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Numerical Analysis of Asymmetric First Price Auctions

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We propose numerical algorithms for solving first price auction problems where bidders draw independent valuations from heterogeneous distributions. The heterogeneity analyzed in this work is what might naturally emerge when subsets of distributionally homogeneous bidders collude. Bid functions and expected revenues are calculated for two special cases. Extensions to more general asymmetric first price auctions are discussed. *Journal of Economic Literature* Classification Numbers: D44, C63, C72, D82.

1. INTRODUCTION

A large body of the literature on auction theory assumes symmetry of beliefs. Buyers have common underlying preferences and draw their signals from a symmetric probability distribution. A few key contributions in this line of research are Maskin and Riley (1984), Matthews (1983), Milgrom and Weber (1982a), and Riley and Samuelson (1981).

The symmetric case offers the related advantage that bid functions and/or expected revenues can often be obtained analytically. Moreover,

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derivation of the bid functions rarely is a prerequisite to revenue calculations, a key feature of many important results.

Most authors are well aware of the restrictive and often unrealistic nature of symmetry assumptions. Relaxing these assumptions results, however, in major analytical complications and presents a formidable challenge. Of course, Myerson's (1981) seminal work stands as an early exception in that he used distributional heterogeneity in deriving his revenue equivalence theorem. However, by invoking the Revelation Principle his work avoided addressing issues such as whether or not Nash equilibrium bid strategies exist for distributionally heterogeneous bidders at a first price auction. The pioneering contributions of Lebrun (1991) and Maskin and Riley (1991, 1992) attempt to fill many of the gaps in the literature with respect to bidder asymmetry.[1] These papers establish uniqueness and existence of the equilibrium at a first price auction when two bidders draw independent and private valuations from heterogeneous distributions. In addition, Maskin and Riley (1991) provide surprising revenue nonequivalence theorems.

Our own interest in asymmetric auctions stems from analysis of collusive behavior among bidders. If all bidders are ex ante homogeneous then collusion among subsets of bidders is very likely to generate asymmetries between participants at an auction.[2] Many challenging issues remain open at this level such as, for example, the viability of coalitions at first price auctions.[3]

In contrast with the symmetric case for first price auctions, expected revenue calculations for the asymmetric case typically require knowledge of the bid functions. The differential equations that characterize these bid functions are mostly untractable, hence numerical techniques can play an important role in the analysis of asymmetric (first price) auctions. However, numerical solutions to the systems of ordinary differential equations (ODE) that result from the first order conditions for the existence of an asymmetric Nash equilibrium are non-trivial to evaluate. Although these solutions belong to a class of "two-point boundary value problems" for

[1] We are not concerned in this paper with asymmetries in the information received by bidders. Previous work in that area includes Engelbrecht-Wiggans *et al.* (1983), Milgrom and Weber (1982b), Weverbergh (1979), and Wilson (1977).

[2] Exceptions are possible. If n bidders form k coalitions that are each of size n/k (where n/k is an integer) then the coalitions (as bidders) will be symmetric. Also, if k of n bidders collude and the others act individually then the resulting $n - k + 1$ bidders could be symmetric if the initial group of n bidders was asymmetric—the "weakest" k bidders collude and in aggregate are distributionally equivalent to any one of the noncolluding bidders.

[3] Although we only refer to first price auctions in this paper the analysis contained herein applies to Dutch auctions as well.

which there exist efficient numerical solution techniques,[4] they all suffer from major pathologies at the origin. First, forward extrapolation produces "nuisance" solutions (linear in our case) that do not satisfy terminal conditions and act as "attractors" on the algorithm. Second, and not unrelated, backward solutions are well-behaved except in neighborhoods of the origin where they become (highly) unstable with the consequence that standard (backwards) "shooting" by interpolation does not work. Naturally, high numerical accuracy becomes essential under such pathological conditions. All together, after a good deal of experimentation, we opted for using backward (piecewise low-order) series expansions for the bid functions. Though the series expansion techniques require more (analytical) input from their users than other purely numerical techniques might, they prove to be numerically stable as well as efficient for our purpose. Their generality will be discussed further in the course of the paper.

The case addressed in the present paper is that of a single-object first price auction where bidders are characterized by a particular kind of distributional heterogeneity. Specifically, the heterogeneity analyzed herein is what might naturally emerge from collusion among a subset of bidders.[5] The coalition submits a single bid at the main auction and behaves asymmetrically relative to the remaining bidders. Two scenarios will be considered in turn. The one which is easiest to handle assumes that the remaining bidders form a countercoalition. In this paper, this is equivalent to the case of two distributionally heterogeneous bidders. Our numerically determined equilibrium bid strategies are consistent with the results of Lebrun (1991) and Maskin and Riley (1991, 1992). Furthermore, we show numerically that the type of bidder asymmetry that might naturally arise from collusion leads to higher expected revenue for the seller in a first price auction as compared to an English auction. Since this type of asymmetry is not covered by Maskin and Riley's (1991) propositions concerning seller's revenue, it suggests an important direction for extension of their work.

Besides the case of two coalitions we also consider the more difficult case in which the remaining bidders act noncooperatively. The ease of

[4] An excellent source for ODE numerical algorithms is Press *et al.* (1986). See, in particular, the Bulirsch–Stoer method in Section 15.4 and the shooting method in Section 16.1.

[5] We do not pose a collusive mechanism in the paper. Strictly speaking, we are formally analyzing only noncooperative behavior by bidders who draw their private valuations independently from heterogeneous distributions. Since we refer to particular distributional types throughout the paper as "coalitions" it might be helpful to think of collusion in the following light. Suppose valuations of coalition members are common knowledge within the coalition. Furthermore, the member with highest value is awarded the item if the coalition wins at the main auction. There are no side-payments within the coalition—winner takes all. Coalition members who do not have the highest value cannot bid at the main auction.

transition from one case to the other illustrates the flexibility of the proposed technique. In addition, it suggests that the analysis of Lebrun (1991) and Maskin and Riley (1991, 1992) can be extended to more than two bidders.[6]

Although the analysis in this paper is in terms of heterogeneously distributed values, it can easily be interpreted as applying to heterogeneous preferences.[7] In particular, we reinterpret the equilibrium bid functions for the first price auction with two coalitions as a first price auction between two bidders with differing degrees of *risk aversion*. In this latter interpretation, the distribution of values is identical and only preferences differ between bidders.

Besides the intrinsic interest in the analysis of heterogeneous preferences we develop the risk aversion interpretation to guide our intuition regarding distributional heterogeneity. It is well-known that first price auctions give greater expected revenue than second price auctions in the presence of symmetric risk aversion—risk averse bidders shade their bids less than risk neutral bidders in the first price auction.[8] The same phenomenon is present in first price auctions when there are two "coalitions" of different size (i.e., n homogeneous bidders form two coalitions of size k and $n - k$). The distributional asymmetry created by the difference in coalition size induces the smaller coalition to shade their bid less than they would in the symmetric case, because shading their bid is more costly in terms of a reduced probability of winning when they face a larger coalition with a more favorable distribution. Although there are other factors relevant to the expected revenue comparison of first price and English auctions, the more aggressive bidding of the disadvantaged coalition seems to be decisive in making the first price auction superior.

The paper is organized as follows: The two models are introduced in Section 2; the two-coalition scenario is solved in Section 3; Section 4 deals with the case where the remaining bidders act noncooperatively; Section 5 offers a comparison to a second price auction, and Section 6 concludes. Additional results and algorithmic details are contained in three technical appendixes.

2. The Model

2.1. *Description*

Our analysis focuses on a single-object first price auction. Specifically, bidders simultaneously submit sealed bids for a single object where the

[6] Maskin and Riley (1992) demonstrate existence of a Nash equilibrium for general n. Uniqueness results are only provided for $n = 2$ when bidders are distributionally heterogeneous.

[7] This point was made by Maskin and Riley (1991).

[8] See Riley and Samuelson (1981) or Riley (1989).

highest bidder wins and pays his bid price. The auctioneer is presumed to be a bid taker who does not act strategically (i.e., no reserve,no entrance fee, etc.). The group of potential bidders comprises n risk neutral individuals who all draw their valuations independently from a uniform distribution on [0, 1]. A subgroup consisting of $k_1 < n$ bidders form a coalition. We assume this coalition acts as one bidder who draws a valuation from the cumulative distribution x^{k_1} where $x \in [0, 1]$.[9]

Consider next the $k_2 = n - k_1$ remaining bidders. Two alternative scenarios will be considered. Under the first scenario we assume that these k_2 bidders form a (counter)coalition which acts as one bidder that draws a valuation from the cumulative distribution x^{k_2} where $x \in [0, 1]$. We are then essentially dealing with a two (aggregate) bidder asymmetric first price auction which constitutes a special case of a more general problem for which Lebrun (1991, Chap. 3) establishes the existence and uniqueness of a Nash equilibrium in pure strategies.

Under the alternative scenario, the k_2 remaining bidders act noncooperatively. Hence $k_2 + 1$ participants will be active at the main auction, of which k_2 draw their valuation from independent uniform distributions on [0, 1] and the last one from the distribution corresponding to the highest of k_1 independent uniforms on [0, 1].

At a more theoretical level the two scenarios naturally constitute the polar cases we might wish to consider in order to define characteristic functions for noninclusive coalitions at first price auctions, but this goes beyond the objectives of the present paper.

2.2. *Notation*

Bid functions are denoted by the greek letter ϕ appropriately subscripted (1 for the k_1-coalition, 2 either for the k_2-coalition or for the symmetric bid function of the k_2 individual bidders, depending on the case considered). Lebrun (1991) has shown that these bid functions are strictly monotone increasing and, therefore, invertible. Inverse bid functions are denoted by the greek letter λ. An obvious necessary condition for (λ_1, λ_2) to be a pair of Nash equilibrium strategies is that they have a common support in the form of an interval $[0, t_*]$, where t_* is the bid associated with a

[9] We are currently exploring the issue of whether or not mechanisms exist that would permit a less than all-inclusive coalition from earning more than non-cooperative expected surplus at a first price auction. Such existence problems appear to be quite untractable, analytically at least. We hope to gain critical insight on these issues by being able to solve numerically such problems as the ones described here. See in particular the tables of results that are provided below as well as the discussion in Section 5.

unit valuation.[10] The (numerical) determination of t_* is a critical component of the problem to be solved.

3. Coalition versus Coalition

3.1. *The Differential Equations*

Let $t = \phi_1(v)$ denote the Nash equilibrium bid submitted by coalition 1 when its highest evaluation is v. Hence t is given by

$$t = \text{Argmax}(v - t)[\lambda_2(t)]^{k_2}. \tag{1}$$

The first-order condition generates the following differential equation:

$$k_2[\lambda_1(t) - t]\lambda_2'(t) = \lambda_2(t). \tag{2}$$

The corresponding equation for coalition 2 is given by

$$k_1[\lambda_2(t) - t]\lambda_1'(t) = \lambda_1(t). \tag{3}$$

The initial conditions are

$$\lambda_1(0) = \lambda_2(0) = 0 \tag{4}$$

and the terminal condition requires the existence of a number $t_* \in (0, 1)$ such that

[10] Some intuition for this well known result is best offered in the contrapositive. Suppose that the upper supports of the bid distributions were $t_2 > t_1$ for the two bidders and the upper supports were bid with zero probability. Then Bidder 2 would always win when drawing valuations that led to bidding in the interval $[t_1, t_2]$. However, when drawing such valuations bidder 2 would always win when bidding exactly t_1 and by doing so he would decrease the amount paid for the item.

Now consider the lower support. Without loss of generality suppose that l_1 and l_2 are the lower supports for bidders 1 and 2 where $0 \leq l_1 \leq l_2 < t_*$. Hence bidder 2 will submit a bid of at least l_2 for any valuation in the interval $[0, l_2]$. In order for this to be optimal it must be the case that he can never win with such a bid. In order to never win it must be the case that the lower support of bidder 1's bid distribution is $l_1 = l_2$ and that this amount is bid with zero probability by bidder 1. But this means that bidder 1 will bid at least l_1 for all valuations, including those that fall in the interval $[0, l_1]$. But by doing so bidder 1 will earn negative surplus when winning. Consequently, the lower support for bidder 1's bid distribution cannot be $l_1 > 0$. The only values of l_1 and l_2 for which this argument fails is $l_1 = l_2 = 0$.

$$\lambda_1(t_*) = \lambda_2(t_*) = 1. \tag{5}$$

We digress momentarily to comment on the relationship of this model of heterogeneous distributions to a model with heterogeneous preferences. Suppose that bidders 1 and 2 each draw their valuation from the uniform distribution over the unit interval. Further suppose that each bidder has a utility function $U(x) = x^{\alpha_i}$ where $0 < \alpha_i \leq 1$. The objective function for a bidder is then

$$(v - t)^{\alpha_i}[\lambda_j(t)].$$

Then if $\alpha_1 = 1/k_2$ and $\alpha_2 = 1/k_1$, Eqs. (2) through (5) also characterize the inverse bid functions for the auction between bidders with heterogeneous risk aversion.[11]

Lebrun (1991) transforms the system consisting of Eqs. (2) and (3) into a Cauchy system of differential equations and, in doing so, is able to establish the existence and uniqueness of a solution, although that solution cannot be found analytically. Furthermore, we have obtained an analytical relationship between λ_1 and λ_2 as well as an analytical expression for t_*. See Appendix A for details. The algorithm discussed below does not require that such expressions be available but they are useful for evaluation of numerical accuracy.

3.2. *Numerical Solution*

Let l_i denote the (right-) derivative of λ_i at the origin,

$$l_i = \lim_{t \to 0^+} \lambda_i'(t). \tag{6}$$

A straightforward application of l'Hospital's rule to Eqs. (2) and (3) yields the following expressions:

$$l_1 = 1 + \frac{1}{k_2}, \qquad l_2 = 1 + \frac{1}{k_1}. \tag{7}$$

Successive derivations of Eqs. (2) and (3) reveal that all higher derivatives of λ_1 and λ_2 are 0 at the origin. It follows that any attempt to evaluate numerically a forward solution to Eqs. (2) and (3) produces a linear solution given by

[11] Chen and Plott (1991) investigate, through analytics and experiments, the issue of heterogeneous risk preferences among bidders.

$$\lambda_i(t) = l_i \cdot t, \tag{8}$$

which is unacceptable since it does not satisfy the terminal condition (5). Furthermore, experimentation indicates that the "nuisance" solution (8) acts as an "attractor" in the sense that forward numerical solutions cannot deviate from it. Hence, our first recommendation is that systems of differential equations such as those consisting of Eqs. (2) and (3) should be solved backward starting from an assumed terminal point using the initial condition (4) as an indicator of whether or not we have used the correct value for t_*.

As shown by Lebrun (1991) and fully confirmed by our numerical results, the solutions to Eqs. (2) and (3) are monotonic in t_*, an important consideration in the design of a backwards "shooting" algorithm. Note, however, that the solutions exhibit an explosive tendency toward minus infinity around the origin as soon as t_* exceeds its equilibrium value. Though this property of the solution enables us to pinpoint t_* with considerable numerical accuracy, it precludes us from applying conventional interpolative shooting techniques.

The partial analytical results which are derived in Appendix A are based upon a transformation of Eqs. (2) and (3) which are reformulated in terms of the auxiliary functions

$$\delta_i(t) = \frac{1}{t} \cdot \lambda_i(t), \qquad i = 1, 2. \tag{9}$$

The transformed system is given by

$$k_2[\delta_1(t) - 1] \cdot [\delta_2(t) + t \cdot \delta_2'(t)] = \delta_2(t) \tag{10}$$

$$k_1[\delta_2(t) - 1] \cdot [\delta_1(t) + t \cdot \delta_1'(t)] = \delta_1(t), \tag{11}$$

together with the following initial and terminal conditions:

$$\delta_i(0) = l_i, \ \delta_i(t_*) = \frac{1}{t_*}, \qquad i = 1, 2 \tag{12}$$

Extensive numerical experimentation with this model, as well as with the one which is discussed in Section 4 below, indicates that the solutions of the transformed system are numerically more "stable," and their accuracy easier to monitor, than those of the initial system.

The solution technique that has proved most efficient for our purpose and which, furthermore, can easily be adapted to other scenarios consists in approximating the δ's by piecewise (low-order) polynomial expansions.[12] All that is required is an efficient algorithm for the evaluation of Taylor series expansions of any order around base points appropriately selected in the interval $(0, t_*]$. Let t_j denote such a base point and

$$\delta_1(t) = \sum_{i=0}^{\infty} a_i(t - t_j)^i \tag{13}$$

$$\delta_2(t) = \sum_{i=0}^{\infty} b_i(t - t_j)^i. \tag{14}$$

Substituting these expressions for the δ's in formulas (10) and (11) and equating the coefficients of $(t - t_j)^n$ across the board yield the following recurring expressions for the a's and b's:

$$\begin{aligned} a_{n+1} = \frac{1}{(n+1)(b_0 - 1)t_j} \cdot \Bigg\{ & \left[\frac{1}{k_1} - (n+1)(b_0 - 1)\right] a_n \\ & - \sum_{i=1}^{n} i \cdot b_{n+1-i} \cdot (a_{i-1} + t_j \cdot a_i) \Bigg\} \end{aligned} \tag{15}$$

$$\begin{aligned} b_{n+1} = \frac{1}{(n+1)(a_0 - 1)t_j} \cdot \Bigg\{ & \left[\frac{1}{k_2} - (n+1)(a_0 - 1)\right] b_n \\ & - \sum_{i=1}^{n} i \cdot a_{n+1-i} \cdot (b_{i-1} + t_j \cdot b_i) \Bigg\}. \end{aligned} \tag{16}$$

These expressions also apply for $n = 0$ under the standard convention that the summations from $i = 1$ to $i = 0$ are set equal to zero. Initial conditions are

$$a_0 = \delta_1(t_j), \qquad b_0 = \delta_2(t_j). \tag{17}$$

In practice it proves convenient to compute directly the individual factor in formulas (15) and (16). Let[13]

[12] It is important to note that the successive derivatives in formulas (13) and (14) will be evaluated analytically, in sharp contrast with the algorithms described in Press *et al.* (1986, Chap. 16) where derivatives are evaluated numerically. The use of analytic derivatives improves numerical accuracy and circumvents the well-known problems associated with the inaccuracy of numerical high-order derivatives.

[13] The a^*'s and b^*'s are functions of both t_j, the base point, and $t - t_j$, but there is no need to account for these arguments in our notation since, in particular, the δ's will be evaluated recursively, one point at a time.

$$a_i^* = a_i \cdot (t - t_j)^i, \qquad b_i^* = b_i \cdot (t - t_j)^i. \tag{18}$$

The recursions for the a^*'s and b^*'s follow from formulas (15) and (16) and are given by

$$\begin{aligned} a_{n+1}^* = & \frac{1}{(n+1)(b_0 - 1)t_j} \cdot \left\{ \left[\frac{1}{k_1} - (n+1)(b_0 - 1) \right] (t - t_j) a_n^* \right. \\ & \left. - \sum_{i=1}^{n} i b_{n+1-i}^* \cdot ((t - t_j) a_{i-1}^* + t_j \cdot a_i^*) \right\} \end{aligned} \tag{19}$$

$$\begin{aligned} b_{n+1}^* = & \frac{1}{(n+1)(a_0 - 1)t_j} \cdot \left\{ \left[\frac{1}{k_2} - (n+1)(a_0 - 1) \right] (t - t_j) b_n^* \right. \\ & \left. - \sum_{i=1}^{n} i a_{n+1-i}^* \cdot ((t - t_j) b_{i-1}^* + t_j \cdot b_i^*) \right\}. \end{aligned} \tag{20}$$

3.3. *The Algorithm*

For the case under consideration, t_* happens to be known analytically (see Appendix A). However, this seems to be the exception rather than the rule. Hence, we opted for an algorithm that includes an iterative search for the equilibrium value of t_*.

A single run of computation requires initializing certain parameters (such as t_*), evaluating the corresponding numerical solution, and then deciding upon whether or not another run is needed.

1. *Initialization.* The parameters to be initialized are below.

(i) t_*: It can be shown that $t_*^{-1} \in (l_1, l_2)$.[14]

(ii) N: The number of (equal length) subintervals of $(0, t_*)$ to be considered. These subintervals are of the form (t_{j-1}, t_j) with $t_0 = 0$ and $t_{N+1} = t_*$.

(iii) p: The order of the Taylor series expansions.

(iv) ε: A small positive number to be used in the evaluation of our convergence criterion. Guidelines for the selection of (N, p, ε) are discussed below.

2. *Numerical Evaluation.* Approximate values for the pairs $(\delta_1(t_{j-1}), \delta_2(t_{j-1}))$ are computed recursively by means of Taylor series expansions

[14] There is no loss of generality in assuming that $l_1 < l_2$ under the current scenario. The case when $l_1 = l_2 = l$ corresponds to a symmetric Nash equilibrium for which the (analytical) solution is known to be $\delta(t) = l$.

of order p around t_j, for j running backward from $j = N + 1$ to $j = 1$. Initial values are $\delta_i(t_{N+1}) = \delta_i(t^*) = 1/t_*$.

3. *Convergence Criterion.* From a theoretical perspective, a reasonable convergence criterion is provided by the following inequality:

$$\frac{1}{2} \cdot \sum_{i=1}^{2} [\delta_i(0) - l_i]^2 \le \varepsilon^2. \tag{21}$$

If (21) holds, then the corresponding sequences of $\delta_i(t_j)$'s constitute our (approximate) numerical solution. Otherwise, t_* is adjusted in whatever direction reduces the left-hand side of (21) (typically the direction is recognized from earlier iterations). In practice, numerical instability in the immediate neighborhood of the origin necessitates fine tuning of the convergence criterion (21). Details are found in Appendix B.

Experimentation with values of k_1 and k_2 running from 1 to 100 suggests that the following values for (N, p, ε) prove more than adequate for all practical purposes:

$N = 10{,}000$

$p = 5$

ε of the order of 10^{-5} to 10^{-8} (see Appendix B).[15]

Computing time for a single run of computation under these values is of the order of 5 sec on a DEC 3100 workstation or a 486/33 PC. Except for start-up costs, computing time is essentially proportional to $\frac{1}{2} \cdot Np \cdot (p + 1)$ and can easily be reduced by a factor of 10 at little loss of precision (as measured, in particular, by ε). Highly accurate numerical evaluation of t_* (6 to 8 correct digits) requires only a few iterations (10 to 20 under our current interactive "rule of thumb" implementation; probably less once an efficient search algorithm is implemented). Values of t_* and ε for all pairs (k_1, k_2) with $k_1 + k_2 = 5$ and $0 < k_1, k_2 < 5$, $N = 10{,}000$, and $p = 5$ are given in Table I. The corresponding bid functions are illustrated in Fig. 1. Inspection of the bid functions shows that the more optimistic party (larger coalition) shades their bid more which is consistent with

[15] Relative errors in the numerical values of the δ functions typically are monotone functions of the distance away from the terminal point. Hence, ε constitutes a conservative approximation to (relative) numerical accuracy on the entire trajectory.

TABLE I

COALITION VERSUS COALITION

k_1	k_2	t_*	ε_*
3	2	0.70769174	0.997×10^{-8}
4	1	0.63737587	0.494×10^{-7}
100	1	0.73910288	0.518×10^{-8}

Maskin and Riley (1991). We evaluate the corresponding expected revenues and comment further on these results in Section 5 below.

4. COALITION VERSUS INDIVIDUAL NON-COLLUSIVE BIDDERS

4.1. *The Differential Equations*

The differential equation for the k_1-coalition remains unchanged and is given by Eqs. (2) and (10). In contrast, each of the remaining k_2 bidders now faces not only the k_1-coalition but also $(k_2 - 1)$ symmetric individual rivals. Thus, the bid t for an individual player with valuation v is given by

$$t = \operatorname{Arg\,max}(v - t) \cdot [\lambda_1(t)]^{k_1} \cdot [\lambda_2(t)]^{k_2-1}. \tag{22}$$

The corresponding differential equation is

$$[\lambda_2(t) - t] \cdot [k_1 \cdot \lambda_1'(t) \cdot \lambda_2(t) + (k_2 - 1) \cdot \lambda_2'(t) \cdot \lambda_1(t)] = \lambda_1(t) \cdot \lambda_2(t) \tag{23}$$

or, after transformation,

$$\begin{aligned} &[\delta_2(t) - 1] \cdot [k_1 \cdot (\delta_1(t) + t \cdot \delta_1'(t)) \cdot \delta_2(t) \\ &\quad + (k_2 - 1) \cdot (\delta_2(t) + t \cdot \delta_2'(t)) \cdot \delta_1(t)] = \delta_1(t) \cdot \delta_2(t). \end{aligned} \tag{24}$$

The initial condition for δ_2 follows by application of l'Hospital rule and is given by

$$\delta_2(0) = 1 + \frac{1}{k_1 + k_2 - 1}. \tag{25}$$

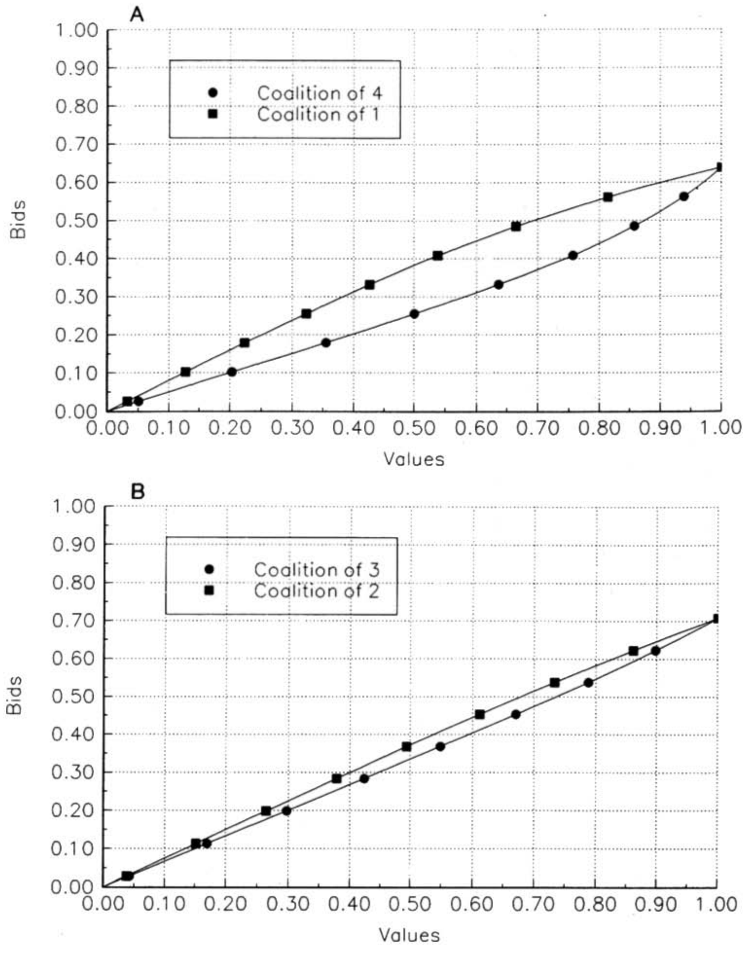


FIG. 1. Coalition vs coalition, $n = 5$: (a) 4 vs 1, (b) 3 vs 2.

Brute force application of Taylor series expansions to Eq. (24) requires evaluating double combinatorial summations. However, computation and programming greatly simplify if the problem is first reformulated in terms of the following auxiliary functions:

$$\delta(t) = ([\delta_1(t)]^{k_1} \cdot [\delta_2(t)]^{k_2-1})^{1/(k_1+k_2-1)} \tag{26}$$

$$\phi_i(t) = \delta_i(t) \cdot \delta(t) \tag{27}$$

$$\phi(t) = \delta_1(t) \cdot \delta_2(t). \tag{28}$$

In particular, Eq. (24) can be rewritten as

$$(k_1 + k_2 - 1) \cdot [\delta_2(t) - 1] \cdot [\delta(t) + t \cdot \delta'(t)] = \delta(t), \tag{29}$$

which is the functional analog to Eq. (11). The Taylor series expansions to be evaluated are now given by

$$\delta_1(t) = \sum_{i=0}^{\infty} a_i \cdot (t - t_j)^i, \qquad \phi_1(t) = \sum_{i=0}^{\infty} \alpha_i \cdot (t - t_j)^i \tag{30}$$

$$\delta_2(t) = \sum_{i=0}^{\infty} b_i \cdot (t - t_j)^i, \qquad \phi_2(t) = \sum_{i=0}^{\infty} \beta_i \cdot (t - t_j)^i \tag{31}$$

$$\delta(t) = \sum_{i=0}^{\infty} c_i \cdot (t - t_j)^i, \qquad \phi(t) = \sum_{i=0}^{\infty} \gamma_i \cdot (t - t_j)^i. \tag{32}$$

The recursive relationships for these coefficients consist of Eq. (16) together with the following additional equations listed in the order in which they are to used at each iterative step,

$$\begin{aligned} c_{n+1} = \frac{1}{(n+1) \cdot (b_0 - 1) \cdot t_j} \cdot \Bigg\{ \left[\frac{1}{k_1 + k_2 - 1} - (n+1) \cdot (b_0 - 1) \right] \cdot c_n \\ - \sum_{i=1}^{n} i \cdot b_{n+1-i} \cdot (c_{i-1} + t_j \cdot c_i) \Bigg\} \end{aligned} \tag{33}$$

$$\begin{aligned} a_{n+1} = c_{n+1} \cdot \frac{\alpha_0}{\beta_0} + \theta \cdot \left(c_{n+1} \cdot \frac{\alpha_0}{\beta_0} - b_{n+1} \cdot \frac{\alpha_0}{\beta_0} \right) \\ + \frac{1}{(n+1) \cdot \beta_0} \cdot \{A + B\}, \end{aligned} \tag{34}$$

where

$$\begin{gathered} A = \sum_{i=0}^{n} i(c_i \gamma_{n+1-i} - a_i \beta_{n+1-i}), \qquad B = \theta \sum_{i=1}^{n} i(c_i \gamma_{n+1-i} - b_i \alpha_{n+1-i}) \\ \alpha_{n+1} = \sum_{i=0}^{n+1} a_i c_{n+1-i}, \qquad \beta_{n+1} = \sum_{i=0}^{n+1} b_i c_{n+1-i}, \qquad \gamma_{n+1} = \sum_{i=0}^{n+1} a_i b_{n+1-i}, \\ \theta = (k_2 - 1)/k_1, \end{gathered} \tag{35}$$

together with the (additional) initial conditions:

$$a_0 = c_0 \cdot \left(\frac{c_0}{b_0} \right)^{\theta} \tag{36}$$

TABLE II

COALITION VERSUS INDIVIDUALS

k_1	k_2	t_*	ε_*
1	4	0.80000000	0.100×10^{-8}
2	3	0.78324204	0.109×10^{-5}
3	2	0.74169876	0.553×10^{-5}
4	1	0.63737587	0.278×10^{-4}
99	2	0.84113794	0.344×10^{-7}

$$\alpha_0 = a_0 c_0, \qquad \beta_0 = b_0 c_0, \qquad \gamma_0 = a_0 b_0. \tag{37}$$

The final transformation into the equivalent of formulas (19) and (20) in Section 3 is straightforward and need not be reproduced here.

4.2. *The Algorithm*

The generalization of the algorithm derived in Section 3 to this more complicated problem is essentially straightforward and requires only minor modifications (essentially the addition of Eqs. (34) and (35) and the relabeling of a couple of variables).

Equally important, though we neither have an analytical expression for t_* nor an analytical relationship between δ_1 and δ_2 as in Section 3, experimentation quickly reveals that all the attractive numerical properties of our earlier algorithm are preserved including, in particular, the amazing numerical accuracy in the determination of t_*.

Results are given in Table II and illustrated in Fig. 2 for all relevant pairs (k_1, k_2) with $k_1 + k_2 = 5$ and $0 < k_1, k_2 < 5$ and are directly comparable to the corresponding results in Table I and Fig. 1. They are further commented upon in Section 5 below.

4.3. *Existence and Uniqueness of a Solution*

The proof in Lebrun (1991) does not cover our second scenario as a special case. However, our numerical algorithm indicates that, within numerical accuracy of the order of 10^{-6} to 10^{-9}, there is one and only one value of t_* (in the relevant interval) for which the first-order conditions are satisfied.[16]

[16] At an anecdotal level, the algorithm described in Section 3 was partially developed at a time when neither Lebrun's proof nor the algebraic results in Appendix A were available. Though rational numbers need not be easily recognizable from their (low-order) decimal expansion, we had acquired the "certainty" that for $k_1 = 2$ and $k_2 = 1$, t_* was equal to 37/64 at a very early stage of our investigation.

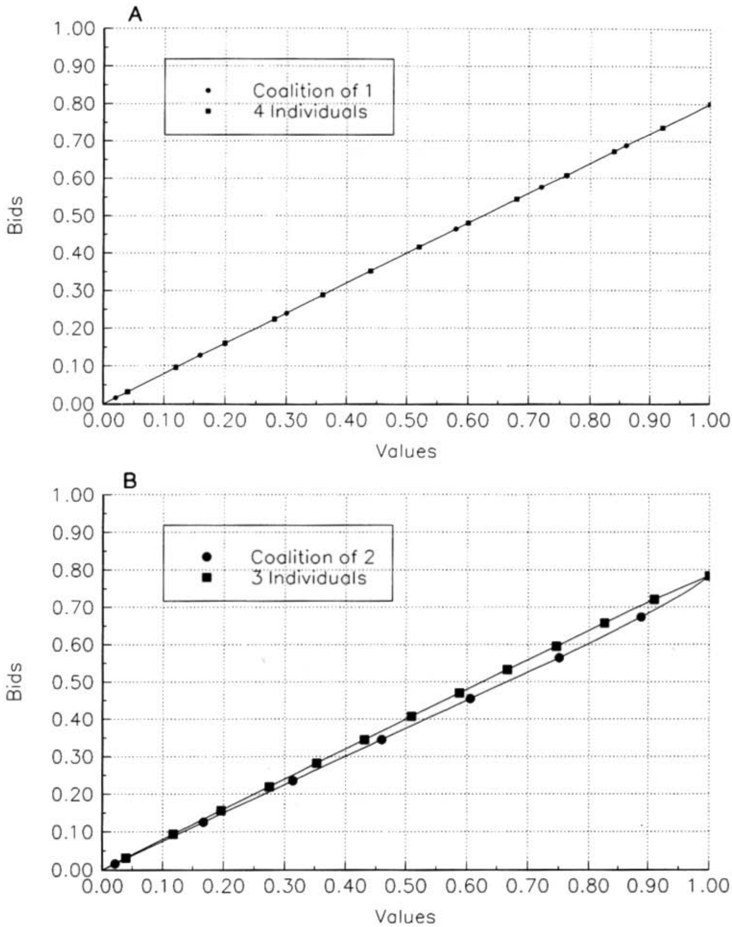
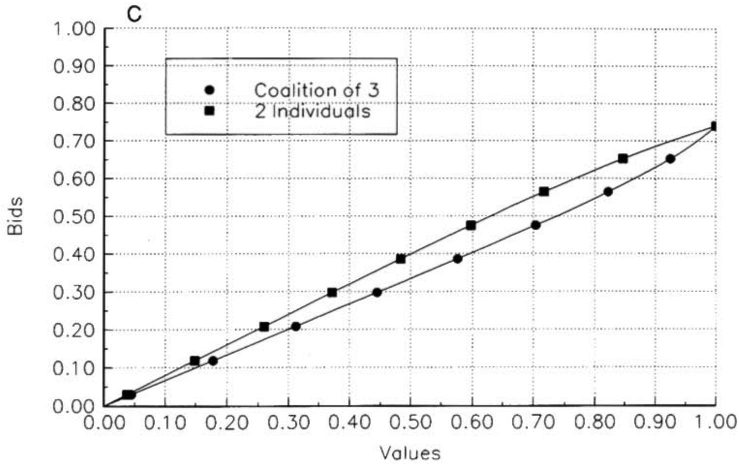


FIG. 2. Coalition vs individuals, $n = 5$: (a) 1 vs 4, (b) 2 vs 3, (c) 3 vs 2, and (d) 4 vs 1.

We can think of numerous generalizations of the two scenarios considered in this paper whose solutions are still going to be (monotonic) functions of a single parameter such as t_*. Numerical "proofs" of the existence and uniqueness of a solution would proceed along the lines discussed here and are likely to prove quite straightforward.

Conversely, our experience with similar iterative procedures in different though related contexts suggests that nonexistence of a Nash equilibrium would manifest itself in the form of cycles in the numerical search for t_* (typically between a value of t_* for which $\delta_1(0)$ is zero but $\delta_2(0)$ is not and another value for which $\delta_2(0)$ is zero).



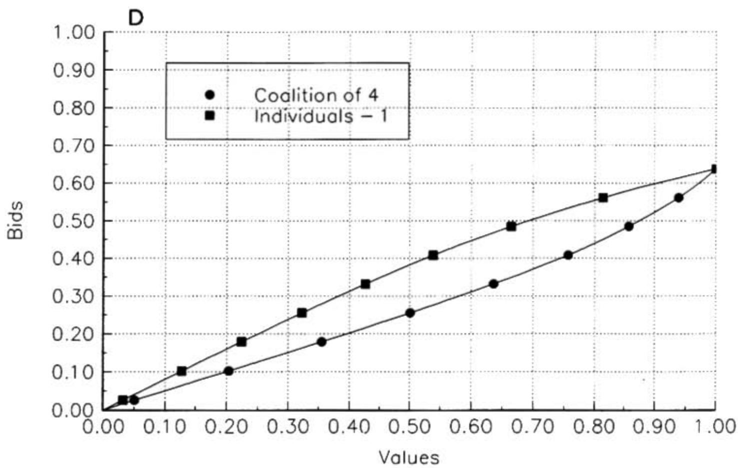


FIG. 2—*Continued*

In short, we are now convinced that there is considerable potential for relatively straightforward numerical investigations of the existence and the uniqueness of Nash equilibrium solutions within a fairly broad range of first price asymmetric auctions.

4.4. *Nonuniform Distribution*

There are numerous interesting scenarios to be considered that require assigning different and/or nonuniform distributions to certain classes of bidders.[17] The extension of our baseline algorithm that is discussed in this subsection applies to either of the two scenarios considered above but, for ease of notation, we restrict our attention to the one discussed in Section 3. We now assume that the bidders in the k_1 and k_2 coalitions draw their (individual) valuation from different distributions with distribution functions F_1 and F_2, respectively. Let

$$\lambda_i^*(t) = F_i(\lambda_i(t)), \qquad i = 1, 2, \tag{38}$$

so that the range of $\lambda_i^*(t)$ remains the interval (0, 1). Since λ_i can easily be retrieved from λ_i^* by inversion of formula (38), using standard interpolation techniques, we now focus our attention on the numerical evaluation of the λ^*'s (to be eventually transformed into δ^*'s, in line with formula (9), a step we do not discuss again here). The counterparts of Eqs. (2) to (5) are now given by

$$k_2 \cdot [F_1^{-1} \cdot (\lambda_1^*(t)) - t] \cdot \lambda_2^{*\prime}(t) = \lambda_2^*(t) \tag{39}$$

$$k_1 \cdot [F_2^{-1} \cdot (\lambda_2^*(t)) - t] \cdot \lambda_1^{*\prime}(t) = \lambda_1^*(t) \tag{40}$$

together with initial and terminal conditions of the form

$$\lambda_1^*(0) = 0, \lambda_i^*(t^*) = 1, \qquad i = 1, 2. \tag{41}$$

Also, the slope of λ_i^* at the origin is given by

$$l_i^* = \lim_{t \to 0^+} \lambda_i^*(t) = f_i(0) \cdot l_i, \tag{42}$$

where $f_i(0)$ is the derivative of F_i at the origin and l_i has been defined in Eq. (7).

Adaptation of our baseline algorithm to Eqs. (39) and (40) requires a few minor modifications as well as a more important one. The latter requires implementation of Taylor series expansions for the function

[17] We have not yet considered extensions to affiliated heterogeneous distributions but this might be an important area for investigation. When antique dealers collude it appears that a primary motivation is to conceal their appraisal expertise from the collectors in attendance—two dealers who know that a highboy is an authentic period piece will reveal this assessment to uninformed collectors if they bid against one another. Preservation of the returns to their private information might be a reason why informed bidders can form less than all-inclusive coalitions that are profitable relative to noncooperative behavior at first price auctions.

TABLE III

AUCTIONEER'S EXPECTED REVENUE AND BIDDERS' EXPECTED SURPLUS (PER CAPITA) AT A FIRST PRICE AUCTION ($n = 5$)

		Coalition vs. Coalition			Coalition vs. Individual		
k_1	k_2	Auct.	k_1	k_2	Auct.	k_1	k_2
1	4	0.5057	0.0860	0.0567	0.6668	0.0335	0.0333
2	3	0.5875	0.0523	0.0467	0.6510	0.0352	0.0371
3	2	0.5875	0.0467	0.0523	0.6089	0.0406	0.0488
4	1	0.5057	0.0567	0.0860	0.5057	0.0567	0.0860
5	0	0.0000	0.1667	N/A	0.0000	0.1667	N/A

Note. Computed by Monte Carlo using 100,000 drawings. Inversion of the λ's in order to retrieve the bid functions themselves proceeds by interpolation—the interval (0, 1) is divided into R subintervals of equal lengths and the bids corresponding to the separation points are computed and stored. The largest Monte Carlo standard error for any entry in the table is 0.00032.

$F_i^{-1}(\lambda_i^*(t))$. An algorithm for efficient recursive evaluation of such expansions is provided in Appendix C.

5. SECOND PRICE VERSUS FIRST PRICE AUCTIONS

At a minimum this research provides a numerical solution for bid functions that would be employed by specific kinds of distributionally heterogeneous bidders at a first price auction. A more optimistic view is that this research provides an insight into a revenue nonequivalence between first price (or Dutch) and second price (or English) auctions when a subset of bidders collude.

Suppose a bidder coalition at a first price auction can be characterized as follows. The membership of the coalition is determined ex ante and the types of each coalition member are common knowledge within the coalition. If the coalition wins the item the coalition member with highest value is awarded the item. There are no side payments within the coalition. In other words, winner takes all. Also, coalition members with values that are not highest within the coalition cannot submit a bid at the main auction. For this description of coalition behavior the expected revenues and surpluses for $k_1 + k_2 = 5$ are reported in Tables III and IV for the first price and second price auctions, respectively.[18] Similar calculations

[18] The reader should be careful to note that these expected revenue calculations do not apply to the risk aversion interpretation. Although both interpretations of our auction problem yield the same bid functions, the order statistics are not the same for the two.

TABLE IV

AUCTIONEER'S EXPECTED REVENUE AND BIDDERS' EXPECTED SURPLUS (PER CAPITA) AT A SECOND PRICE AUCTION (n = 5)

		Coalition vs. Coalition			Coalition vs. Individuals		
k_1	k_2	Auct.	k_1	k_2	Auct.	k_1	k_2
1	4	0.4667	0.0333	0.0833	0.6667	0.0333	0.0333
2	3	0.5833	0.0417	0.0556	0.6501	0.0417	0.0333
3	2	0.5833	0.0556	0.0417	0.6001	0.0556	0.0333
4	1	0.4667	0.0833	0.0333	0.4667	0.0833	0.0333
5	0	0.0000	0.1667	N/A	0.0000	0.1667	N/A

Note. All calculations are analytic.

for the first price and second price auctions are reported in Table V for $k_1 + k_2 = 101$ (for large k_1 only).[19] Bid functions are depicted in Fig. 3a for $k_1 = 100$ and in Fig. 3b for $k_1 = 99$ ($k_2 = 2$ individual bidders). Note that the left-hand side of Tables III–V concern the case of a coalition facing a complementary coalition, while the right-hand side concerns the case of a coalition facing individual noncooperative bidders.

In comparing Tables III and IV and the calculations within Table V, note that conditional on the size of the coalition the auctioneer's expected

TABLE V

AUCTIONEER'S EXPECTED REVENUE AND BIDDERS' EXPECTED SURPLUS (PER CAPITA)—n = 101

	k_1	k_2	Auct.	k_1	k_2
Second Price*	100	1	0.4999	0.0049	0.0001
	99	2	0.6665	0.0033	0.0001
First Price**	100	1	0.6578	0.0025	0.0412
	99	2	0.7787	0.0015	0.0159
First Price or Second Price*	101	0	0.0000	0.0098	N/A

* Calculated analytically.

** Based upon 1 million drawings. (100 vs 1), The largest standard deviation for an element in the row is 0.000024. (99 vs 2) The largest standard deviation for an element in the row is 0.000008.

[19] Besides their theoretical interest, high k scenarios provide a useful test of numerical accuracy since experimentation indicates that, everything else being constant, accuracy decreases as k_1 and/or k_2 increases.

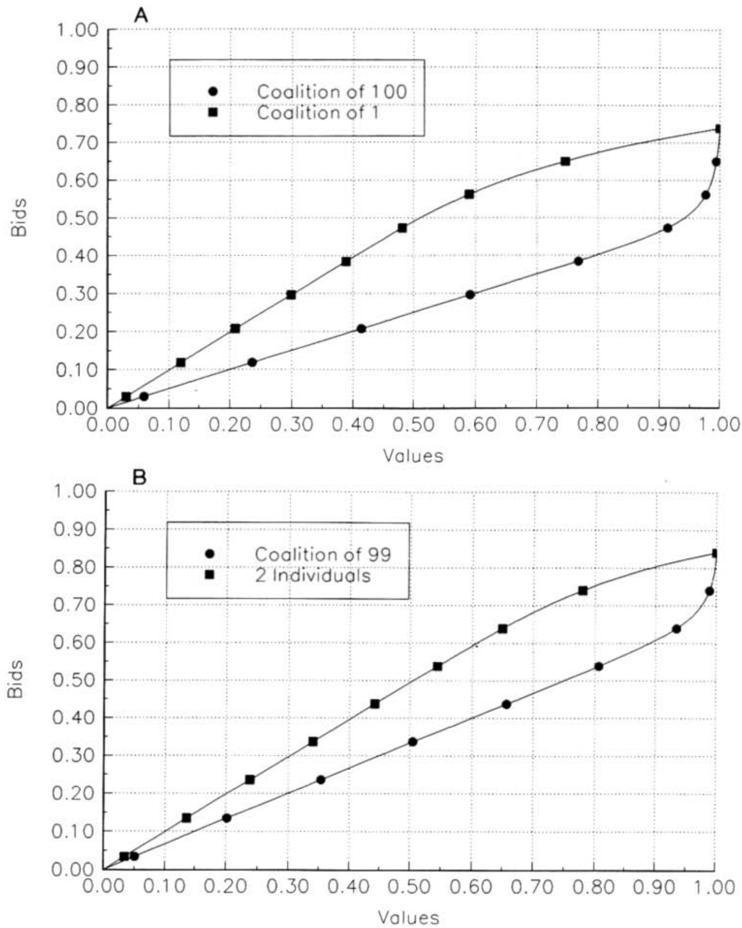


FIG. 3. Coalition vs individuals, $n = 101$: (a) 100 vs 1, (b) 99 vs 2.

revenue at the first price auction is always greater than or equal to expected revenue at the second price auction. We take this to be evidence supporting our conjecture that the first price auction is less susceptible to collusion than the second price auction. We argue in the following paragraphs that the calculations for the first price auction are a lower bound on the expected revenue to the auctioneer, while those for the second price auction reflect its actual performance.

A variety of issues are relevant in assessing our conjecture. First, does the revenue comparison hold up for arbitrary distributions? Maskin and Riley (1991) show for the two bidder case that with certain kinds of distributional heterogeneity the English auction dominates the first price. They provide sufficient conditions on the valuation distributions for dominance of the first price auction.[20] The distributional heterogeneity considered in this paper is not covered by their conditions.[21] However, Maskin and Riley (1991) intuitively characterize distributions that will lead to revenue dominance of the first price auction. "When the asymmetry results because one buyer places sufficiently higher probability on high valuations then it is the sealed bid auction which dominates" (p. 29). It is certainly the case that the valuation distributions for coalitions in this paper possess this property. Intuition for dominance of the first price auction in our case is available by recalling that our problem is equivalent to one where at least one of the bidders is risk averse. The revenue superiority of the first price auction with risk averse bidders is a well known result from auction theory.

The numerics show that expected revenue is higher for first price auctions, *given* the size of the coalition. But, are collusive agreements easier to reach in first price auctions? The results in Tables III and IV suggest that this is unlikely. Note that when the coalition of k_1 faces k_2 individuals at the second price auction coalition members always do better than individual bidders and the difference grows as k_1 grows. In contrast, at the first price auction the k_2 individual bidders always do as well[22] or better than the coalition and the difference grows as k_1 grows. This argument is strengthened by the results displayed in Fig. 3 and Table V concerning the case of $k_1 + k_2 = 101$. One hundred colluding bidders generate a very large external benefit for the single noncolluding bidder at a first price auction. In fact, the outside bidder's expected surplus exceeds what he would obtain as a member of an all-inclusive coalition![23]

[20] Maskin and Riley's (1991) Proposition 3.6 states:
Sufficient Conditions for the Superiority of the High Bid Auction: If

$$F_1(v)/F_2(v) \text{ is increasing in } v \tag{a}$$

and for all v, w, $w > v$

$$F_1(w) < F_2(v) \Rightarrow F_1'(w) < F_2'(v), \tag{b}$$

then expected revenue is higher in the sealed high bid auction that in the open auction.

[21] Our distributions violate condition (b) of Maskin and Riley's (1991) Proposition 3.6 as stated in the previous footnote. This illustrates a central role of numerical analysis—suggesting the direction of new theoretical analysis.

[22] For the case of a coalition $k_1 = 1$ facing $k_2 = 4$ individual bidders at a first price auction we know that per capita revenue is identical for all bidders and the advantage to k_1 in Table III is not significant.

[23] The result is analogous to that of Salant *et al.* (1983) where a merger between two of three Cournot competing firms benefits the firm who did not participate in the merger.

This observation leads to the third issue. Is participation in the coalition at a first price auction individually rational? Individual defection is not profitable for $k_1 + k_2 = 5$. Specifically, any coalition member considering leaving a coalition of size k_1 to become one of $k_2 + 1$ noncooperative bidders who face a coalition of size $k_1 - 1$ finds this to be unprofitable. Of course, it would also be unprofitable to defect from the coalition if the consequence would be that everyone bid noncooperatively. However, as noted in the last paragraph, for $k_1 = 101$ and $k_2 = 0$ individual defection is profitable. Table V also reveals that defection from a coalition of $k_1 = 100$ is profitable too. It is important to note that if defection results in everyone bidding noncooperatively then participation in the coalition is still individually rational. However, given that individual defection is not profitable for $k_1 = 5$ in Table III but is for $k_1 = 101$ in Table V, it seems reasonable to conjecture that small all-inclusive coalitions might be feasible at first price auctions while large all-inclusive coalitions might be infeasible.

Suppose we relax the assumption that valuations are known within the coalition. Intuitively, imposition of the incentive compatibility constraint cannot increase expected surplus for the coalition—the coalition cannot do worse by solving an unconstrained problem instead of a constrained one. Without providing explicit details, Graham *et al.* (1990) identify a mechanism that a coalition can employ at a second price auction (called a second price preauction knockout, or "PAKT") to elicit valuations from members that result in the same auctioneer revenues and bidder surpluses as reported in Tables IV and V (for the second price auction). Clearly, any incentive-compatible valuation elicitation mechanism used by a coalition at a first price auction could not do better than what is reported in Tables III and V (for the first price auction). In this light, these first price coalition calculations are as favorable as possible from the coalition's viewpoint.

Finally, have we disadvantaged coalitions at first price auctions in our comparisons by not allowing for side payments among coalition members? This is unlikely. Again, suppose types are known within the coalition and the highest valued bidder within the coalition will win if the coalition wins. The coalition will need to charge the winner a premium above the price paid at the main auction in order to generate revenues to fund the side payments. However, if a coalition member can submit a bid as a noncooperative bidder, then by bidding ε above the bid that the coalition would bid on his behalf he can secure the item for himself whenever the coalition would have won for him and not share any of his surplus with other bidders. In other words, the participation constraint seems to drive us toward the absence of side payments.

6. Conclusion

We have proposed an algorithm for evaluating bid functions for specific kinds of distributionally heterogeneous bidders at a first price auction. The algorithm has proven to be very efficient and numerically stable "despite" singularities at the origin. It also appears to be generalizable to a broader class of asymmetric first price auction problems though, in the present paper, we have limited outselves to outlining the principles underlying such generalizations. Numerical examples have been provided for auctions involving 5 and, in addition, 101 players. We believe that such calculations improve our initial understanding of a broad range of (collusion related) issues that have proved hitherto analytically untractable.

We emphasize the usefulness of numerical methods in probing and formulating conjectures. For example, the four graphs in Fig. 2 seem to demonstrate a smooth transition between cases. However, consider the coalition of 2 versus 3 individual bidders. When our analysis of this case was first completed there were absolutely no results in the literature regarding the existence of equilibria in the four bidder case, much less its smoothness. Also, consider the case of a coalition of 3 versus 2 individuals in the same figure. Prior to generating the equilibrium bid functions we did not have any result to suggest that there was a common terminal point (the logic of footnote 10 regarding the common terminal point does not cover all relevant cases with three bidders). Since producing the bid functions we have demonstrated that the terminal point for the three bidders is the same.[24]

[24] Let t_2^* and t_1^* denote the terminal bids for the individuals and coalition, respectively. The nontrivial case to eliminate is $t_2^* > t_1^*$. This is conceivable since each individual bidder faces another individual bidder and the coalition. Note that with $v = 1$ Eq. (2) equals 0 for $t = t_1^*$. Hence, $\forall t > t_1^*$, we have

$$k_2(1 - t)\lambda_2'(t) - \lambda_2(t) < 0. \tag{A}$$

Let $v_1^* = \lambda_2(t_1^*)$. For individuals with $v > v_1^*$ there is no competition from the coalition. So, the objective function is

$$(v - t) \cdot (\lambda_2(t))^{k_2 - 1}.$$

The first-order condition evaluated at t_2^* is

$$(k_2 - 1)(1 - t_2^*)\lambda_2'(t_2^*) = \lambda_2(t_2^*). \tag{B}$$

Substituting (B) into (A) yields

$$\lambda_2(t_2^*)\frac{1}{k_2 - 1} < 0$$

which is not possible. Hence, $t_1^* = t_2^*$. We are grateful to Dan Levin for encouraging us to produce this result.

APPENDIX A: ANALYTICAL RESULTS

Partial analytical results are available for the system of differential equations defined by formulas (2) to (5). Consider the transformed system as given in formulas (10) to (12), deleting the argument t for ease of notation. Equations (10) and (11) imply the relationship

$$\frac{\delta_1'}{\delta_1}\cdot\left(\frac{1}{k_2(\delta_1-1)}-1\right)=\frac{\delta_2'}{\delta_2}\cdot\left(\frac{1}{k_1(\delta_2-1)}-1\right) \tag{43}$$

or, equivalently,

$$\frac{1}{k_2}\cdot\delta_1'\cdot\left(\frac{1}{\delta_1-1}-\frac{k_2+1}{\delta_1}\right)=\frac{1}{k_1}\cdot\delta_2'\cdot\left(\frac{1}{\delta_2-1}-\frac{k_1+1}{\delta_2}\right). \tag{44}$$

Integration, followed by exponentiation, generates the identity

$$\left(\frac{\delta_1-1}{\delta_1^{k_2+1}}\right)^{k_1}=C\cdot\left(\frac{\delta_2-1}{\delta_2^{k_1+1}}\right)^{k_2} \tag{45}$$

where C is a constant. If we let t tend toward zero, accounting for the initial condition in Eq. (12), we find that

$$C=\frac{(1+k_1)^{k_2}}{(1+k_2)^{k_1}}\cdot\left(\frac{k_2(1+k_1)}{k_1(1+k_2)}\right)^{k_1k_2}. \tag{46}$$

If we then replace t by t_* and use the terminal condition in Eq. (12), we find that

$$t_*=1-C^{1/(k_1-k_2)}. \tag{47}$$

The numerical values which are reported in Table I coincide up to 8 significant digits with these theoretical values. Routine calculations reveal that the limit of t_* is $\frac{3}{4}$ for $k_2 = 1$ as $k_1 \to \infty$. The bid functions in Fig. 3a are close to this limit ($t_* = 0.739$ for $k_1 = 100$ and $k_2 = 1$).

APPENDIX B: CONVERGENCE CRITERION

The application of the convergence criterion (21) requires additional qualification. As k_1 and k_2 increase we observe greater numerical instability in the immediate neighborhood of the origin. On the other hand, convergence toward the limiting values l_1 and l_2 takes place increasingly faster (in line with the fact that the gap between l_1 and l_2 gets smaller) and, in fact, is achieved for all practical purposes before the numerical solutions start diverging. Convergence being monotone we can easily adapt the criterion (21) in order to cope with local instability around the origin. Specifically, we evaluate the distance

$$\varepsilon_j=\{[\delta_1(t_j)-l_1]^2+[\delta_2(t_j)-l_2]^2\}^{1/2} \tag{48}$$

for each intermediate point t_j in (0, 1). Our modified convergence criterion is then given by

$$\operatorname*{Min}_{j} \varepsilon_j^2 \leq \varepsilon^2. \tag{49}$$

Furthermore, once a value t_* has been found for which (49) holds, the δ's are assigned their limiting values for all values of t which are less than that which minimizes ε_j^2. In practice we search for a t_* that solves the optimization problem

$$\varepsilon_*^2 = \operatorname*{Min}_{t_*} (\operatorname*{Min}_{j} \varepsilon_j^2) \tag{50}$$

up to the required accuracy and then use the corresponding ε_* to decide upon the critical value of t below which the δ's are set at their limiting value. The values which are reported for ε in Tables I and II correspond to such pairs (ε_*, t_*).

APPENDIX C: A CHAIN RULE FOR TAYLOR SERIES EXPANSION

Efficient evaluation of Taylor series expansion for such functions as $F_i^{-1}(\lambda_i^*(t))$ in Eqs. (39) and (40) necessitate the implementation of an appropriate chain rule. Let us briefly discuss such a rule, using a self-explanatory set of notations distinct from that used in the rest of the paper. That rule follows immediately from a lemma which is probably well-known but is included here for completeness since we have found no references for it.

LEMMA. *Let*

$$f(u) = \sum_{i=0}^{\infty} f_i (u - u_0)^i \tag{51}$$

$$g(t) = \sum_{i=0}^{\infty} g_i (t - t_0)^i, \tag{52}$$

together with $u_0 = g(t_0)$. *Then*

$$f(g(t)) = \sum_{i=0}^{\infty} a_i (t - t_0)^i, \tag{53}$$

where $a_0 = f_0$ *and for* $i \geq 1$

$$a_i = \sum_{l=1}^{i} f_l \cdot \theta_{l,i} \tag{54}$$

and where the θ *are evaluated recursively as follows:*

$$\theta_{1,1} = g_1 \tag{55}$$

$$\theta_{l,i} = \sum_{j=1}^{i+1-l} g_j \cdot \theta_{l-1,i-j}, \qquad 1 \leq l \leq i. \tag{56}$$

Proof. The proof exploits the uniqueness of the Taylor series expansion in (53), thereby avoiding explicit chain rule derivations of $f(g(t))$. We have

$$\begin{aligned} f(g(t)) &= \sum_{l=0}^{\infty} f_l (g(t) - g(t_0))^l \\ &= \sum_{l=0}^{\infty} f_l \left[\sum_{j=1}^{\infty} g_j (t - t_0)^j \right]^l . \end{aligned} \tag{57}$$

Hence a_i is indeed given by formula (54) where $\theta_{l,i}$ denotes the coefficient of $(t - t_0)^i$ in the lth power of the factor in brackets in the r.h.s. of Eq. (57), whence $\theta_{l,i} = 0$ for $l > i$. The proof follows from the fact that

$$\sum_{i=l}^{\infty} \theta_{l,i} (t - t_0)^i = \left[\sum_{j=l-1}^{\infty} \theta_{l-1,j} (t - t_0)^j \right] \cdot \left[\sum_{k=1}^{\infty} g_k (t - t_0)^k \right]. \tag{58}$$

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