V_n$ the support of $\sigma_n(\cdot|v_n)$ is optimal, i.e. is contained in $\phi_n(v_n, \sigma)$.

Lemma 3.2. *The correspondence Φ_n is upper semi-continuous with images that are nonempty, compact, and convex.*

Proof. Since $\phi_n(\cdot, \sigma)$ is an upper semi-continuous correspondence, it admits a measurable selection, which is then a pure strategy that belongs to $\Phi_n(\sigma)$. Therefore, $\Phi_n(\sigma)$ is a nonempty set. Also it is clearly compact and convex. Upper semi-continuity follows from the upper semi-continuity of ϕ_n. □

Now define the joint optimal-reply correspondence $\Phi^* : \Sigma \twoheadrightarrow \Sigma$ to be the product of the maps Φ_n. The Fan-Glicksberg extension of the Kakutani fixed-point theorem assures the existence of a fixed point of Φ^*. Clearly, each fixed point of Φ^* is an equilibrium of the game G^*. Observe, however, that Φ^* has a trivial fixed point in which every bidder bids v_* regardless of his value. Nevertheless, if Φ^* has another fixed point for which the strategies of the bidders have no mass points in the distributions of their bids then it is an equilibrium of the auction game G.

Theorem 3.3. *Suppose σ^* is a fixed point of Φ^* such that for each bidder n and each bid $b \in B$, $\sigma^*_n(\{ b \}|\cdot)$ is zero λ-a.e. on V_n. Then σ^* is an equilibrium of the auction game G.*

Proof. Let $H^* = \beta(\sigma^*)$. The supposition implies that for each bidder n and value v_n the conditional distribution $H^*_n(\cdot|v_n)$ of the highest bid among n's opponents has no mass points. The probability that any bid b_n by n is the same as the highest bid among his opponents is therefore zero. Hence, regardless of the tie-breaking rule in the auction game G, bidder n's conditional payoff function $[v_n - b_n]H^*_n(b_n|v_n)$ is the same as the conditional payoff function $\pi_n(v_n, b_n, \sigma^*)$ in the modified game G^*, and his optimal replies are the same. In the auction game G the behavioral strategy $\sigma^*_n(\cdot|v_n)$ is therefore an optimal reply by n conditional on each of his values v_n. Thus σ^* is an equilibrium of the auction game G. □

4. A Basic Existence Result for Behavioral Strategies

In this section we construct a sequence of well-behaved perturbations of the modified game G^* that induce perturbations of the optimal-reply correspondence Φ^*. We then show that limit points of equilibria of these perturbed games are equilibria of G^* with no atoms in the distributions of players' bids. Hence these limit points are equilibria of the auction game G.

Interpret a conditional belief $H_n(\cdot|v_n)$ for bidder n as a function defined for bids outside B in the obvious way: $H(b|v_n)$ is zero for $b < v_*$ and one for $b > v^*$. Given $H_n(\cdot|v_n)$, for each positive integer k and each bid b define $\bar{H}^k_n(b|v_n)$ to be the expectation of $H_n(\cdot|v_n)$ using the uniform distribution of bids on the interval $[b - 1/k, b + 1/k]$, i.e.[7]

$$\bar{H}^k_n(b|v_n) = [2/k] \int_{b-1/k}^{b+1/k} H_n(c|v_n)\, dc\,.$$

For each k define a perturbed payoff function $\pi^k_n : V_n \times B \times \Sigma$ as follows. If $H_n = \beta_n(\sigma)$ then

$$\pi^k_n(v_n, b_n, \sigma) = [v_n - b_n]\bar{H}^k_n(b_n|v_n)\,.$$

Denote the game with this perturbed payoff function by G^k.

As in the previous section, one can now define an optimal reply correspondence $\phi^k_n : V_n \times \Sigma \to B$ by letting $\phi^k_n(v_n, \sigma)$ be the set of maximizers of $\pi^k_n(v_n, b_n, \sigma)$. Also let Φ^k_n be the set

[7]The arguments go through if one uses an atomless distribution whose density is symmetric around the mean b, like the normal distribution, or an asymmetric one that is not skewed to the right.

of behavioral strategies that mix over these optimal replies. The previous characterizations of ϕ_n and Φ_n apply to ϕ_n^k and Φ_n^k as well. Hence the joint optimal-reply correspondence $\Phi^k : \Sigma \twoheadrightarrow \Sigma$ has a fixed point σ^k that is an equilibrium of G^k and induces beliefs $H^k = \beta(\sigma^k)$. As before, $\bar{H}_n^k(b|v_n)$ is the expectation of $H_n^k(\cdot|v_n)$ with respect to the uniform distribution on $[b - 1/k, b + 1/k]$.

For an increasing subsequence of k's, the equilibrium strategy profiles σ^k and beliefs H^k converge to limit points, say to σ^* and H^*. Obviously, $H^* = \beta(\sigma^*)$. We now prove existence of an equilibrium of the auction game G by showing that σ^* satisfies the supposition in Theorem 3.3.

Theorem 4.1. *[Existence Theorem] The strategy profile σ^* is an equilibrium of the auction game G, with H^* describing bidders' beliefs.*

Proof. The proof begins with four lemmas that verify that σ^* an equilibrium of the game G^*, i.e. it is an optimal reply to itself when the payoff function is π. A fifth lemma verifies that there are no atoms in H^*, which establishes that σ^* is an equilibrium of the auction game G.

For each k and bidder n, let $\theta_n^k : V_n \to \mathbb{R}$ be the function that assigns to each value v_n his equilibrium payoff for the equilibrium σ^k of the game G^k. The equicontinuity property of the conditional densities $f_n(\cdot|v_n)$ implies the first lemma.

Lemma 4.2. *The family of functions θ_n^k is equicontinuous.*

Lemma 4.3. *For any bidder n, suppose (v_n^k, b^k) converges to (v_n, b). Then*

$$H_n^*(b_n|v_n) \geqslant \limsup_k \bar{H}_n^k(b_n^k|v_n^k) \geqslant \liminf_k \bar{H}_n^k(b_n^k|v_n^k) \geqslant H_n^*(b^-|v_n)\,.$$

Proof. Given any $\varepsilon > 0$, fix two bids $b_1 < b < b_2$ that are points of continuity of H_n^* such that $H_n^*(b_1|v_n) > H_n^*(b^-|v_n) - \varepsilon/2$ and $H_n^*(b_2|v_n) < H_n^*(b|v_n) + \varepsilon/2$. Since the H_n^k's are converging to H_n^*, and b_1 and b_2 are points of continuity of H_n^*, for all large k, $H_n^k(b_1|v_n^k) > H_n^*(b^-|v_n) - \varepsilon$ and $H_n^k(b_2|v_n^k) < H_n^*(b|v_n) + \varepsilon$. Thus for all such k and $b_1 < b' < b_2$ then $H_n^*(b^-|v_n) - \varepsilon < H_n^k(b'|v_n^k) < H_n^*(b|v_n^k) + \varepsilon$. For large k, $[b_n^k - 1/k, b_n^k + 1/k] \subset [b_1, b_2]$ so $\bar{H}_n^k(b_n^k|v_n^k)$ is in the interval $(H_n^*(b^-|v_n) - \varepsilon, H_n^*(b|v_n) + \varepsilon)$. Since ε was arbitrary, the lemma follows. □

Let $\theta_n^*(v_n)$ be the maximum payoff to bidder n when his value is v_n and his belief is H_n^*.

Lemma 4.4. *For each bidder n, θ_n^k converges pointwise to θ_n^*. Therefore, a subsequence of θ_n^k converges uniformly to θ_n^*.*

Proof. Suppose $\theta_n^k(v_n)$ does not converge to $\theta_n^*(v_n)$ for some value v_n. Then replacing the sequence with some convergent subsequence, $\lim_k \theta_n^k(v_n)$ exists and is different from $\theta_n^*(v_n)$. Take a subsequence of bids b_n^k for v_n that achieves $\theta_n^k(v_n)$ and, if necessary by going to a further subsequence, that converges to some b_n. By the previous lemma, $H_n^*(b_n|v_n) \geqslant \limsup_k \bar{H}_n^k(b_n^k|v_n)$. Therefore, $\lim_k \theta_n^k(v_n) \leqslant [v_n - b_n]H_n^*(b_n|v_n) \leqslant \theta_n^*(v_n)$. This implies that $\lim_k \theta_n^k(v_n) < \theta_n^*(v_n)$. Hence, there exists a bid c that is a point of continuity of $H_n^*(\cdot|v_n)$ and such that $\lim_k \theta_n^k(v_n) < [v_n - c]H_n^*(c|v_n)$. Applying the previous lemma to c, one gets $\lim_k \bar{H}_n^k(c|v_n) = H_n^*(c|v_n)$. For large k, therefore, c is a better bid than b_n^k against σ^k, which is a contradiction.

To prove the second statement, observe that the sequence θ_n^k is bounded. Since it is equicontinuous, it is totally bounded. Hence, it admits a Cauchy subsequence. Such a subsequence must necessarily converge to θ_n^* from the first statement of the lemma. □

Hereafter assume that the selected subsequence is such that θ_n^k converges uniformly to θ_n^*.

Lemma 4.5. *For each bidder n and almost all values $v_n \in V_n$, $\pi_n(v_n, b, \sigma^*) = \theta_n^*(v_n)$ for every bid b in the support of $\sigma_n^*(\cdot|v_n)$.*

Proof. Fix a bidder n and let X_n be the set of (v_n, b) such that $b \leqslant v_n$ and $\pi_n(v_n, b, \sigma^*) = \theta_n^*(v_n)$, i.e. X_n is the graph of the optimal-reply correspondence of n in reply to the strategy profile σ^*. By the upper semi-continuity of the optimal-reply correspondence, X_n is a closed subset of $V_n \times B$. We claim that for each $(v_n, b) \notin X$ there exists an open neighborhood $W_n \times C$ such that $\int_{W_n} \sigma_n^*(C|w_n)dw_n = 0$, which then proves the lemma. To prove this claim, we can choose an open measurable neighborhood $W_n \times C$ of (v_n, b) such that its closure is contained in $(V_n \times B) \setminus X$. Observe that for all large k, $\pi_n^k(w_n, c, \sigma^k) < \theta_n^k$ for all $(w_n, c) \in W_n \times C$. Indeed, otherwise there exists a sequence (w_n^k, c_n^k) converging to some (w_n, c_n) in the closure of $W_n \times C$ and such that $\pi_n^k(w_n^k, c_n^k, \sigma^k) = \theta_n^k$. Using Lemma 4.3 and the convergence of θ_n^k, one obtains

$$(w_n - c)H_n^*(c|w_n) \geq \limsup_k (w_n^k - c_n^k)\bar{H}_n^k(c_n^k|w_n^k) = \lim_k \theta_n^k(w_n^k) = \theta_n(w_n),$$

which contradicts the definition of $W_n \times C$. Since the bids in C are suboptimal for values in W_n in reply to σ^k for all large k, one obtains $\int_{W_n} \sigma_n^k(C|w_n)\, dw_n = 0$ for all such k, and then by the lower semi-continuity of the indicator function on the open set C, $\int_{W_n} \sigma_n^*(C|w_n)\, dw_n = 0$, which proves the claim. □

Lemma 4.5 concludes the first part of the proof of the theorem. For the payoff function π, each player n's strategy σ_n^* is an optimal reply to σ^* and therefore σ^* is an equilibrium of the modified game G^*. The second part proves that no bidder's strategy induces an atom in the distribution of his bids.

Lemma 4.6. *For each bidder n and bid b, $\sigma^*_n(\{b\}|v_n) = 0$ for almost every value $v_n \in V_n$.*

Proof. Suppose to the contrary that $\sigma^*_n(\{b\}|v_n) > 0$ for some bidder n, bid b, and a.e. value v_n in a subset $W_n \subset V_n$ with positive Lebesgue measure. Then from (3) of Lemma 2.1 it follows that $\mathrm{H}^*_m(\{\, b\,\}|v_m) > 0$ for all $m \neq n$ and $v_m \in V_m$. By Lemma 4.5, moreover, $\theta^*_n(v_n) = [v - b]H^*_n(b|v_n)$ for all v_n outside a set of measure zero in W_n. The proof begins by verifying three claims.

Claim 1. We claim that for each bidder m, $\theta^*_m(v_m) > 0$ iff $v_m > v_*$. To prove this observe that for each value v_m his feasible set of bids in G^* is the set $B(v_m)$ of bids $v_* \leqslant b \leqslant v_m$, hence $\theta^*_m(v_*) = 0$. If $v_m > v_*$ then he gets a positive payoff if he bids slightly less than v_m since opponents do not bid more than their values, which might all be less than v_m. •

Claim 2. We claim that there exists $\delta_0 > 0$ such that $\sigma^*_m((b - \delta_0, b)|\cdot) = 0$ a.e. for all bidders $m \neq n$. This claim is vacuous if $b = v_*$. Therefore suppose $b > v_*$ and suppose there does not exist such a δ_0. Then there exists a sequence of δ^i's converging to zero such that $\sigma^*_m((b - \delta^i, b)|\cdot)$ is positive on a set of positive measure in V_m for some bidder $m \neq n$. By going to a subsequence, we can assume that this m is constant along the sequence. Now for each i there exists v^i_m and some $b^i \in (b - \delta^i, b)$ in the support $\sigma^*_m(\cdot|v^i_m)$ and such that $\theta^*_m(v^i_m) = \pi_m(v^i_m, b^i, \sigma^*) = [v^i_m - b^i]H^*_m(b^i|v^i_m)$. Take a subsequence of v^i_m's converging to some v_m to get that $\theta^*_m(v_m) = \lim_i \theta^*_m(v^i_m) = [v_m - b]H^*_m(b^-|v_m) < [v_m - b]H^*_m(b|v_m)$, which contradicts the fact that $\theta^*_m(v_m)$ is the maximum payoff for v_m against σ^*. •

Claim 3. We claim that there exists some bidder $m \neq n$ such that $\sigma^*_m(b|v_m) > 0$ for a subset of values v_m in V_m with positive measure. To prove this claim, assume to the contrary that bidder n is the only one bidding b with positive probability. Then b is a point of continuity of $H^*_n(\cdot|v_n)$ for all v_n. As seen in Claim 2, none of n's opponents bids in the interval $(b - \delta_0, b)$ under σ^*. Therefore, $H^*_n(b'|v_n)$ is constant for b' in the interval $(b - \delta_0, b]$ for each v_n. This implies that for each $v_n \geqslant b$, $\pi_n(v_n, b - \delta_0, \sigma^*) > \pi_n(v_n, b, \sigma^*)$, while values $v_n < b$ do not bid b anyway. Therefore, the bid b does not attain $\theta^*_n(v_n)$ for any value v_n. By Lemma 4.5, this implies that W_n has measure zero, which is the desired contradiction. •

For each player n let W_n be the set of n's values for which n bids b with positive probability. Let N^* be the subset of bidders n for whom W_n has positive measure. By Claim 3 there are at least two bidders in N^*. We can assume that the δ_0 in Claim 2 is such that no bidder bids in the interval $(b - \delta_0, b)$ under σ^*.

Let $\varepsilon < (1/12)\min_{n,v_n} \mathrm{H}^*_n(\{b\}|v_n)$. For each bidder $n \in N^*$ choose a closed interval V^ε_n of the form $[v, v^*]$ such that: (i) $W_n \cap V^\varepsilon_n$ has a positive measure; (ii) for each bidder m and value $v_m \in V_m$, $F_m(\cup_{n\in N^*\setminus\{m\}} V_n \setminus V^\varepsilon_n | v_m) < \varepsilon$.[8]

Let α be the infimum over $n \in N^*$, $v \in V^\varepsilon_n \cap W_n$ of $(v - b)$. Observe that $\alpha > 0$ by the fact that θ^*_n is strictly positive on V^ε_n and the bid b is an optimal reply for values in W_n. Choose a bid b_0 such that b_0 is a point of continuity of beliefs, $b - \delta_0 < b_0 < b$, and $b - b_0 < 3\varepsilon\alpha$. Since no bidder bids in the interval $(b - \delta_0, b)$, $H^*_n(b_0|v_n) = H^*_n(b^-|v_n)$ for all n and v_n.

For each bidder n the function $y_n : [0, 1] \to \mathbb{R}$ given by $y_n(\delta') = \int_{V_n\setminus W_n} \sigma^*_n([b_0, b+\delta']|v_n)\, dv_n$ is a decreasing function with $y_n(0) = 0$ since no value v_n of n bids in $[b_0, b)$ with positive probability under σ^*_n and values not in W_n bid b with zero probability. Therefore, there exists $\delta_1 > 0$ such that for every bidder m,

$$\int_{V_n\setminus W_n} \sigma^*_n([b, b + \delta_1]|v_n)) f_m(v_n|v_m)\, dv_n < \varepsilon/2N\,.$$

Choose the bid $b_1 < b + \delta_1$ such that it is a point of continuity of the belief distribution for all bidders and $H^*_n(b_1|v_n) \leqslant H^*_n(b|v_n) + \varepsilon$ for all bidders n and values v_n. For large k, since b_0 and b_1 are points of continuity of the belief distributions, for each m and v_m, $\bar{H}^k_m(b_0|v_m) \leqslant H^*_m(b_0|v_m)+\varepsilon = H^*_m(b^-|\varepsilon)+\varepsilon$ and $\bar{H}^k_m(b_1|v_m) \geqslant H^*_m(b_1|v_m)-\varepsilon \geqslant H^*_m(b|v_m)-\varepsilon$. By the continuity and monotonicity of $\bar{H}^k_m$, there exists a unique bid $c^k_m(v_m)$ in (b_0, b_1) such that $\bar{H}^k_m(c^k(v_m)|v_m) = H^*_m(b^-|v_m) + .5H^k_m(\{b\}|v_m) + 3\varepsilon$.

Let c^k be the infimum over m and v_m of $c^k_m(v_m)$. Obviously $b_0 < c^k < b_1$. We claim that for each bidder $m \in N^*$, for no value in $W_m \cap V^\varepsilon_m$ would m bid in the interval $(b_0, c^k]$ for large k. Indeed, for any bid c in that interval, $\bar{H}^k_m(c|v_m) \leqslant H^*_m(b^-|v_m) + .5H^k_m(\{b\}|v_m) + 3\varepsilon$ and hence

$$\pi^k_m(c, v_m, \sigma^k) \leqslant [v - b_0]\bar{H}^k_m(c|v_m) \leqslant [v - b + (b - b_0)][H^*_m(b|v_m) - 3\varepsilon] \leqslant \theta^*_n(v_m) - 3\varepsilon\alpha + (b - b_0)\,.$$

By definition $(b - b_0) < 3\varepsilon\alpha$. Therefore the bid c is not an optimal reply to σ^k. Hence no bidder m bids in the interval $(b_0, c^k]$ when k is large.

Consider a bidder n for whom there exists a value v_n such that $c^k = c^k_n(v_n)$. For large k, $H^k_n(b_0|v_n) < H_n(b^-|v_n) + \varepsilon$ and $H^k_n(b_1|v_n) < H^*_n(b_1|v_n) + \varepsilon < H^*_n(b|v_n) + 2\varepsilon$. The probability that for some bidder $m \neq n$ with value in $V_m \setminus W_m$ bids in the interval $[b_0, b_1]$ is smaller than $\varepsilon/2$ for large k, since this holds at the limit strategy σ^*. By the above argument, for each $m \neq n$ in N^*, the values in W_m with bids in the range (b, c^k) are in $V_m \setminus V^\varepsilon_m$ and the probability of such an event is smaller than $\varepsilon/2$ by construction. This implies that for large

[8] This is the first point in the proof where the assumption that F is nonatomic is invoked. Without this assumption, in particular if there is an atom at $b = v_*$, there could be a tie with positive probability.

k, $H_n^k(c|v_n)$ is no more than $H_n^*(b^-|v_n) + 2\varepsilon$ on (b_0, c^k), and no more than $H_n^*(b|v_n) + 2\varepsilon$ on $[c^k, b_1)$. But the expectation $H_n^k(c^k|v_n)$ is strictly smaller than $H_n^*(b^-|v_n) + .5H_n^*(b|v_n) + 3\varepsilon$, which contradicts the definition of $c_n^k(v_n)$. Thus the initial supposition is falsified and Lemma 4.6 is proved. □

In sum, Lemma 4.5 shows that σ^* is an equilibrium of G^*, and Lemma 4.6 shows that it does not induce an atom in the distribution of any player's bids. Together with Theorem 3.3, these prove that σ^* is also an equilibrium of the auction game G, which completes the proof of Theorem 4.1. □

5. Concluding Remark

A valuable corollary of the existence theorem is that it identifies some equilibria of the auction game with atomless bid distributions as essential solutions of a well-defined fixed-point problem — a characterization that has been absent from previous work on this topic. A possible advantage is that stronger results might be obtained by invoking index theory. The Leray-Schauder index for fixed points of a map on a Banach space seems especially useful because it is obtained from the index of fixed points of nearby finite-dimensional perturbations of the map. In particular, essential fixed points are approximated by essential fixed points of finite-dimensional perturbations.

In later work we intend to examine whether the method in this paper can be applied to establish that an auction game has an essential equilibrium in pure strategies.

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