Substitute Goods and Multi-Item Auctions

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What’s New and What’s Not

- Much of this presentation is a summary of known results about the important consequences for auctions of two conditions:
  - “goods are substitutes”
  - the coalitional value function is “bidder-submodular.”
- The relation between these two conditions is “known but opaque.”
- What is *new* is direct mathematical treatment that connects the two conditions tightly.

“Gross Substitutes”

SUBSTITUTE GOODS

Substitute Goods

- Each bidder *n* is described by a continuous, non-decreasing valuation function v^n that specifies the value to the bidder of any bundle x^n of goods.
- Our analysis makes use of Hotelling profit functions:

$$\pi^n(p) \equiv \max_{x^n \geq 0, x^n \in X} v^n(x^n) - p \bullet x^n$$

- By Hotelling's lemma, goods are substitutes *if and only if* π is submodular.

Implications for Equilibrium

- **Theorem**. If goods are ("strong") substitutes for all bidders, then for every supply vector *y* there exists a price vector *p* such that *y* is in demand set at *p*.
- Applies to both divisible and indivisible goods.
  - Milgrom and Strulovici (2009): "strong substitutes"
  - Milgrom (2000)
  - Gul and Stacchetti (1999)
  - Kelso and Crawford (1982)

Aggregate Objective Function

- Let $v^S(y) \equiv \max_x \sum_{i \in S} v^i(x^i)$ subject to $\sum_{i \in S} x^i \leq y$,
  - Market demand can be computed by maximizing this function at any given prices. This works for both divisible and indivisible goods.
- Define $\pi^S(p) \equiv \sum_{i \in S} \pi^i(p)$.

Equilibrium Existence $\Rightarrow$ Duality

- **Theorem**. For the discrete case as well as the concave case, for all S, y, $v^S(y) = \min_{p \geq 0} \sum_{i \in S} \pi^i(p) + p \bullet y$.

- **Proof**. Fix y and let $\bar{p}$ be the corresponding market clearing price vector. Then, $y \in \text{argmax}_z \, v^S(z) - \bar{p} \bullet z$. Hence, $\pi^S(\bar{p}) = v^S(y) - \bar{p} \bullet y$, so $v^S(y) \geq \min_{p \geq 0} \pi^S(p) + p \bullet y$. By definition of π^S, for all p, $v^S(y) \leq \pi^S(p) + p \bullet y$. So, $v^S(y) = \min_p \pi^S(p) + p \bullet y = \min_p \sum_{i \in S} \pi^i(p) + p \bullet y$. **QED**

This treatment based on Ausubel and Milgrom (2002)

CORE AND BIDDER SUBMODULARITY

The Core

- Informally, the core consists of the "*competitive*" outcomes of an auction – the ones in which no group of bidders is willing to pay more to the seller for the items on offer.
- A feasible, individually rational auction outcome is *blocked* and hence *not* in the core if there is some coalition (necessarily including the seller) that could deviate to do better.
  - The seller would be paid more.
  - The deviating bidders would get better deals.

Maximal Core Payoffs

- Ausubel-Milgrom and Bikchandani-Ostroy.
  - Player 0 is the seller
- Theorem. For any set of bidders N, bidder i's Vickrey payoff is her maximum payoff over all core imputations:

$$w(N) - w(N - i) = \max\left\{\pi^i : \pi \in Core(N, w)\right\}$$

Proof Sketch

- Any imputation with a higher payoff for *i* is blocked by the coalition $N\backslash i$.
- The imputation where *i* gets $w(N)-w(N\backslash i)$, other bidders get 0, and the seller gets the rest is in the core.

Bidder Submodularity

- For any set of bidders S, the value of that coalition alone is zero, but the value if the seller with goods endowment $\bar{x}$ is also included is:

$$w(S) = \max_x \sum_{j \in S} v^j(x^j)$$

$$\text{subject to } \sum_{j \in S} x^j \leq \bar{x}$$

- "Bidder submodularity" means that the function *w* is submodular, that is,

$$(\forall S, T \subset N) \; w(S) + w(T) \geq w(S \cap T) + w(S \cup T)$$

Two Equivalent Formulations

- *w* is bidder submodular if and only if

$$\left(\forall S, i, j \notin S\right)$$
$$w(S+i) - w(S) \geq w(S+i+j) - w(S+j) \text{ or equivalently,}$$
$$w(S+i+j) - w(S) \geq \left(w(S+i+j) - w(S+j)\right) + \left(w(S+j+i) - w(S+i)\right)$$

- These say roughly that
  - Expanding the set of bidders (by adding j) reduces the marginal contribution of each existing bidder i
  - The marginal contribution of the pair *ij* to the coalition *S+i+j* is greater than the sum of their individual contributions

Vickrey Monotonicity

- In a Vickrey auction, each bidder's payoff is its marginal contributions to the coalition consisting of the seller and all other bidders: $w(S+j) - w(S)$.
- ***Theorem.*** For all sets of bidders S, each bidder's Vickrey payoffs falls weakly when an additional bidder enters the auction if and only if w is bidder submodular.

When is Vickrey in the Core?

- **Theorem**. The Vickrey auction outcome with bidders *S* is in Core(*S*,*w*) for all $S \subseteq N$ if and only if *w* is bidder submodular.
- Informal arguments. In a Vickrey auction…
  - … the outcome is efficient and, if *w* is bidder submodular, then each coalition of bidders *N\S* is paid no more than its incremental contribution, so the coalition *S* cannot block.
  - …if *w* is NOT bidder submodular, then some pair of bidders {i,j} gets MORE than its incremental contribution to some *S+i+j*, so coalition *S* can block.

Finally! What’s New!

CONNECTING SUBSTITUTES TO BIDDER SUBMODULARITY

A Coalitional Value Calculation

- What is the value of a coalition consisting of a bidder with valuation *v* and another with an additive or linear valuation *p*?

$$w(v,p) = \max_{x \geq 0} v(x) + p \bullet (\bar{x} - x)$$
$$= \pi(p) + p \bullet \bar{x}$$

- *This ties dual functions directly to coalitional value functions.*

…and a second Calculation

- What is the value of a coalition consisting of a bidder with valuation *v* and two others with additive valuations *p* and *p′*?

$$w(v,p,p') = \max_{x \geq 0} v(x) + (p \vee p') \bullet (\bar{x} - x)$$
$$= \pi(p \vee p') + (p \vee p') \bullet \bar{x}$$

- *This ties a lattice notion in the dual directly to coalitional value functions.*

Substitutes and Coalitional Values

- Let V be the set of goods valuations from which bidder valuations are drawn.
  - Let V_{add} be the set of additive goods valuations.
  - Let V_{sub} be the substitutable valuations.
- Given a profile of valuations $v \in V^N$, let w_v be the corresponding coalition value function.
- ***Theorem.*** (Substitutes & Bidder-Submodularity) Suppose that $V_{\text{add}} \subseteq V$. Then, V includes only substitutes goods valuations ($V \subseteq V_{sub}$) *if and only if* for all $v \in V^N$, w_v is bidder-submodular.

“Direct” Proof Ideas

- The definition of the coalitional value function is naturally given in terms of a maximization.
- Duality allows the maximization to be expressed as a dual minimization.
- Submodularity of the dual profit functions easily implies submodularity of the coalitional value functions.
- Hotelling’s lemma connects submodularity of the dual profit function to substitutes in demand.

Proof: Direct

- Suppose that $V \subseteq V_{sub}$. Think of a set S as represented by an indicator vector of 0's and 1's. Then for all $v \in V^N$,

$$\begin{aligned} w(S) &= \max_{x \geq 0} \sum_{i \in S} v^i(x^i) \text{ subject to } \sum_{i \in S} x^i \leq \bar{x} \\ &= \min_{p \geq 0} \max_x \sum_{i \in S} v^i(x^i) - p \cdot \left(\sum_{i \in S} x^i - \bar{x} \right) \\ &= \min_{p \geq 0} \sum_{i \in S} \pi^i(p) + p \cdot \bar{x} \\ &= \min_{p \geq 0} \sum_{i \in N} S^i \pi^i(p) + p \cdot \bar{x} \\ &= \min_{p \geq 0} F(p, S) \end{aligned}$$

- Since v is substitutable, π is decreasing and sub-modular, so F is submodular. Thus, w is submodular.

Proof: Converse

- Suppose the goods are not substitutes, so v is not submodular. Then there exist price vectors p and p' such that $\pi(p \wedge p') + \pi(p \vee p') > \pi(p') + \pi(p)$. But,

Coalition	*Value*
$v, p \wedge p'$	$\pi(p \wedge p') + (p \wedge p') \bullet \bar{x}$
$v, p \wedge p', p$	$\pi(p) + p \bullet \bar{x}$
$v, p \wedge p', p'$	$\pi(p') + p' \bullet \bar{x}$
$v, p \wedge p', p, p'$	$\pi(p \vee p') + (p \vee p') \bullet \bar{x}$

Recalling that $(p \vee p') \bullet \bar{x} + (p \wedge p') \bullet \bar{x} = p \bullet \bar{x} + p' \bullet \bar{x}$, it follows that the coalitional value function is not submodular. **QED**

Vickrey & Substitutes Summation

- Theorem.
  If $V_{\text{add}} \subseteq V$, then the five conditions below are equivalent.

1. $V \subseteq V_{sub}$
2. For all $v \in V^N$, $\pi_{w_v}^{Vickrey} \in Core(N, w_v)$
3. For all $v \in V^N$, $\pi_{w_v}^{n,Vickrey}(S)$ is non-increasing in S.
4. For all $v \in V^N$, losing bidders in the Vickrey auction have no profitable joint deviation.
5. For all $v \in V^N$, no Vickrey bidder can gain by the use of shills.

- Proof: See Milgrom (2004) chapter 8.

END